FUNCTIONS AND Baire spaces

Zbigniew Duszyński

Abstract. Some results concerning functions that preserve Baire spaces in the context of images and preimages are obtained in Section IV of [R. C. Haworth, R. A. McCoy, Baire spaces, Dissertationes Math., 151, PWN, Warszawa 1977]. In Section 2 of the present paper further results in this direction are offered. In Sections 3 and 4 a few theorems related to some other problems on baireness of topological spaces are proposed.

1. Preliminaries

Throughout, topological spaces are denoted by \((X, \tau)\) or \((Y, \sigma)\). Let \(S\) be a subset of a space \((X, \tau)\). The **interior** and the **closure** of \(S\) in \((X, \tau)\) will be denoted by \(\text{int}_\tau(S)\) (or \(\text{int}(S)\)) and \(\text{cl}_\tau(S)\) (or \(\text{cl}(S)\)) respectively. The set \(S\) is said to be **semi-open** [12] (resp. **preopen** [13], **semi-closed** [3], **pre-closed**, **regular open**) in \((X, \tau)\), if \(S \subset \text{cl}(\text{int}(S))\) (resp. \(S \subset \text{int}(\text{cl}(S))\), \(S \supset \text{int}(\text{cl}(S))\), \(S \supset \text{cl}(\text{int}(S))\), \(S = \text{int}(\text{cl}(S))\)).

The collection of all semi-open (resp. preopen, semi-closed, pre-closed, closed, clopen) subsets of \((X, \tau)\) shall be denoted by \(\text{SO}(X, \tau)\) (resp. \(\text{PO}(X, \tau)\), \(\text{SC}(X, \tau)\), \(\text{PC}(X, \tau)\), \(\text{c}(X, \tau)\), \(\text{CO}(X, \tau)\)). A set \(S \in \text{SO}(X, \tau)\) if and only if there exists an \(O \in \tau\) such that \(O \subset S \subset \text{cl}(S)\) [12]. Every nonvoid semi-open set contains a nonvoid open subset [3, Remark 1.2]. The intersection of all \(F \in \text{SC}(X, \tau)\) with \(S \subset F\) is called the **semi-closure** [3] of \(S\) in \((X, \tau)\). The union of all \(U \in \text{SO}(X, \tau)\) with \(U \subset S\) is called the **semi-interior** [3] of \(S\) in \((X, \tau)\). The semi-closure and semi-interior of \(S\) are denoted respectively by \(\text{scl}(S)\) (or \(\text{scl}_\tau(S)\)) and \(\text{sint}(S)\) (or \(\text{sint}_\tau(S)\)). We have \(S \in \text{SO}(X, \tau)\)

---

**Key words and phrases.** Continuous, Contra-semicontinuous, Semi-open, \(\delta\)-open function, Baire space.

**2010 Mathematics Subject Classification.** 54C08.

**Received:** July 07, 2010.
(resp. \( S \in SC(X, \tau) \)) if and only if \( S = \text{sint}(S) \) (resp. \( S = \text{scl}(S) \)) \cite[Theorem 1.4]{3}. Also, \( \text{scl}(S) = \text{int}(\text{cl}(S)) \) if and only if \( S \in \text{PO}(X, \tau) \) \cite[Proposition 2.7(a)]{11}. If \( O \in \tau \) and \( S \in \text{SO}(X, \tau) \) then \( O \cap S \in \text{SO}(X, \tau) \) \cite[Theorem 1.9]{3}. A space \((X, \tau)\) is \textbf{Baire} if each nonempty set \( S \in \tau \) is of the 2nd category (equivalently: \( \text{cl}(\bigcap_{n \in \mathbb{N}} O_n) = X \) for any family \( \{O_n\}_{n \in \mathbb{N}} \subset \tau \) with \( \text{cl}(O_n) = X \) for each \( n \); or: \( \text{cl}(\bigcap_{n \in \mathbb{N}} A_n) = X \) for any family \( \{A_n\}_{n \in \mathbb{N}} \subset \text{SO}(X, \tau) \) with \( \text{cl}(A_n) = X \) for each \( n \), see \cite{8}). A space \((X, \tau)\) is said to be \textbf{S-connected} \cite{16} if there are no nonempty sets \( A_1, A_2 \in \text{SO}(X, \tau) \) such that \( A_1 \cap A_2 = \emptyset \) and \( A_1 \cup A_2 = X \). In the opposite case \((X, \tau)\) is called \textbf{S-disconnected}.

