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SOME UNIQUENESS RESULTS ON MEROMORPHIC FUNCTIONS SHARING TWO OR THREE SETS

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Abstract. In the paper we study the uniqueness of meromorphic functions and prove some theorems which are the improvements of some results earlier given by Yi, Jank and Terglane and a recent result of the first author. Examples are provided to show that some assumptions are sharp.

1. Introduction Definitions and Results

In the paper by meromorphic function we always mean a function which is meromorphic in the open complex plane $\mathbb{C}$. We use the standard notations and definitions of the value distribution theory available in [4]. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r)=o(T(r))$ as $r\to\infty$, outside a possible exceptional set of finite linear measure.

If for some $a\in\mathbb{C}\cup\{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S)=\bigcup_{a\in S}\{z: f(z)-a=0\}$, where each zero is counted according to its multiplicity. We denote by $\overline{E}_f(S)$ the set contains the same points as that of $E_f(S)$ but without counting multiplicities. If

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$E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

Let m be a positive integer or infinity and $a \in \mathbb{C} \cup \{\infty\}$. We denote by $E_{m)}(a; f)$ the set of all a-points of f with multiplicities not exceeding m, where an a-point is counted according to its multiplicity. If for some $a \in \mathbb{C} \cup \{\infty\}$, $E_{\infty)}(a; f)$=$E_{\infty)}(a; g)$ we say that f, g share the value a CM. For a set S of distinct elements of $\mathbb{C}$ we define $E_{m)}(S, f) = \bigcup_{a\in S} E_{m)}(a, f)$.

In the paper unless otherwise stated we denote by S_1, S_2 and S_3 the following three sets $S_1 = \{1, \omega, \omega^2, \dots, \omega^{n-1}\}$, $S_2 = \{0\}$ and $S_3 = \{\infty\}$, where $\omega = cos\frac{2\pi}{n} + isin\frac{2\pi}{n}$ and n is a positive integer.

Improving and extending all the previous results (c.f. [2], [3], [14]) related to the problem of uniqueness of two meromorphic functions f, g for which $E_f(S_i) = E_g(S_i)$, where $i = 1, 2, 3$.

Yi [15] and independently Tohge [12] proved the following theorem.

Theorem A. *Let f and g be two non-constant meromorphic functions such that $E_f(S_i) = E_g(S_i)$, where $i = 1, 2, 3$. If $n \geq 2$ then one of the following hold:*

$$f \equiv tg, \tag{1.1}$$

where $t^n = 1$ or

$$f.g \equiv s, \tag{1.2}$$

where 0, ∞ are lacunary values of f and g, and $s^n = 1$.

In 1991 Jank and Terglane [10] improved Theorem A as follows.

Theorem B. *Let f and g be two non-constant meromorphic functions such that $E_f(S_1) = E_g(S_1)$, $E_f(S_2) = E_g(S_2)$ and $\overline{E}_f(S_3) = \overline{E}_g(S_3)$. If $n \geq 2$ then f, g satisfy* (1.1) *or* (1.2).

In 1997 H.X.Yi [17] proved the following theorems.

Theorem C. *Let f and g be two non-constant meromorphic functions such that $E_f(S_1) = E_g(S_1)$, $\overline{E}_f(S_2) = \overline{E}_g(S_2)$ and $E_f(S_3) = E_g(S_3)$. If $n \geq 2$ then f, g satisfy* (1.1) *or* (1.2).

Theorem D. *Let f and g be two non-constant meromorphic functions such that $E_f(S_1) = E_g(S_1)$, $\overline{E}_f(S_2) = \overline{E}_g(S_2)$ and $\overline{E}_f(S_3) = \overline{E}_g(S_3)$. If $n \geq 3$ then f, g satisfy* (1.1) *or* (1.2).

In the paper we relax the nature of sharing the sets in the above mentioned theorems.

In 1997, H.X.Yi and L.Z.Yang [19] proved the following result.

Theorem E. *Let f and g be two non-constant meromorphic functions such that $E_f(S_1) = E_g(S_1)$ and $\overline{E}_f(S_3) = \overline{E}_g(S_3)$. If $n \geq 6$ then then f, g satisfy* (1.1) *or* (1.2)*.*

In 2001 Lahiri introduced the idea of weighted sharing of values and sets in [6], [7]. In the following definition we explain the notion.

Definition 1.1. [6], [7] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [6] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.

Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$.

In 2006 Lahiri and Banerjee [8] have improved Theorem E by relaxing the nature of sharing the sets with the idea of weighted sharing of values and sets which we have just discussed.

Recently the first author [1] have also investigated the problem of uniqueness of two meromorphic functions sharing the two sets S_1 and S_3 and improved and supplemented the results of Yi-Yang [19] and Lahiri-Banerjee [8].

In [1] the first author proved the following theorem.

