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γ-COMPACTNESS IN L-TOPOLOGICAL SPACES

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Abstract. The concepts of γ-compactness, countable γ-compactness, the γ-Lindelöf property are introduced in L-topological spaces by means of γ-open L-sets and their inequalities when L is a complete DeMorgan algebra. These definitions do not rely on the structure of the basis lattice L and no distributivity in L is required.

1. Introduction

Fuzzy sets theory [27], a recent generalization of classical set theory, has attracted the attention of researchers working in various areas including topology, which has had a seminal influence in the development of this new theory.

The concept of compactness of an I-topological space was first introduced by Chang [4] in terms of open cover. Chang's compactness has been greatly extended to the variable-basis case by Rodabaugh [9], and it can be regarded as a successful definition of compactness in poslat topology from the categorical point of view (see [9, 16]). Moreover, Gantner et al. introduced α-compactness [6], Lowen introduced fuzzy compactness, strong fuzzy compactness and ultra-fuzzy compactness [13, 14], Chadwick [3] generalized Lowen's compactness, Liu introduced Q-compactness [11], Li introduced strong Q-compactness [10] which is equivalent to strong fuzzy compactness in [14], Wang and Zhao introduced N-compactness [25, 28], and Shi introduced S^*-compactness [20].

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Recently, Shi presented a new definition of fuzzy compactness in L-topological spaces [23, 19] by means of open L-sets and their inequality where L is a complete DeMorgan algebra. This new definition doesn't depend on the structure of L. When L is completely distributive, it is equivalent to the notion of fuzzy compactness in [12, 13, 26].

In L-topology, the weaker forms of open L-sets, which were constructed by the compositions of different combinations of the closure and interior operator, have been studied by several mathematicians. In general topology, the class of b-open (or γ-open) sets was presented in [1]. In 1996, Hanafy [8] defined the class of γ-open L-sets (in the case $L = [0, 1]$) as an extension of b-open sets to L-topology.

In this paper, following the lines of [19, 20, 23], we will introduce the γ-compactness in L-topological spaces by means of γ-open L-sets and their inequality. We also introduce countable γ-compactness and the γ-Lindelöf property in L-topology. These definitions do not rely on the structure of the basis lattice L and no distributivity in L is required.

2. Preliminaries

Throughout this paper $(L, \leq, \bigwedge, \bigvee, ')$ is a complete DeMorgan algebra, X is a nonempty set. L^X is the set of all L-fuzzy sets (or L-sets, for short) on X. The smallest element and the largest element in L^X are denoted by $\chi_\emptyset$ and χ_X, respectively. We often don't distinguish a crisp subset A of X and its character function χ_A.

A complete lattice L is a complete Heyting algebra if it satisfies the following infinite distributive law: For all $a \in L$ and all $B \subset L$, $a \wedge \bigvee B = \bigvee\{a \wedge b \,|\, b \in B\}$.

An element a in L is called a prime element if $a \geq b \wedge c$ implies $a \geq b$ or $a \geq c$. An element a in L is called co-prime if a' is prime [7]. The set of non-unit prime elements in L is denoted by $P(L)$. The set of non-zero co-prime elements in L is denoted by $M(L)$.

The binary relation $\prec$ in L is defined as follows: for a, $b \in L$, $a \prec b$ if and only if for every subset $D \subseteq L$, the relation $b \leq sup\, D$ always implies the existence of $d \in D$ with $a \leq d$ [5]. In a completely distributive DeMorgan algebra L, each element b is a sup of $\{a \in L | a \prec b\}$. A set $\{a \in L | a \prec b\}$ is called the greatest minimal family of b in the sense of [12, 26], denoted by $\beta(b)$, and $\beta^*(b) = \beta(b) \cap M(L)$. Moreover, for $b \in L$, we define $\alpha(b) = \{a \in L | a' \prec b'\}$ and $\alpha^*(b) = \alpha(b) \cap P(L)$.