\section{Continuous and semi-open injections}

\textbf{Lemma 2.1.} If an injection \( f : (X, \tau) \to (Y, \sigma) \) is continuous and closed, then for any subset \( S \subset X \) we have

\begin{equation}
(2.1) \quad \text{int}_\tau \left( \text{cl}(f(S)) \right) \subset f \left( \text{int}_\tau(\text{cl}(S)) \right).
\end{equation}

\textbf{Proof.} As \( f \) is closed we have \( f^{-1}\left( \text{int}(\text{cl}(f(S))) \right) \subset f^{-1}\left( \text{int}(f(\text{cl}(S))) \right) \). But \( f \) is also continuous, so \( f^{-1}\left( \text{int}(\text{cl}(f(S))) \right) \subset \text{int}\left( f^{-1}(f(\text{cl}(S))) \right) = \text{int}(\text{cl}(S)). \) Consequently,

\[ f\left( f^{-1}\left( \text{int}(\text{cl}(f(S))) \right) \right) \subset f\left( \text{int}(\text{cl}(S)) \right). \]

Since \( f \) is closed, \( \text{int}(\text{cl}(f(S))) \subset f(X) \). Thus, by injectivity of \( f \) the inclusion (2.1) holds. \hfill \Box

A function \( f : (X, \tau) \to (Y, \sigma) \) is said to be \textbf{semi-open} \cite{2} if \( f(S) \in \text{SO}(Y, \sigma) \) for every \( S \in \tau \).

\textbf{Theorem 2.1.} Let an injection \( f : (X, \tau) \to (Y, \sigma) \) be continuous, semi-open, and closed. If \((Y, \sigma)\) is Baire, then so is \((X, \tau)\).

\textbf{Proof.} Suppose \( G \in \tau \) is of first category in \((X, \tau)\). Then \( f(G) = \bigcup_{k=1}^{\infty} f(N_k) \) where for each \( k \in \mathbb{N} \) the sets \( N_k \) are nowhere dense in \((X, \tau)\). Using Lemma 2.1 we get that the images \( f(N_k) \) are nowhere dense in \((Y, \sigma)\), \( k \in \mathbb{N} \). Hence the semi-open set \( f(G) \subset Y \) is of first category. Thus there exists a nonempty set \( O \in \sigma \), \( O \subset f(G) \), which is of first category. It contradicts the assumption that \((Y, \sigma)\) is Baire. \hfill \Box
Recall the following result: if a function $f : (X, \tau) \to (Y, \sigma)$ is continuous and open, then for any subset $S \subset X$ we have
\[
\text{int}_\sigma \left( \text{cl}_\sigma \left( f(S) \right) \right) \supset f \left( \text{int}_\tau (\text{cl}_\tau (S)) \right).
\]
So, in view of Lemma 2.1 we get what follows.

**Proposition 2.1.** If an injection $f : (X, \tau) \to (Y, \sigma)$ is continuous, open, and closed, then for any subset $S \subset X$
\[
\text{int}_\sigma \left( \text{cl}_\sigma \left( f(S) \right) \right) = f \left( \text{int}_\tau (\text{cl}_\tau (S)) \right).
\]

**Corollary 2.1.** Let an injection $f : (X, \tau) \to (Y, \sigma)$ be continuous, open, and closed. Then, for any subset $S \subset X$,
\begin{enumerate}[(a)]  
  \item $S$ is nowhere dense in $(X, \tau)$ if and only if $f(S)$ is nowhere dense in $(Y, \sigma)$,  
  \item if $S$ is regular open in $(X, \tau)$, then $f(S)$ is regular open in $(Y, \sigma)$. \end{enumerate}

The following continuity-like property will be useful in the sequel.

**Definition 2.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be $s$-**perfectly continuous** if the preimage $f^{-1}(V) \in \text{CO} (X, \tau)$ for each $V \in \text{SO} (Y, \sigma)$.

Every $s$-perfectly continuous function is perfectly continuous ($f^{-1}(V) \in \text{CO} (X, \tau)$ for each $V \in \sigma$, see [1]), but the converse does not hold in general.