Theorem F. *Let f and g be two non-constant meromorphic functions such that $E_{2)}(S_1, f) = E_{2)}(S_1, g)$, $E_f(S_3, 0) = E_g(S_3, 0)$ and $n \geq 8$ then f, g satisfy* (1.1) *or* (1.2)*.*

In this paper we shall improve Theorem F by showing that the assumption $n \geq 8$ can be replaced by $n \geq 7$.

Following theorems are the main results of the paper.

Theorem 1.1. *If* $E_{m)}(S_1, f) = E_{m)}(S_1, g)$, $E_f(S_2, 0) = E_g(S_2, 0)$, $E_f(S_3, k) = E_g(S_3, k)$, *where* $k(2m - 17) > 12$ *and* $n \geq 2$ *then* f, g *satisfy one of* (1.1) *or* (1.2).

Theorem 1.2. *If* $E_{m)}(S_1, f) = E_{m)}(S_1, g)$, $E_f(S_2, p) = E_g(S_2, p)$, $E_f(S_3, 0) = E_g(S_3, 0)$, *where* $p(2m - 17) > 12$ *and* $n \geq 2$ *then* f, g *satisfy one of* (1.1) *or* (1.2).

Theorem 1.3. *If* $E_{7)}(S_1, f) = E_{7)}(S_1, g)$, $E_f(S_2, 0) = E_g(S_2, 0)$, $E_f(S_3, 0) = E_g(S_3, 0)$ *and* $n \geq 3$ *then* f, g *satisfy one of* (1.1) *or* (1.2).

Theorem 1.4. *If* $E_{2)}(S_1, f) = E_{2)}(S_1, g)$, $E_f(S_3, 0) = E_g(S_3, 0)$ *and* $n \geq 7$ *then* f, g *satisfy* (1.1) *or* (1.2).

Example 1.1. Let $f(z) = (1 - e^z)^3$ and $g(z) = 3(e^{-z} - e^{-2z})$ and $S_1 = \{1\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$. Clearly f and g share (S_1, ∞), $(S_2, 0)$, (S_3, ∞), but neither condition (1.1) nor (1.2) is satisfied. So the condition $n \geq 2$ in Theorem 1.1 is the best possible.

Example 1.2. Let $f(z) = \dfrac{(1 - 3e^z)}{(1 - e^z)^3}$ and $g(z) = \dfrac{(1 - 3e^z)}{4(1 - e^z)}$ and and S_1, S_2, S_3 be same as defined in Example 1.1. Clearly f and g share (S_1, ∞), (S_2, ∞), $(S_3, 0)$, but neither condition (1.1) nor (1.2) is satisfied. So the condition $n \geq 2$ in Theorem 1.2 is the best possible.

Example 1.3. Let $f(z) = \dfrac{(e^{2z} + 1)^2}{2e^z(e^{2z} - 1)}$ and $g(z) = \dfrac{2ie^z(e^{2z} + 1)}{(e^{2z} - 1)^2}$ and $S_1 = \{-1, 1\}$, $S_2 = \{0\}$, $S_3 = \{\infty\}$. Clearly f and g share (S_1, ∞), $(S_2, 0)$, $(S_3, 0)$, but neither condition (1.1) nor (1.2) is satisfied. So the condition $n \geq 3$ in Theorem 1.3 is the best possible.

Corollary 1.1. *When* $k = \infty$ *and* $p = \infty$ *both Theorem 1.1, Theorem 1.2 hold for* $m \geq 9$.

We explain some definitions and notations which are used in the paper.

Definition 1.3. [5] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a-points of f whose multiplicities are not greater (less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f \mid\leq m)(\overline{N}(r,a;f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f \mid< m)$, $N(r,a;f \mid> m)$, $\overline{N}(r,a;f \mid< m)$ and $\overline{N}(r,a;f \mid> m)$ are defined analogously.

Definition 1.4. We denote by $\overline{N}(r,a;f \mid= k)$ the reduced counting function of those a-points of f whose multiplicity is exactly k, where $k \geq 2$ is an integer.

Definition 1.5. Let f and g be two non-constant meromorphic functions such that f and g share a value a IM where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r,a;f)$ $(\overline{N}_L(r,a;g))$ the counting function of those a-points of f and g where $p > q$ $(q > p)$, each a-point is counted only once.

Definition 1.6. Let f and g be two non-constant meromorphic functions and m be a positive integer such that $E_{m)}(a;f) = E_{m)}(a;g)$ where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity $p > 0$, an a-point of g with multiplicity $q > 0$. We denote by $\overline{N}_L^{(m+1}(r,a;f)$ $(\overline{N}_L^{(m+1}(r,a;g))$ the counting function of those common a-points of f and g where $p > q$ $(q > p)$, each a-point is counted only once.