For $a \in L$ and $A \in L^X$, we use the following notations from [18].

$$A_{[a]} = \{x \in X | A(x) \geq a\}, \ A^{(a)} = \{x \in X | A(x) \not\leq a\},$$
$$A_{(a)} = \{x \in X | \ a \in \beta(A(x))\}.$$

An L-topological space (or L-space, for short) is a pair $(X, \mathcal{T})$, where $\mathcal{T}$ is a subfamily of L^X which contains $\chi_\emptyset$; χ_X and is closed for any suprema and finite infima. $\mathcal{T}$ is called an L-topology on X. Members of $\mathcal{T}$ are called open L-sets and their complements are called closed L-sets.

Definition 2.1. [12, 26] An L-space $(X, \mathcal{T})$ is called weakly induced if $\forall a \in L$, $A \in L^X$, it follows that $A^{(a)} \in [\mathcal{T}]$, where $[\mathcal{T}]$ denotes the topology formed by all crisp sets in $\mathcal{T}$.

Definition 2.2. [12, 26] For a topological space (X, τ), let $\omega_L(\tau)$ denote the family of all lower semi-continuous maps from (X, τ) to L, i.e., $\omega_L(\tau) = \{A \in L^X; A^{(a)} \in \tau, \quad a \in L\}$. Then $\omega_L(\tau)$ is an L-topology on X; in this case, $(X, \omega_L(\tau))$ is called topologically generated by (X, τ). A topologically generated L-space is also called an induced L-space.

Definition 2.3. [22] Let $(X, \mathcal{T})$ be an L-space, $a \in L_0$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a β_a-cover of G if for any $x \in X$, it follows that $a \in \beta(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x))$. $\mathcal{U}$ is called a strong β_a-cover of G if $a \in \beta(\bigwedge_{x \in X}(G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x)))$.

Definition 2.4. [22] Let $(X, \mathcal{T})$ be an L-space, $a \in L_0$ and $G \in L^X$. A family $\mathcal{U} \subseteq L^X$ is called a Q_a-cover of G if for any $x \in X$, it follows that $G'(x) \vee \bigvee_{A \in \mathcal{U}} A(x) \geq a$.

It is obvious that a strong β_a-cover of G is a β_a-cover of G, and a β_a-cover of G is a Q_a-cover of G. For $a \in L$ and a crisp subset $D \subset X$, we define $a \wedge D$ and $a \vee D$ as follows:

$$(a \wedge D)(x) = \begin{cases} a, & x \in D; \\ 0, & x \notin D. \end{cases} \quad (a \vee D)(x) = \begin{cases} 1, & x \in D; \\ 0, & x \notin D. \end{cases}$$

Theorem 2.1. [18] *For an L-set $A \in L^X$, the following facts are true:*

(1) $A = \bigvee_{a \in L}(a \wedge A_{(a)}) = \bigvee_{a \in L}(a \wedge A_{[a]})$.
(2) $A = \bigwedge_{a \in L}(a \vee A^{(a)}) = \bigwedge_{a \in L}(a \vee A^{[a]})$.

Theorem 2.2. [18] *Let $(X, \omega_L(\tau))$ be the L-space topologically generated by (X, τ) and $A \in L^X$. Then the following facts hold:*

(1) $cl(A) = \bigvee_{a\in L}(a \wedge (A_{(a)})^-) = \bigvee_{a\in L}(a \wedge (A_{[a]})^-)$;
(2) $cl(A)_{(a)} \subset (A_{(a)})^- \subset (A_{[a]})^- \subset cl(A)_{[a]}$;
(3) $cl(A) = \bigwedge_{a\in L}(a \vee (A^{(a)})^-) = \bigwedge_{a\in L}(a \vee (A^{[a]})^-)$;
(4) $cl(A)^{(a)} \subset (A^{(a)})^- \subset (A^{[a]})^- \subset cl(A)^{[a]}$;
(5) $int(A) = \bigvee_{a\in L}(a \wedge (A_{(a)})^\circ) = \bigvee_{a\in L}(a \wedge (A_{[a]})^\circ)$;
(6) $int(A)_{(a)} \subset (A_{(a)})^\circ \subset (A_{[a]})^\circ \subset int(A)_{[a]}$;
(7) $int(A) = \bigwedge_{a\in L}(a \vee (A^{(a)})^\circ) = \bigwedge_{a\in L}(a \vee (A^{[a]})^\circ)$;
(8) $int(A)^{(a)} \subset (A^{(a)})^\circ \subset (A^{[a]})^\circ \subset int(A)^{[a]}$;