**Example 2.1.**
\begin{enumerate}[(a)]  
  \item Let $X = \{a, b, c\} = Y$, $\tau = \{\emptyset, X, \{a\}, \{b, c\}\}$, $\sigma = \{\emptyset, Y, \{a\}\}$. The function $\text{id}_X : (X, \tau) \to (X, \sigma)$ is perfectly continuous, but it is not $s$-perfectly continuous: consider $\{a, b\} \in \text{SO} (X, \sigma)$.  
  \item If we put in (a) $Y = \{a, b, c, d\}$, then injective but not surjective $\text{id}_X$ is perfectly continuous and not $s$-perfectly continuous, as well. \end{enumerate}

Recall that every perfectly continuous function is continuous, but the converse is not true, see [15, p.249]. Next, we shall establish

**Theorem 2.2.** Let an injection $f : (X, \tau) \to (Y, \sigma)$ be $s$-perfectly continuous and semi-open. If $(X, \tau)$ is compact, $(Y, \sigma)$ is Hausdorff and Baire, then $(X, \tau)$ is Baire.

In order to prove this result we shall need some lemmas. The second one is well-known.
Lemma 2.2. [14, Theorem 5] Let, for a \((Y, \sigma), Y_0 \in \text{SO}(Y, \sigma)\). Then for every \(A \subset Y_0, A \in \text{SO}(Y, \sigma)\) if and only if \(A \in \text{SO}(Y_0, \sigma_{Y_0})\).

Lemma 2.3. If a function \(f : (X, \tau) \to (Y, \sigma)\) is continuous, \((X, \tau)\) is compact, and \((Y, \sigma)\) is Hausdorff, then for each subset \(S \subset X\) one has
\[
f\left(\overline{S}\right) = \overline{f(S)}.
\]

Recall that a collection of subsets of a space \((X, \tau)\) is called a \textit{pseudo-cover} [9, p.13], if its union is dense in \((X, \tau)\). A pseudo-cover is said to be semi-open (resp. open) if all its members are semi-open (resp. open) in \((X, \tau)\).

Lemma 2.4 ([8] for semi-open and [9, Theorem 1.21] for open case). If there exists a semi-open (or open) pseudo-cover of \((X, \tau)\) whose members are Baire subspaces of \((X, \tau)\), then \((X, \tau)\) is a Baire space as well.

Lemma 2.5 ([8], [9, Proposition 1.14], [7, p.256 Problem 2]). Any semi-open (hence open) subset of a Baire space is Baire too.

\textit{Proof of Theorem 2.2.} In virtue of Lemma 2.4 it is enough to show that each point of \((X, \tau)\) has an open neighbourhood which is Baire in \((X, \tau)\). So, let \(x \in X\) be arbitrarily chosen. Then \(f(x) \in f(X) \in \text{SO}(Y, \sigma)\). Since \((Y, \sigma)\) is Hausdorff, there exists \(G_x \subseteq Y\) with \(f(x) \in G_x \in \sigma\). Hence \(f(x) \in G_x \cap f(X) = V_x \in \text{SO}(Y, \sigma)\), consequently by Lemma 2.2, \(V_x \in \text{SO}\left(f(X), \sigma_{f(X)}\right)\). Now consider a subset \(f^{-1}(V_x) \in \tau\). Since \(f\) is continuous, then the restriction \(f \upharpoonright f^{-1}(V_x) = f_x : f^{-1}(V_x) \to V_x\) is continuous too. Thus
\[
f_x\left(\overline{\tau_{f^{-1}(V_x)}(S)}\right) \subset \overline{\sigma_{V_x}(f_x(S))}
\]
for any \(S \subset f^{-1}(V_x)\). Let \(\{O_n\}_{n \in \mathbb{N}} \subset \tau_{f^{-1}(V_x)}\) be an arbitrary family of sets dense in \((f^{-1}(V_x), \tau_{f^{-1}(V_x)})\). We shall show that
\begin{equation}
\overline{\bigcap_{n \in \mathbb{N}} O_n} = f^{-1}(V_x).
\end{equation}

By continuity of \(f_x\) each \(f_x(O_n) \in \text{SO}(Y, \sigma), n \in \mathbb{N}\), is dense in \((V_x, \sigma_{V_x})\). Obviously (by Lemma 2.2), \(f_x(O_n) \in \text{SO}(V_x, \sigma_{V_x})\) for each \(n\). Since, by assumption, \((Y, \sigma)\) is Baire, then \((V_x, \sigma_{V_x})\) is Baire as well (see Lemma 2.5). So, \(\overline{\bigcap_{n \in \mathbb{N}} f_x(O_n)} = V_x\). But \(f^{-1}(V_x) \in c (X, \tau)\) and hence it is compact. Applying now Lemma 2.3 we get
\[
f_x\left(\overline{\bigcap_{n \in \mathbb{N}} O_n}\right) = \overline{\sigma_{V_x}\left(f_x\left(\bigcap_{n \in \mathbb{N}} O_n\right)\right)} = \overline{\sigma_{V_x}\left(\bigcap_{n \in \mathbb{N}} f_x(O_n)\right)} = V_x.
\]
Therefore (2.2) follows. \(\square\)
A brief insight into the foregoing proof leads to the next result.