Definition 1.7. Let z_0 be a 1-point of f with multiplicity p, a 1-point of g with multiplicity q. We denote by $\overline{N}_E^{(m+1}(r,1;f)$ the counting function of those 1-points of f and g where $p = q \geq m+1$, each point in this counting function is counted only once.

Definition 1.8. [7] We denote by $N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2)$.

Definition 1.9. Let m be a positive integer. Also let z_0 be a zero of $f(z) - a$ of multiplicity p and a zero of $g(z) - a$ of multiplicity q. We denote by $\overline{N}_{f\geq m+1}(r,a;f \mid g \neq a)$ $(\overline{N}_{g\geq m+1}(r,a;g \mid f \neq a))$ the reduced counting functions of those a-points of f and g for which $p \geq m+1$ and $q = 0$ $(q \geq m+1$ and $p = 0)$.

Definition 1.10. [6], [7] Let f, g share $(a,0)$. We denote by $\overline{N}_*(r,a;f,g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$ and $\overline{N}_*(r,a;f,g) = \overline{N}_L(r,a;f) + \overline{N}_L(r,a;g)$.

Definition 1.11. For $E_{m)}(a;f)=E_{m)}(a;g)$ we define $\overline{N}_{\otimes}(r,a;f,g)$ as follows

$$\begin{aligned}
& \overline{N}_{\otimes}(r,a;f,g)\\
= & \overline{N}_{L}^{(m+1}(r,a;f)+\overline{N}_{L}^{(m+1}(r,a;g)+\overline{N}_{f\geq m+1}(r,a;f\mid g\neq a)\\
& +\overline{N}_{g\geq m+1}(r,a;g\mid f\neq a)\\
\leq & \overline{N}(r,a;f\mid\geq m+1)+\overline{N}(r,a;g\mid\geq m+1).
\end{aligned}$$

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. Henceforth we shall denote by H, U and V the following three functions.

$$H=\left(\frac{F''}{F'}-\frac{2F'}{F-1}\right)-\left(\frac{G''}{G'}-\frac{2G'}{G-1}\right),$$

$$U=\frac{F'}{F-1}-\frac{G'}{G-1}$$

and

$$V=\frac{F'}{F-1}-\frac{F'}{F}-\left(\frac{G'}{G-1}-\frac{G'}{G}\right)=\frac{F'}{F(F-1)}-\frac{G'}{G(G-1)}.$$

Lemma 2.1. [11] *For $E_{m)}(1;F)=E_{m)}(1;G)$ and $H\not\equiv 0$ then*

$$N(r,1;F\mid=1)=N(r,1;G\mid=1)\leq N(r,H)+S(r,F)+S(r,G).$$

Lemma 2.2. [9] *If $N(r,0;f^{(k)}\mid f\neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r,0;f^{(k)}\mid f\neq 0)\leq k\overline{N}(r,\infty;f)+N(r,0;f\mid<k)+k\overline{N}(r,0;f\mid\geq k)+S(r,f).$$

Lemma 2.3. *Let $E_{m)}(1;f)=E_{m)}(1;g)$ and $3\leq m<\infty$. Then*

$$\begin{aligned}
& \overline{N}(r,1;f\mid=2)+2\overline{N}(r,1;f\mid=3)+\ldots(m-1)\overline{N}(r,1;f\mid=m)\\
& +m\overline{N}_{E}^{(m+1}(r,1;f)+m\overline{N}_{L}^{(m+1}(r,1;f)+(m+1)\overline{N}_{L}^{(m+1}(r,1;g)\\
& +m\overline{N}_{g\geq m+1}(r,1;g\mid\ f\neq 1)\\
\leq & N(r,1;g)-\overline{N}(r,1;g).
\end{aligned}$$

Proof. Since $E_{m)}(1;f) = E_{m)}(1;g)$, we note that common zeros of $f-1$ and $g-1$ up to multiplicity m are same. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $q = m+1$ the possible values of p are as follows (i) $p = m+1$ (ii) $p \geq m+2$ (iii) $p = 0$. Similarly when $q = m+2$ the possible values of p are (i) $p = m+1$ (ii) $p = m+2$ (iii) $p \geq m+3$ (iv) $p = 0$. If $q \geq m+3$ we can similarly find the possible values of p. Now the lemma follows from above explanation. □

Lemma 2.4. *Let $E_{2)}(1;f) = E_{2)}(1;g)$. Then*

$$\overline{N}(r,1;f\mid=2) + 2\overline{N}_E^{(3}(r,1;f) + 2\overline{N}_L^{(3}(r,1;f) + 2\overline{N}_L^{(3}(r,1;g) + 2\overline{N}_{g\geq 3}(r,1;g\mid\ f\neq 1)$$
$$\leq N(r,1;g) - \overline{N}(r,1;g).$$