where $(A_{(a)})^-$ and $(A_{(a)})^\circ$ denote respectively the closure and the interior of $A_{(a)}$ in (X, τ) and so on, $cl(A)$ and $int(A)$ denote respectively the closure and the interior of A in $(X, \omega_L(\tau))$.

Definition 2.5. [22] Let $(X, \mathcal{T})$ be an L-space, $a \in L_1$ and $G \in L^X$. A family $\mathcal{A} \subseteq L^X$ is said to be:

(1) an a-shading of G if for any $x \in X$, $(G'(x) \vee \bigvee_{A\in\mathcal{A}} A(x)) \not\leq a$.
(2) a strong a-shading of G if $\bigwedge_{x\in X}(G'(x) \vee \bigvee_{A\in\mathcal{A}} A(x)) \not\leq a$.
(3) an a-remote family of G if for any $x \in X$, $(G(x) \wedge \bigwedge_{B\in\mathcal{A}} B(x)) \not\geq a$.
(4) a strong a-remote family of G if $\bigvee_{x\in X}(G(x) \wedge \bigwedge_{B\in\mathcal{A}} B(x)) \not\geq a$.

Definition 2.6. [22] Let $a \in L_0$ and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is said to have a weak a-nonempty intersection in G if $\bigvee_{x\in X}(G(x) \wedge \bigwedge_{A\in\mathcal{U}} A(x)) \geq a$. $\mathcal{U}$ is said to have the finite (countable) weak a-intersection property in G if every finite (countable) subfamily $\mathcal{P}$ of $\mathcal{U}$ has a weak a-nonempty intersection in G.

Definition 2.7. [22] Let $a \in L_0$ and $G \in L^X$. A subfamily $\mathcal{U}$ of L^X is said to be a weak a-filter relative to G if any finite intersection of members in $\mathcal{U}$ is weak a-nonempty in G. A subfamily $\mathcal{B}$ of L^X is said to be a weak a-filterbase relative to G if

$$\{A \in L^X; \quad \text{there exists } B \in \mathcal{B} \text{ such that } B \leq A\}$$

is a weak a-filter relative to G.

For a subfamily $\Phi \subseteq L^X$, 2^{Φ} denotes the set of all finite subfamilies of Φ and $2^{[\Phi]}$ denotes the set of all countable subfamilies of Φ.

Definition 2.8. Let G be an L-set of an L-space $(X, \mathcal{T})$. G is called:

(i) semiopen L-set [2] if $G \leq cl(int(G))$,

(ii) preopen L-set [15] if $G \leq int(cl(G))$,
(iii) α-open L-set [17] if $G \leq int(cl(int(g)))$.

Definition 2.9. Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then G is called fuzzy compact [23, 19] (resp. semicompact [21], P-compact [24], α-compact [22]) if for every family $\mathcal{U} \subset L^X$ of open L-sets (resp. semiopen L-sets, preopen L-sets, α-open L-sets), it follows that

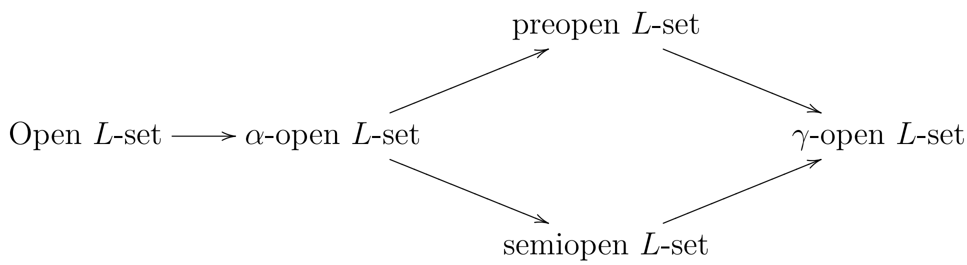
$$\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\right)\leq\bigvee_{\psi\in 2^{\mathcal{U}}}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\psi}A(x)\right).$$