**Theorem 2.3.** Let an injection $f : (X, \tau) \to (Y, \sigma)$ be perfectly continuous and open. If $(X, \tau)$ is compact, $(Y, \sigma)$ is Hausdorff and Baire, then $(X, \tau)$ is Baire.

Recall that a function $f : (X, \tau) \to (Y, \sigma)$ is called a **semi-homeomorphism** [4] if it is bijective, pre-semi-open (i.e., $f(U) \in \text{SO}(Y, \sigma)$ for every $U \in \text{SO}(X, \tau)$), and irresolute ($f^{-1}(V) \in \text{SO}(X, \tau)$ for every $V \in \text{SO}(Y, \sigma)$). Each homeomorphism is a semi-homeomorphism, but not conversely [4, Theorem 1.9 and Example 1.2]. The following interesting result is a consequence of [9, Theorem 4.1 and Proposition 4.3]; the details are left to the reader (we make use of [3, Remark 1.2]).

**Proposition 2.2.** Let $f : (X, \tau) \to (Y, \sigma)$ be a semi-homeomorphism. Then $(X, \tau)$ is a Baire space if and only if $(Y, \sigma)$ is Baire.

3. Contra-continuity

A function $f : (X, \tau) \to (Y, \sigma)$ is said to be **contra-continuous** [5] if $f^{-1}(V) \in c(X, \tau)$ for every $V \in \sigma$ (or, equivalently, $f^{-1}(F) \in \tau$ for every $F \in c(Y, \sigma)$).

**Theorem 3.1.** Let a surjection $f : (X, \tau) \to (Y, \sigma)$ be contra-continuous and open. Then $(Y, \sigma)$ is a Baire space.

**Proof.** Suppose $B \in \sigma, B \neq \emptyset$, is of first category in $(Y, \sigma)$. So, $B = \bigcup_{n \in \mathbb{N}} B_n$ where each $B_n$ is nowhere dense in $(Y, \sigma)$. By contra-continuity and openness of $f$ we calculate as follows.

\[
    f^{-1}(B) = f^{-1}\left( \bigcup_{n \in \mathbb{N}} B_n \right) = \bigcup_{n \in \mathbb{N}} f^{-1}(B_n) \subset \bigcup_{n \in \mathbb{N}} f^{-1}(\text{cl}(B_n)) = \\
    = \bigcup_{n \in \mathbb{N}} \text{int}\left( f^{-1}(\text{cl}(B_n)) \right) \subset \bigcup_{n \in \mathbb{N}} f^{-1}(\text{int}(\text{cl}(B_n))) = \emptyset.
\]

A contradiction completes the proof. \(\square\)

Let us remark that from the above proof it follows, under assumptions of Theorem 3.1, that each nonempty subset of a space $(Y, \sigma)$ is of second category.

**Definition 3.1.** A function $f : (X, \tau) \to (Y, \sigma)$ is said to be **s-contra-precontinuous** if the preimage $f^{-1}(V) \in \text{PC}(X, \tau)$ for each $V \in \text{SO}(Y, \sigma)$ (equivalently $f^{-1}(F) \in \text{PO}(X, \tau)$ for each $F \in \text{SC}(Y, \sigma)$).
Each $s$-contra-precontinuous function is contra-precontinuous ($f^{-1}(V) \in \text{PC}(X, \tau)$ for each $V \in \sigma$ [10]) and not conversely.

**Example 3.1.** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{b\}\}$, $\sigma = \{\emptyset, X, \{a\}\}$. Then $\text{id}_X : (X, \tau) \to (X, \sigma)$ is contra-precontinuous, but not $s$-contra-precontinuous: consider $V = \{a, b\} \in \text{SO}(X, \sigma)$.

**Theorem 3.2.** Let a surjection $f : (X, \tau) \to (Y, \sigma)$ be $s$-contra-continuous, continuous and open. Then $(Y, \sigma)$ is a Baire space.