Proof. Since $E_{2)}(1;f) = E_{2)}(1;g)$, we note that the simple and double 1-points of f and g are same. Let z_0 be a 1-point of f with multiplicity p and a 1-point of g with multiplicity q. If $q = 3$ the possible values of p are as follows (i) $p = 3$ (ii) $p \geq 4$ (iii) $p = 0$. Similarly when $q = 4$ the possible values of p are (i) $p = 3$ (ii) $p = 4$ (iii) $p \geq 5$ (iv) $p = 0$. If $q \geq 5$ we can similarly find the possible values of p. Now the lemma follows from above explanation. □

Lemma 2.5. *Let $E_{m)}(1;F) = E_{m)}(1;G)$ and F, G share $(\infty;0)$. Also let $H \not\equiv 0$. Then*

$$\begin{aligned} N(r,\infty;H) &\leq \overline{N}(r,0;F\mid\geq 2) + \overline{N}(r,0;G\mid\geq 2) + \overline{N}_*(r,\infty;F,G) \\ &\quad + \overline{N}_{\otimes}(r,1;F,G) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Proof. The proof of the lemma can be carried out in the line of the proof of Lemma 4 [8]. So we omit it. □

Lemma 2.6. *Let $E_{m)}(1;F) = E_{m)}(1;G)$ and F, G share $(0,p)$ and $(\infty;k)$. Also let $H \not\equiv 0$. Then*

$$\begin{aligned} N(r,\infty;H) &\leq \overline{N}_*(r,0;F;G) + \overline{N}_*(r,\infty;F,G) + \overline{N}_{\otimes}(r,1;F,G) \\ &\quad + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'). \end{aligned}$$

Proof. We omit the proof since the proof can be carried out in the line of proof of Lemma 2.5. □

Henceforth we assume

$$F = f^n \quad and \quad G = g^n. \tag{2.1}$$

Lemma 2.7. *Let F, G be given by (2.1) and $H \not\equiv 0$. If $E_{m)}(1;F) = E_{m)}(1;G)$, f, g share (∞, k), $(0,p)$, where $3 \leq m < \infty$. Then*

$$\begin{aligned} & nT(r,f) \\ & \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;g) + \overline{N}_*(r,0;f,g) \\ & \quad + \overline{N}_*(r,\infty;f,g) - m(r,1;G) - \overline{N}(r,1;F \mid= 3) - \ldots \\ & \quad - (m-2)\overline{N}(r,1;F \mid= m) - (m-2)\overline{N}_L^{(m+1}(r,1;F) \\ & \quad - (m-1)\overline{N}_L^{(m+1}(r,1;G) - (m-1)\overline{N}_E^{(m+1}(r,1;F) \\ & \quad + 2\overline{N}_{F\geq m+1}(r,1;F \mid G \neq 1) - (m-1)\overline{N}_{G\geq m+1}(r,1;G \mid F \neq 1) \\ & \quad + S(r,f) + S(r,g). \end{aligned}$$

Similar expressions also hold for g.

Proof. By the second fundamental theorem we get

$$\begin{aligned} & T(r,F) + T(r,G) \\ \leq \; & \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) \\ & + \overline{N}(r,1;F) + \overline{N}(r,1;G) - N_0(r,0;F') - N_0(r,0;G') \\ & + S(r,F) + S(r,G). \end{aligned} \tag{2.2}$$

Using Lemmas 2.1, 2.3 and 2.6 we see that

$$\begin{aligned} & \overline{N}(r,1;F) + \overline{N}(r,1;G) \\ \leq \; & N(r,1;F \mid= 1) + \overline{N}(r,1;F \mid= 2) + \overline{N}(r,1;F \mid= 3) + \ldots + \overline{N}(r,1;F \mid= m) \\ & + \overline{N}_E^{(m+1}(r,1;F) + \overline{N}_L^{(m+1}(r,1;F) + \overline{N}_L^{(m+1}(r,1;G) + \overline{N}_{F\geq m+1}(r,1;F \mid G \neq 1) \\ & + \overline{N}(r,1;G) \\ \leq \; & \overline{N}_*(r,0;f,g) + \overline{N}_*(r,\infty;f,g) + \overline{N}_\otimes(r,1;F,G) + \overline{N}(r,1;F \mid= 2) + \ldots \\ & + \overline{N}(r,1;F \mid= m) + \overline{N}_E^{(m+1}(r,1;F) + \overline{N}_L^{(m+1}(r,1;F) + \overline{N}_L^{(m+1}(r,1;G) \\ & + \overline{N}_{F\geq m+1}(r,1;F \mid G \neq 1) + T(r,G) - m(r,1;G) + O(1) - \overline{N}(r,1;F \mid= 2) \\ & - 2\overline{N}(r,1;F \mid= 3) - (m-1)\overline{N}(r,1;F \mid= m) - \ldots - m\overline{N}_E^{(m+1}(r,1;F) \\ & - m\overline{N}_L^{(m+1}(r,1;F) - (m+1)\overline{N}_L^{(m+1}(r,1;G) - m\overline{N}_{G\geq m+1}(r,1;G \mid F \neq 1) \\ & + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G') + S(r,F) + S(r,G) \end{aligned} \tag{2.3}$$