Lemma 2.1. [19] *Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces, where L is a complete Heyting algebra, $f : X \to Y$ be a mapping, $f_L^{\to} : L^X \to L^Y$ is the extension of f. Then for any $P \subset L^Y$, we have that*

$$\bigvee_{y\in Y}\left(f_L^{\to}(G)(y)\wedge\bigwedge_{B\in\mathcal{P}}B(y)\right)=\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}f_L^{\leftarrow}(B)(x)\right).$$

Definition 2.10. [8] An L-set G in an L-space $(X, \mathcal{T})$ is called γ-open L-set if $G \leq cl(int(G)) \vee int(cl(G))$. G is called γ-closed L-set if G' is γ-open L-set.

Remark 2.1. We could know from [8] that the relationship between γ-open L-set and those mentioned in the Definition 2.8 could be clarified as follows:



3. Definition and characterizations of γ-compactness

Definition 3.1. Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is called (countably) γ-compact if for every (countable) family $\mathcal{U} \subseteq L^X$ of γ-open L-sets, it follows that

$$\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\right)\leq\bigvee_{\psi\in 2^{\mathcal{U}}}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\psi}A(x)\right).$$

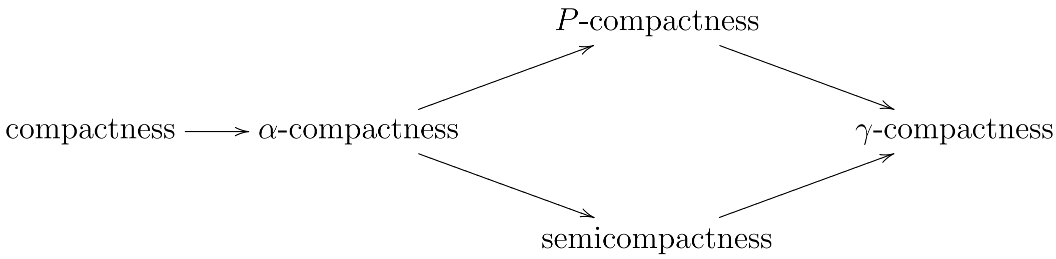
Definition 3.2. Let $(X, \mathcal{T})$ be an L-space. $G \in L^X$ is said to have the γ-Lindelöf property (or be a γ-Lindelöf L-set) if for every family $\mathcal{U}$ of γ-open L-sets, it follows that

$$\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\mathcal{U}}A(x)\right)\leq\bigvee_{\psi\in 2^{[\mathcal{U}]}}\bigwedge_{x\in X}\left(G'(x)\vee\bigvee_{A\in\psi}A(x)\right).$$

Remark 3.1.

(i) γ-compactness implies countable γ-compactness and the γ-Lindelöf property. Moreover, an L-set having the γ-Lindelöf property is γ-compact if and only if it is countably γ-compact.

(ii) We can clarify the relationship between γ-compactness and those listed in the Definition 2.9 as follows:



Theorem 3.1. *Let $(X,\mathcal{T})$ be an L-space. $G\in L^X$ is (countably) γ-compact if and only if for every (countable) family $\mathcal{B}$ of γ-closed L-sets, it follows that*

$$\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{B}}B(x)\right)\geq\bigwedge_{\vartheta\in 2^{\mathcal{B}}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\vartheta}B(x)\right).$$

Proof. Straightforward. □

Theorem 3.2. *Let $(X,\mathcal{T})$ be an L-space. $G\in L^X$ has the γ-Lindelöf property if and only if for every family $\mathcal{B}$ of γ-closed L-sets, it follows that*

$$\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{B}}B(x)\right)\geq\bigwedge_{\vartheta\in 2^{[\mathcal{B}]}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\vartheta}B(x)\right).$$