**Proof.** Let $\emptyset \neq B = \bigcup_{n \in \mathbb{N}} B_n \in \sigma$, where $B_n$ is nowhere dense in $(Y, \sigma)$ for every $n \in \mathbb{N}$. Since each nowhere dense set is semi-closed [3, Theorem 1.3], using our assumption we calculate as follows:

$$f^{-1}(B) = \bigcup_{n \in \mathbb{N}} f^{-1}(B_n) \subset \bigcup_{n \in \mathbb{N}} \text{int} \left( \text{cl} \left( f^{-1}(B_n) \right) \right) \subset \bigcup_{n \in \mathbb{N}} \text{int} \left( \text{cl} \left( f^{-1}(B_n) \right) \right) \subset \bigcup_{n \in \mathbb{N}} f^{-1}(\text{int} \left( \text{cl} \left( B_n \right) \right)) = \emptyset.$$

Thus $\emptyset \neq f^{-1}(B) = \emptyset$, a contradiction. $\square$

Observe also that under assumptions of Theorem 3.2 every nonempty subset of $(Y, \sigma)$ is of second category. As an example of a Baire space with such a property, it is enough to consider $X = \{a, b, c, d\}$ with

$$\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}.$$

A function $f : (X, \tau) \to (Y, \sigma)$ is called **contra-semicontinuous** [6] if $f^{-1}(V) \in \text{SC}(X, \tau)$ for every $V \in \sigma$ (equivalently, $f^{-1}(F) \in \text{SO}(X, \tau)$ for every $F \in c(Y, \sigma)$). In order to prove our next result we need the following improvement of Baire category theorem.

**Lemma 3.1.** [8] Let $(X, \tau)$ be a Baire space. If $\{A_n\}_{n \in \mathbb{N}} \subset \text{SC}(X, \tau)$ is a family covering $X$, then at least one $A_n$ must contain a set from $\tau$; i.e., have a nonvoid interior.

**Theorem 3.3.** Let $(X, \tau)$ be a Baire space and let $(Y, \sigma)$ be Lindelöf. If there is a contra-semicontinuous bijection $f : (X, \tau) \to (Y, \sigma)$, then $(X, \tau)$ is $S$-disconnected.

**Proof.** As $(Y, \sigma)$ is Lindelöf, by definition, it is Hausdorff. So, for every $y \in Y$ there is a $U_y \in \sigma$ with $y \in U_y \subset Y$ and in turn (lindelöfness) $Y = \bigcup_{k=1}^{\infty} U_{y_k}$ for a certain
functions and baire spaces

We have a function \( f \). Let a space \( X, \tau \). The notation is the same as in the proof of [9, Theorem 4.11]. Suppose that \( f \) is \( \delta \)-open. If there exists a residual subset \( Z \) of \((X, \tau)\) be of second category in itself, and a surjection \( f : (X, \tau) \to (Y, \sigma) \) is of first category in itself; i.e., \( X = \bigcup_{n \in \mathbb{N}} F_n \) for some nowhere dense closed sets \( F_n \) in \((X, \tau)\), \( n \in \mathbb{N} \). We set \( M(F_n) = \{ y \in Y : \text{int}_{f^{-1}(y)}(f^{-1}(y) \cap F_n) \neq \emptyset \} \), \( n \in \mathbb{N} \). Fix a countable base \( \{ U_i \}_{i \in \mathbb{N}} \) for \((X, \tau)\) and set also \( M_i^n = \{ y \in Y : \emptyset \neq f^{-1}(y) \cap U_i \subset F_n \} \).
We have $M(F_n) = \bigcup_{i \in \mathbb{N}} M^n_i$, $n \in \mathbb{N}$. Choose arbitrary $n$ and consider a nonempty $M^n_i$, $i \in \mathbb{N}$. For any $y \in M^n_i$ we have $f^{-1}(y) \cap U_i \subset f(F_n)$. Consequently, $M^n_i \cap f(U_i) \subset f(F_n)$. Therefore, $\bigcup_{i \in \mathbb{N}} M^n_i \cap \bigcup_{i \in \mathbb{N}} f(U_i) = M(F_n) \cap Y = M(F_n) \subset f(F_n)$. By assumption the set $M(F_n)$ is nowhere dense and so $M = \bigcup_{n \in \mathbb{N}} M(F_n)$ has the first category in $(Y, \sigma)$. The remaining steps of the proof are the same as at the end of the proof of [9, Theorem 4.1].

**Corollary 4.1.** Theorem 4.2 is true if to assume all $f^{-1}(z)$, $z \in Z$, are Baire subspaces (instead of the second category assumption).

**REFERENCES**


**Institute of Mathematics,\nCasimir the Great University,\nPlac Wejssenhoffa 11, 85-072 Bydgoszcz,\nPoland\nE-mail address: imath@ukw.edu.pl**