$$\begin{aligned}
\leq\ & \overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)+T(r,G)-m(r,1;G)-\overline{N}(r,1;F\mid=3)\\
&-2\overline{N}(r,1;F\mid=4)-\ldots-(m-2)\overline{N}(r,1;F\mid=m)-(m-2)\overline{N}_L^{(m+1}(r,1;F)\\
&-(m-1)\overline{N}_L^{(m+1}(r,1;G)-(m-1)\overline{N}_E^{(m+1}(r,1;F)\\
&-(m-1)\overline{N}_{G\geq m+1}(r,1;G\mid F\neq 1)+2\overline{N}_{F\geq m+1}(r,1;F\mid G\neq 1)\\
&+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,F)+S(r,G).
\end{aligned}$$

Using (2.3) in (2.2), the lemma follows. □

Lemma 2.8. *Let F, G be given by* (2.1) *and $H\not\equiv 0$. If $E_{2)}(1;F)=E_{2)}(1;G)$, f, g share $(\infty,0)$. Then*

$$\begin{aligned}
&nT(r,f)\\
\leq\ & N_2(r,0;F)+\overline{N}(r,\infty;f)+N_2(r,0;G)+\overline{N}(r,\infty;g)+\overline{N}_*(r,\infty;f,g)\\
&-m(r,1;G)-\overline{N}_E^{(3}(r,1;F)+2\overline{N}_{F\geq 3}(r,1;F\mid G\neq 1)-\overline{N}_{G\geq 3}(r,1;G\mid F\neq 1)\\
&+S(r,f)+S(r,g).
\end{aligned}$$

Similar expressions also hold for g.

Proof. We omit the proof since using Lemmas 2.1, 2.4 and 2.5 the proof can be carried out in the line of proof of Lemma 2.7. □

Lemma 2.9. [17] *Let F, G be given by (2.2). If F, G share $(0,0)$ and $U\equiv 0$ then $F\equiv G$.*

Lemma 2.10. [17] *Let F, G be given by (2.2). If F, G share $(\infty,0)$ and $V\equiv 0$ then $F\equiv G$.*

Lemma 2.11. *Let F, G be given by* (2.1) *and $F\not\equiv G$. If $E_{m)}(S_1;f)=E_{m)}(S_1;g)$, $E_f(S_2,p)=E_g(S_2,p)$ and $E_f(S_3,k)=E_g(S_3,k)$, where $1\leq m<\infty$, $0\leq p<\infty$, $0\leq k<\infty$ then*

$$\begin{aligned}
&(n-1)\overline{N}(r,0;f\mid=1)+(2n-1)\overline{N}(r,0;f\mid=2)+\ldots\\
&+\left(np+n-1-\frac{1}{nk+n-1}\right)\overline{N}(r,0;f\mid\geq p+1)\\
\leq\ & \frac{nk+n}{nk+n-1}\overline{N}_\otimes(r,1;F,G)+S(r).
\end{aligned}$$

Proof. Since $F\not\equiv G$ we have from Lemmas 2.9 and 2.10 that $U\not\equiv 0$ and $V\not\equiv 0$. According to the statement of the lemma it is clear that $E_{m)}(1;F)=E_{m)}(1;G)$

and F, G share $(0;np)$, $(\infty;nk)$ and so a zero (pole) of F with multiplicity $r \geq np+1(\geq nk+1)$ is a zero (pole) of G with multiplicity $s \geq np+1(\geq nk+1)$ and vice versa. We note that F and G have no zero (pole) of multiplicity q where $np < q < np+n(nk < q < nk+n)$. Hence we get from the definition of U

$$\begin{aligned}
(2.4)\qquad & (n-1)N(r,0;f\mid=1)+(2n-1)\overline{N}(r,0;f\mid=2)+\ldots \\
& +(np+n-1)\,\overline{N}(r,0;f\mid\geq p+1) \\
= \; & (n-1)\overline{N}(r,0;F\mid=n)+(2n-1)\overline{N}(r,0;F\mid=2n)+\ldots \\
& +(np+n-1)\overline{N}(r,0;F\mid\geq np+n) \\
\leq \; & N(r,0;U) \\
\leq \; & T(r,U)+O(1) \\
\leq \; & N(r,\infty;U)+S(r) \\
\leq \; & \overline{N}_*(r,\infty;F,G)+\overline{N}_{\otimes}(r,1;F,G)+S(r) \\
\leq \; & \overline{N}(r,\infty;F\mid\geq nk+n)+\overline{N}_{\otimes}(r,1;F,G)+S(r).
\end{aligned}$$