Proof. Straightforward. □

Theorem 3.3. *Let $(X,\mathcal{T})$ be an L-space and $G\in L^X$. Then the following conditions are equivalent:*

(1) *G is a (countably) γ-compact.*

(2) *For any $a\in L_1$, each (countable) γ-open strong a-shading $\mathcal{U}$ of G has a finite subfamily which is a strong a-shading of G.*

(3) *For any $a\in L_0$, each (countable) γ-closed strong a-remote family $\mathcal{P}$ of G has a finite subfamily which is a strong a-remote family of G.*

(4) *For any $a \in L_0$, each (countable) family of γ-closed L-sets which has the finite weak a-intersection property in G has a weak a-nonempty intersection in G.*
(5) *For each $a \in L_0$, every γ-closed (countable) weak a-filterbase relative to G has a weak a-nonempty intersection in G.*

Theorem 3.4. *Let $(X, \mathcal{T})$ be an L-space and $G \in L^X$. Then the following conditions are equivalent:*

(1) *G has the γ-Lindelöf property.*
(2) *For any $a \in L_1$, each γ-open strong a-shading $\mathcal{U}$ of G has a countable subfamily which is a strong a-shading of G.*
(3) *For any $a \in L_0$, each γ-closed strong a-remote family $\mathcal{P}$ of G has a countable subfamily which is a strong a-remote family of G.*
(4) *For any $a \in L_0$, each family of γ-closed L-sets which has the countable weak a-intersection property in G has a weak a-nonempty intersection in G.*

4. Properties of (countable) γ-compactness

Theorem 4.1. *Let L be a complete Heyting algebra. If both G and H are (countably) γ-compact, then $G \vee H$ is (countably) γ-compact.*

Proof. For any (countable) family $\mathcal{B}$ of γ-closed L-sets, we have by Theorem 3.1 that

$$\begin{aligned}
&\bigvee_{x\in X}\left((G\vee H)(x)\wedge \bigwedge_{B\in\mathcal{B}} B(x)\right)\\
=\ &\left\{\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\mathcal{B}} B(x)\right)\right\}\vee\left\{\bigvee_{x\in X}\left(H(x)\wedge \bigwedge_{B\in\mathcal{B}} B(x)\right)\right\}\\
\geq\ &\left\{\bigwedge_{\vartheta\in 2^{\mathcal{B}}}\bigvee_{x\in X}\left(G(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right)\right\}\\
\vee\ &\left\{\bigwedge_{\vartheta\in 2^{\mathcal{B}}}\bigvee_{x\in X}\left(H(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right)\right\}\\
=\ &\bigwedge_{\vartheta\in 2^{\mathcal{B}}}\bigvee_{x\in X}\left((G\vee H)(x)\wedge \bigwedge_{B\in\vartheta} B(x)\right).
\end{aligned}$$

This shows that $G \vee H$ is (countably) γ-compact. □

Analogously we have the following result.

Theorem 4.2. *Let L be a complete Heyting algebra. If both G and H have γ-Lindelöf property, then $G \vee H$ has γ-Lindelöf property.*

Theorem 4.3. *If G is (countably) γ-compact and H is γ-closed, then $G \wedge H$ is (countably) γ-compact.*

Proof. For any (countable) family $\mathcal{B}$ of γ-closed L-sets, we have by Theorem 3.1 that

$$\begin{aligned}
\bigvee_{x\in X}\left((G\wedge H)(x)\wedge\bigwedge_{B\in\mathcal{B}}B(x)\right) &= \bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{B}\cup\{H\}}B(x)\right)\\
&\geq \bigwedge_{\vartheta\in 2^{\mathcal{B}\cup\{H\}}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\vartheta}B(x)\right)\\
&= \left\{\bigwedge_{\vartheta\in 2^{\mathcal{B}}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\vartheta}B(x)\right)\right\}\\
&\wedge \left\{\bigwedge_{\vartheta\in 2^{\mathcal{B}}}\bigvee_{x\in X}\left(G(x)\wedge H(x)\wedge\bigwedge_{B\in\vartheta}B(x)\right)\right\}\\
&= \left\{\bigwedge_{\vartheta\in 2^{\mathcal{B}}}\bigvee_{x\in X}\left(G(x)\wedge H(x)\wedge\bigwedge_{B\in\vartheta}B(x)\right)\right\}\\
&= \left\{\bigwedge_{\vartheta\in 2^{\mathcal{B}}}\bigvee_{x\in X}\left((G\wedge H)(x)\wedge\bigwedge_{B\in\vartheta}B(x)\right)\right\}.
\end{aligned}$$