In a similar argument as above we get from the definition of V

$$\begin{aligned}
(2.5)\qquad & (nk+n-1)\overline{N}(r,\infty;F\mid\geq nk+n) \\
\leq \; & N(r,0;V) \\
\leq \; & N(r,\infty;V)+S(r) \\
\leq \; & \overline{N}(r,0;F\mid\geq np+n)+\overline{N}_{\otimes}(r,1;F,G)+S(r).
\end{aligned}$$

Using (2.5) in (2.4) the lemma follows. □

Lemma 2.12. *Let F, G be given by (2.1) and $F \not\equiv G$. If $E_{m)}(S_1;f)=E_{m)}(S_1;g)$, $E_f(S_2,p)=E_g(S_2,p)$ and $E_f(S_3,k)=E_g(S_3,k)$, where $1\leq m<\infty$, $0\leq p<\infty$, $0\leq k<\infty$ then*

$$\begin{aligned}
& (n-1)\overline{N}(r,\infty;f\mid=1)+(2n-1)\overline{N}(r,\infty;f\mid=2)+\ldots \\
& +\left(nk+n-1-\frac{1}{np+n-1}\right)\overline{N}(r,\infty;f\mid\geq k+1) \\
\leq \; & \frac{np+n}{np+n-1}\overline{N}_{\otimes}(r,1;F,G)+S(r).
\end{aligned}$$

Proof. We omit the proof since it can be carried out in the line of proof of Lemma 2.11. □

Lemma 2.13. [1] *Let F, G be given by* (2.1) *and* $V \not\equiv 0$. *If* f, g *share* (∞, k), *where* $0 \leq k < \infty$, *and* $E_{m)}(1;F)=E_{m)}(1;G)$, *then*

$$\begin{aligned}(nk+n-1)\,\overline{N}(r,\infty;f \mid\geq k+1) &= (nk+n-1)\overline{N}(r,\infty;F \mid\geq nk+n)\\ &\leq \frac{m+1}{m}[\overline{N}(r,0;f)+\overline{N}(r,0;g)]\\ &\quad + \frac{2}{m}\,\overline{N}(r,\infty;f)+S(r,f)+S(r,g).\end{aligned}$$

Lemma 2.14. [16] *If* $H \equiv 0$ *then* $T(r,G)=T(r,F)+O(1)$. *Also if* $H \equiv 0$ *and*

$$\limsup_{r\longrightarrow\infty,\ r\in I} \frac{\overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,\infty;G)}{T(r,F)} < 1$$

where $I \subset (0,1)$ *is a set of infinite linear measure, then* $F \equiv G$ *or* $F.G \equiv 1$.

Lemma 2.15. [18] *If* $H \equiv 0$, *then* F, G *share* $(1,\infty)$. *If further* F, G *share* $(\infty,0)$ *then* F, G *share* (∞,∞).

Lemma 2.16. *Let* F, G *be given by* (2.1) *and* $n \geq 2$. *Also let* $E_{m)}(1;F)=E_{m)}(1;G)$. *If* f, g *share* $(0,0)$, (∞,k), *where* $0 \leq k < \infty$ *and* $H \equiv 0$. *Then* f, g *satisfy one of* (1.1) *or* (1.2).

Proof. Since $H \equiv 0$ we get from Lemma 2.15 that F and G share $(1,\infty)$ and (∞,∞). So $\overline{N}_{\otimes}(r,1;F,G)=\overline{N}_{*}(r,\infty;F,G)\equiv 0$. If possible let us suppose (1.1) is not satisfied. Then clearly $F \not\equiv G$. Since $F \not\equiv G$ we have from Lemmas 2.9 and 2.10 respectively $U \not\equiv 0$ and $V \not\equiv 0$. Hence

$$\begin{aligned}(n-1)\overline{N}(r,0;f) &= (n-1)\overline{N}(r,0;g)\\ &\leq N(r,0;U)\\ &\leq N(r,\infty;U)+S(r)\\ &\leq \overline{N}_{*}(r,\infty;F,G)+\overline{N}_{\otimes}(r,1;F,G)+S(r)\\ &= S(r).\end{aligned}$$

and

$$\begin{aligned}(n-1)\overline{N}(r,\infty;f) &= (n-1)\overline{N}(r,\infty;g)\\ &\leq N(r,0;V)\\ &\leq N(r,\infty;V)+S(r)\\ &\leq \overline{N}_{*}(r,0;f,g)+\overline{N}_{\otimes}(r,1;F,G)+S(r).\\ &= S(r).\end{aligned}$$

Since $n \geq 2$ we have from above $\overline{N}(r,0;f) = \overline{N}(r,0;g) = S(r)$ and $\overline{N}(r,\infty;f) = \overline{N}(r,\infty;g) = S(r)$. Hence using Lemma 2.14 we get the conclusion of the lemma. □

Lemma 2.17. *Let F, G be given by (2.1), $E_{m)}(1;F) = E_{m)}(1;G)$, $1 \leq m < \infty$. Then*

(i) $\overline{N}(r,1;F \mid\geq m+1) \leq \frac{1}{m}\left[\overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\otimes}(r,0;f')\right] + S(r,f)$,

(ii) $\overline{N}(r,1;G \mid\geq m+1) \leq \frac{1}{m}\left[\overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_{\otimes}(r,0;g')\right] + S(r,g)$, *where* $N_{\otimes}(r,0;f') = N(r,0;f' \mid f \neq 0, \omega_1, \omega_2 \ldots \omega_n)$.

Proof. We prove only (i).
Using Lemma 2.2 we see that

$$\begin{aligned} & \overline{N}(r,1;F \mid\geq m+1) \\ \leq\ & \frac{1}{m}\left(N(r,1;F) - \overline{N}(r,1;F)\right) \\ \leq\ & \frac{1}{m}\left[\sum_{j=1}^{n}\left(N(r,\omega_j;f) - \overline{N}(r,\omega_j;f)\right)\right] \\ \leq\ & \frac{1}{m}\left(N(r,0;f' \mid f \neq 0) - N_{\otimes}(r,0;f')\right) \\ \leq\ & \frac{1}{m}\left[\overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_{\otimes}(r,0;f')\right] + S(r,f). \end{aligned}$$

This proves the lemma. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then $E_{m)}(1;F) = E_{m)}(1;G)$ and f, g share $(0,0)$ and $(\infty;k)$. We consider the following cases.

Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. Noting that f and g share $(0,0)$ and $(\infty;k)$ implies $\overline{N}_*(r,0;f,g) \leq \overline{N}(r,0;f) = \overline{N}(r,0;g)$ and $\overline{N}_*(r,\infty;f,g) \leq \overline{N}(r,\infty;f \mid\geq k+1) = \overline{N}(r,\infty;g \mid\geq k+1)$, using Lemma 2.7, Lemma 2.11 with $p=0$, (2.5) with $k=0$ and $p=0$ and Lemma 2.12 with $p=0$ and Lemma 2.17 we obtain

$$\begin{aligned} & nT(r,f) + nT(r,g) \\ \leq\ & 3\,\overline{N}(r,0;f) + 3\,\overline{N}(r,0;g) + 2\,\overline{N}(r,\infty;f) + 2\,\overline{N}(r,\infty;g) \\ & +2\,\overline{N}_*(r,\infty;f,g) - (m-3)\,\overline{N}_{\otimes}(r,1;F,G) + S(r,f) + S(r,g) \end{aligned} \tag{3.1}$$

$$
\begin{aligned}
&\leq \frac{6n(k+1)}{(n-1)(nk+n-1)-1}\,\overline{N}_{\otimes}(r,1;F,G)\\
&\quad+\left[\frac{4}{(n-1)}+\frac{2n}{(n-1)(nk+n-1)-1}\right.\\
&\quad\left.+\frac{4n(k+1)}{(n-1)\{(n-1)(nk+n-1)-1\}}\right]\overline{N}_{\otimes}(r,1;F,G)\\
&\quad-(m-3)\overline{N}_{\otimes}(r,1;F,G)+S(r,f)+S(r,g)\\
&\leq \left[\frac{n(n-1)(6k+8)+4n(k+1)}{(n-1)\{(n-1)(nk+n-1)-1\}}+\frac{4}{(n-1)}+3-m\right]\overline{N}_{\otimes}(r,1;F,G)\\
&\quad+S(r,f)+S(r,g)\\
&\leq \left[\frac{n(n-1)(6k+8)+4n(k+1)}{(n-1)\{(n-1)(nk+n-1)-1\}}+\frac{4}{(n-1)}+3-m\right]\Big\{\overline{N}(r,1;F\geq m+1)\\
&\quad+\overline{N}(r,1;G\geq m+1)\Big\}+S(r,f)+S(r,g)\\
&\leq \frac{2}{m}\left[\frac{n(n-1)(6k+8)+4n(k+1)}{(n-1)\{(n-1)(nk+n-1)-1\}}+\frac{4}{(n-1)}+3-m\right]T(r,f)\\
&\quad+\frac{2}{m}\left[\frac{n(n-1)(6k+8)+4n(k+1)}{(n-1)\{(n-1)(nk+n-1)-1\}}+\frac{4}{(n-1)}+3-m\right]T(r,g)\\
&\quad+S(r,f)+S(r,g).
\end{aligned}
$$

From (3.1) we see that

$$
\begin{aligned}
(3.2)\qquad &\left(n+2-\frac{6}{m}-\frac{8}{m(n-1)}-\frac{n(12k+16)+\frac{8n(k+1)}{n-1}}{m(n-1)(nk+n-1)-m}\right)T(r,f)\\
&+\left(n+2-\frac{6}{m}-\frac{8}{m(n-1)}-\frac{n(12k+16)+\frac{8n(k+1)}{n-1}}{m(n-1)(nk+n-1)-m}\right)T(r,g)\\
&\leq S(r,f)+S(r,g).
\end{aligned}
$$