This shows that $G \wedge H$ is (countably) γ-compact. □

Theorem 4.4. *If G has the γ-Lindelöf property and* H *is γ-closed, then $G \wedge H$ has the γ-Lindelöf property.*

Proof. Similar to Theorem 4.3. □

Definition 4.1. Let $(X, \mathcal{T}_1)$ and $(Y, \mathcal{T}_2)$ be two L-spaces. A map $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ is called γ-irresolute iff $f_L^{\leftarrow}(G)$ is γ-open for each γ-open L-set G.

Theorem 4.5. *Let L be a complete Heyting algebra and let $f : (X, \mathcal{T}_1) \to (Y, \mathcal{T}_2)$ be a γ-irresolute map. If G is a γ-compact (or a countably γ-compact, a γ- Lindelöf) L-set in $(X, \mathcal{T}_1)$, then so is $f_L^{\rightarrow}(G)$ in $(Y, \mathcal{T}_2)$.*

Proof. Suppose that $\mathcal{P}$ is a family of γ-closed L-sets, then

$$\begin{aligned}\bigvee_{y\in Y}\left(f_L^{\rightarrow}(G)(y)\wedge\bigwedge_{B\in\mathcal{P}}B(y)\right) &= \bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}f_L^{\leftarrow}(B)(x)\right)\\ &\geq \bigwedge_{\vartheta\in 2^{\mathcal{P}}}\bigvee_{x\in X}\left(G(x)\wedge\bigwedge_{B\in\mathcal{P}}f_L^{\leftarrow}(B)(x)\right)\\ &= \bigwedge_{\vartheta\in 2^{\mathcal{P}}}\bigvee_{y\in Y}\left(f_L^{\leftarrow}(G)(y)\wedge\bigwedge_{B\in\mathcal{P}}B(y)\right).\end{aligned}$$

Therefore $f_L^{\rightarrow}(G)$ is γ-compact. □

Theorem 4.6. *Let L be a complete Heyting algebra and let $f:(X,\mathcal{T}_1)\to(Y,\mathcal{T}_2)$ be a γ-continuous map. If G is a γ-compact (a countably γ-compact, a γ-Lindelöf) L-set in $(X,\mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is a compact (countably compact, Lindelöf) L-set in $(Y,\mathcal{T}_2)$.*

Proof. Straightforward. □

Definition 4.2. Let $(X,\mathcal{T}_1)$ and $(Y,\mathcal{T}_2)$ be two L-spaces. A map $f:(X,\mathcal{T}_1)\to(Y,\mathcal{T}_2)$ is called strongly γ-irresolute if $f_L^{\leftarrow}(G)$ is open in $(X,\mathcal{T}_1)$ for every γ-open L-set G in $(Y,\mathcal{T}_2)$.

It is obvious that a strongly γ-irresolute map is γ-irresolute and continuous. Analogously we have the following result.

Theorem 4.7. *Let L be a complete Heyting algebra and $f:(X,\mathcal{T}_1)\to(Y,\mathcal{T}_2)$ be a strongly γ-irresolute map. If G is a compact (countably compact, Lindelöf) L-set in $(X,\mathcal{T}_1)$, then $f_L^{\rightarrow}(G)$ is a γ-compact (a countably γ-compact, a γ-Lindelöf) L-set in $(Y,\mathcal{T}_2)$.*

Proof. Straightforward. □

5. Good Extension

Theorem 5.1. *Let $(X,\mathcal{T})$ be an L-space and $G\in L^X$. Then the following conditions are equivalent:*

1. *G is γ-compact.*
2. *For any $a\in L_0$ ($a\in M(L)$), each γ-closed strong a-remote family of G has a finite subfamily which is an a-remote (a strong a-remote) family of G.*

(3) *For any* $a \in L_0$ *(*$a \in M(L)$*) and any* γ*-closed strong* a*-remote family* $\mathcal{P}$ *of* G*, there exists a finite subfamily* $\mathcal{F}$ *of* $\mathcal{P}$ *and* $b \in \beta(a)$ *(*$b \in \beta^*(a)$*) such that* $\mathcal{F}$ *is a (strong)* b*-remote family of* G*.*

(4) *For any* $a \in L_1$ *(*$a \in P(L)$*), each* γ*-open strong* a*-shading of* G *has a finite subfamily which is an* a*-shading (a strong* a*-shading) of* G*.*

(5) *For any* $a \in L_1$ *(*$a \in P(L)$*) and any* γ*-open strong* a*-shading* $\mathcal{U}$ *of* G*, there exists a finite subfamily* $\mathcal{V}$ *of* $\mathcal{U}$ *and* $b \in \beta(a)$ *(* $b \in \beta^*(a)$*) such that* $\mathcal{V}$ *is a (strong)* b*-shading of* G*.*

(6) *For any* $a \in L_0$ *(*$a \in M(L)$*), each* γ*-open strong* β_a*-cover of* G *has a finite subfamily which is a (strong)* β_a*-cover of* G*.*

(7) *For any* $a \in L_0$ *(* $a \in M(L)$*) and any* γ*-open strong* β_a*-cover* $\mathcal{U}$ *of* G*, there exists a finite subfamily* $\mathcal{V}$ *of* $\mathcal{U}$ *and* $b \in L$ *(*$b \in M(L)$*) with* $a \in \beta(b)$ *such that* $\mathcal{V}$ *is a (strong)* β_b*-cover of* G*.*

(8) *For any* $a \in L_0$ *(*$a \in M(L)$*) and any* $b \in \beta(a) \setminus \{0\}$*, each* γ*-open* Q_a*-cover of* G *has a finite subfamily which is a* Q_b*-cover of* G*.*

(9) *For any* $a \in L_0$ *(*$a \in M(L)$*) and any* $b \in \beta(a) \setminus \{0\}$ *(* $b \in \beta^*(a)$*), each* γ*-open* Q_a*-cover of* G *has a finite subfamily which is a (strong)* Q_b*-cover of* G*.*

Analogously we also can present characterization of countable γ-compactness and the γ-Lindelöf property. Now we consider the goodness of γ-compactness.

Lemma 5.1. *Let* $(X, \omega(L))$ *be generated topologically by* (X, τ)*. If* A *is a* γ*-open set in* (X, τ)*, then* χ_A *is* γ*-open* L*-set in* $(X, \omega_L(\tau))$*. If* B *is a* γ*-open* L*-set in* $(X, \omega_L(\tau))$*, then* $B_{(a)}$ *is* γ*-open set in* (X, τ) *for every* $a \in L$*.*

Proof. Let A be a γ-open set in (X, τ), then $A \subseteq (A^\circ)^- \cup (A^-)^\circ$. Thus we have

$$\begin{aligned} \chi_A &\leq \chi_{(A^\circ)^- \cup (A^-)^\circ} = \chi_{(A^\circ)^-} \vee \chi_{(A^-)^\circ} \\ &= cl(\chi_{(A^\circ)}) \vee int(\chi_{(A^-)}) = cl(int(\chi_A)) \vee int(cl(\chi_A)). \end{aligned}$$

This shows that χ_A is γ-open in $(X, \omega_L(\tau))$. If B is a γ-open L-set in $(X, \omega_L(\tau))$, then $B \leq cl(int(B)) \vee int(cl(B))$. From Theorem 2.2, we have

$$\begin{aligned} B_{(a)} &\subseteq [cl(int(B)) \vee int(cl(B))]_{(a)} \subseteq cl(int(B))_{(a)} \cup int(cl(B))_{(a)} \\ &\subseteq ((int(B))_{(a)})^- \cup (cl(B)_{(a)})^\circ \subseteq (((B_{(a)})^\circ))^- \cup ((B_{(a)})^-)^\circ. \end{aligned}$$

This shows that $B_{(a)}$ is a γ-open set in (X, τ). □

The next two theorems show that γ-compactness, countable γ-compactness and the γ-Lindelöf property are good extensions.