Since $n \geq 2$ and $k(2m-17) > 12$, (3.2) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then the theorem follows from Lemma 2.16. □

Proof of Theorem 1.2. We omit the proof since it can be carried out in the line of proof of Theorem 1.1. □

Proof of Theorem 1.3. Let F, G be given by (2.1). Then $E_{7)}(1;F) = E_{7)}(1;G)$ and f, g share $(0,0)$ and $(\infty;0)$. We consider the following cases.

Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. Noting that f and g share $(0,0)$ and $(\infty;0)$ implies $\overline{N}_{*}(r,0;f,g) \leq \overline{N}(r,0;f) = \overline{N}(r,0;g)$ and $\overline{N}_{*}(r,\infty;f,g) \leq \overline{N}(r,\infty;f) = \overline{N}(r,\infty;g)$, using Lemma 2.7, Lemmas 2.11 and 2.12 with $p = 0$, $k = 0$ and Lemma 2.17 we obtain

$$
\begin{aligned}
(3.3) \qquad & nT(r,f) + nT(r,g) \\
\leq\; & 3\,\overline{N}(r,0;f) + 3\,\overline{N}(r,0;g) + 3\,\overline{N}(r,\infty;f) + 3\,\overline{N}(r,\infty;g) \\
& -4\,\overline{N}_{\otimes}(r,1;F,G) + S(r,f) + S(r,g) \\
\leq\; & \frac{6}{n-2}\,\overline{N}_{\otimes}(r,1;F,G) + \frac{6}{n-2}\,\overline{N}_{\otimes}(r,1;F,G) - 4\overline{N}_{\otimes}(r,1;F,G) \\
& + S(r,f) + S(r,g) \\
\leq\; & \left[\frac{12}{(n-2)} - 4\right]\overline{N}_{\otimes}(r,1;F,G) + S(r,f) + S(r,g) \\
\leq\; & \frac{2}{7}\left(\frac{20-4n}{n-2}\right)T(r,f) + \frac{2}{7}\left(\frac{20-4n}{n-2}\right)T(r,g) \\
& + S(r,f) + S(r,g).
\end{aligned}
$$

From (3.3) we see that

$$
(3.4) \quad \left(n - \frac{40-8n}{7(n-2)}\right)T(r,f) + \left(n - \frac{40-8n}{7(n-2)}\right)T(r,g) \leq S(r,f) + S(r,g).
$$

Since $n \geq 3$, (3.4) leads to a contradiction.

Case 2. Let $H \equiv 0$. Then the theorem follows from Lemma 2.16. □

Proof of Theorem 1.4. Let F, G be given by (2.1). Then $E_{2)}(1;F) = E_{2)}(1;G)$ and f, g share $(\infty;0)$. We consider the following cases.

Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. So from Lemma 2.10 we get $V \not\equiv 0$. Hence using Lemmas 2.8, 2.13 with $m = 2$ and $k = 0$ and Lemma 2.17 we obtain

$$
\begin{aligned}
(3.5) \qquad & nT(r,f) + nT(r,g) \\
\leq\; & 4\overline{N}(r,0;f) + 4\overline{N}(r,0;g) + 2\overline{N}(r,\infty;f) + 2\overline{N}(r,\infty;g) \\
& + 2\overline{N}_{*}(r,\infty;f,g) + \overline{N}_{F\geq 3}(r,1;F \mid G \neq 1) \\
& + \overline{N}_{G\geq 3}(r,1;G \mid F \neq 1) + S(r,f) + S(r,g) \\
\leq\; & 4\overline{N}(r,0;f) + 4\overline{N}(r,0;g) + \frac{1}{2}\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} \\
& + 7\overline{N}(r,\infty;f) + S(r,f) + S(r,g) \\
\leq\; & \left(\frac{9}{2} + \frac{21}{2(n-2)}\right)\{\overline{N}(r,0;f) + \overline{N}(r,0;g)\} + S(r,f) + S(r,g).
\end{aligned}
$$

From (3.5) we see that

$$\begin{aligned}(3.6)\qquad & \left(n-\frac{9}{2}-\frac{21}{2(n-2)}\right)T(r,f)+\left(n-\frac{9}{2}-\frac{21}{2(n-2)}\right)T(r,g)\\ \leq\ & S(r,f)+S(r,g).\end{aligned}$$

Since $n \geq 7$, (3.6) leads to a contradiction.

Case 2. Let $H \equiv 0$. Since n (≥ 7) from Lemma 2.14 it follows that f and g satisfy one of (1.1) or (1.2). □

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