Theorem 5.2. *Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). Then $(X, \omega_L(\tau))$ is (countably) γ-compact if and only if (X, τ) is (countably) γ-compact.*

Proof. (**Necessity**) Let $\mathcal{A}$ be a γ-open cover (a countable γ-open cover) of (X, τ). Then $\{\chi_A : A \in \mathcal{A}\}$ is a family of γ-open L-sets in $(X, \omega_L(\tau))$ with

$$\bigwedge_{x\in X} (\bigvee_{A\in\mathcal{U}} \chi_A(x)) = 1.$$

From (countable) γ-compactness of $(X, \omega_L(\tau))$ we know that

$$1 \geq \bigvee_{\psi\in 2^{\mathcal{U}}} \bigwedge_{x\in X} (\bigvee_{A\in\psi} \chi_A(x)) \geq \bigwedge_{x\in X} (\bigvee_{A\in\mathcal{U}} \chi_A(x)) = 1.$$

This implies that there exists $\psi \in 2^{[\mathcal{U}]}$ such that $\bigwedge_{x\in X}(\bigvee_{A\in\psi} \chi_A(x)) = 1$. Hence ψ is a cover of (X, τ). Therefore (X, τ) is (countably) γ-compact.

(**Sufficiency**) Let $\mathcal{U}$ be a (countable) family of γ-open L-sets in $(X, \omega_L(\tau))$ and let $\bigwedge_{x\in X}(\bigvee_{B\in\mathcal{U}} B(x)) = a$. If $a = 0$, then we obviously have

$$\bigwedge_{x\in X} (\bigvee_{B\in\mathcal{U}} B(x)) \leq \bigvee_{\psi\in 2^{\mathcal{U}}} \bigwedge_{x\in X} (\bigvee_{A\in\psi} B(x)).$$

Now we suppose that $a \neq 0$. In this case, for any $b \in \beta(a) \setminus \{0\}$ we have

$$b \in \beta(\bigwedge_{x\in X} (\bigvee_{B\in\mathcal{U}} B(x))) \subseteq \bigcap_{x\in X} \beta(\bigvee_{B\in\mathcal{U}} B(x)) = \bigcap_{x\in X} \bigcup_{B\in\mathcal{U}} \beta(B(x)).$$

By Lemma 5.1 this implies that $\{B_{(b)} : B \in \mathcal{U}\}$ is a γ-open cover of (X, τ). From (countable) γ-compactness of (X, τ) we know that there exists $\psi \in 2^{\mathcal{U}}$ such that $\{B_{(b)} : B \in \psi\}$ is a cover of (X, τ). Hence $b \leq \bigvee_{x\in X}(\bigwedge_{B\in\psi} B(x))$. Furthermore we have

$$b \leq \bigwedge_{x\in X} (\bigvee_{B\in\psi} B(x)) \leq \bigvee_{\psi\in 2^{\mathcal{U}}} \bigwedge_{x\in X} (\bigvee_{B\in\psi} B(x)).$$

This implies that

$$\bigwedge_{x\in X} (\bigvee_{B\in\mathcal{U}} B(x)) = a = \bigvee\{b : b \in \beta(a)\} \leq \bigvee_{\psi\in 2^{\mathcal{U}}} \bigwedge_{x\in X} (\bigvee_{B\in\psi} B(x)).$$

Therefore $(X, \omega_L(\tau))$ is (countably) γ-compact. □

Analogously we have the following theorem.

Theorem 5.3. *Let $(X, \omega_L(\tau))$ be generated topologically by (X, τ). Then $(X, \omega_L(\tau))$ has the γ-Lindelöf property if and only if (X, τ) has the γ-Lindelöf property.*

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