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COUNTEREXAMPLES FOR PROPERTIES OF LINE GRAPHS OF GRAPHS OF DIAMETER TWO

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Abstract. Some errors in a recently published paper [Ramane et al., Distance spectra and distance energies of iterated line graphs of regular graphs, *Publ. Inst. Math.* **85** (2009) 39–46] are pointed out. These pertain to conditions under which the iterated line graphs of a graph of diameter two have also diameter two.

Let G be a connected simple graph with vertex set $\mathbf{V}(G) = \{v_1, v_2, \ldots, v_n\}$. The *distance* between the vertices v_i and v_j, $v_i, v_j \in \mathbf{V}(G)$, is equal to the length (= number of edges) of a shortest path starting at v_i and ending at v_j (or vice versa). The diameter $diam(G)$ of G is equal to the maximum distance between two vertices of G.

The line graph $L(G)$ of G is the graph whose vertex set is in one-to-one correspondence with the edge set of the graph G and where two vertices of $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common. For $k = 1, 2, \ldots,$ the k-th iterated line graph of G is $L^k(G) = L(L^{k-1}(G))$, where $L^0(G) \equiv G$ and $L^1(G) \equiv L(G)$.

In a recent paper [1] the distance spectra and distance energies of regular graphs and their iterated line graphs were analyzed. For this, it was necessary to establish certain properties of graphs of diameter two. Of these results we re-state here:

Theorem 4. *If $diam(G) \leq 2$ and if none of the three graphs F_1, F_2, and F_3, depicted in Figure 1, is an induced subgraph of G, then $diam(L(G)) \leq 2$.*

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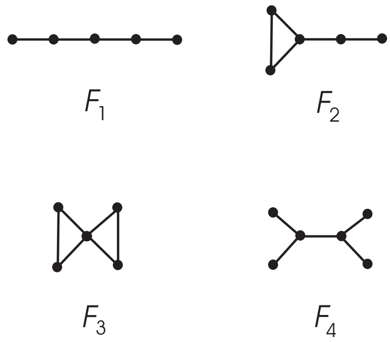

FIGURE 1. The "forbidden" induced subgraphs

Theorem 5. *If diam*$(G) \leq 2$ *and if none of the four graphs depicted in Figure 1 is an induced subgraph of* G*, then none of the four graphs of Figure 1 is an induced subgraph of* $L(G)$*.*

Theorem 6. *If diam*$(G) \leq 2$ *and if none of the four graphs depicted in Figure 1 is an induced subgraph of* G*, then for* $k \geq 1$*,*

(a) *diam*$(L^k(G)) \leq 2$ *and*

(b) *none of the four graphs of Figure 1 is an induced subgraph of* $L^k(G)$*.*

Theorem 4 happens to be correct.

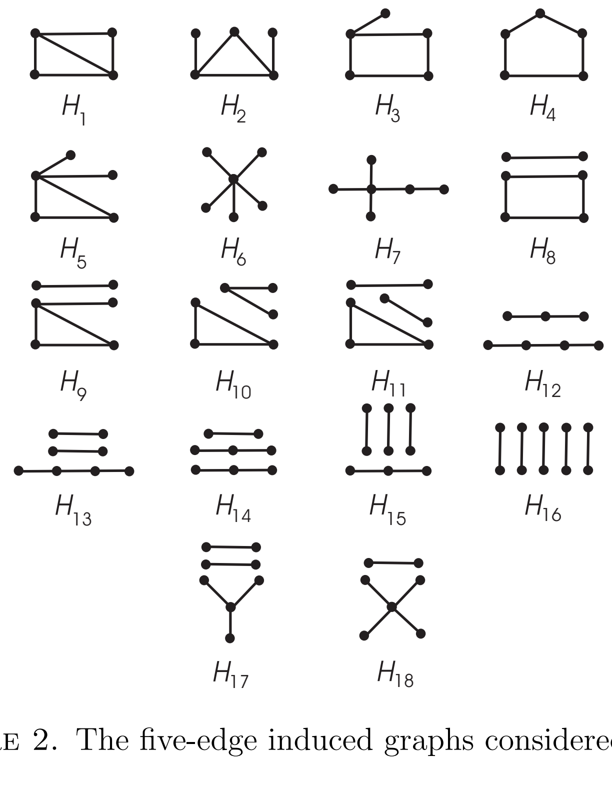

FIGURE 2. The five-edge induced graphs considered in [1]

In the proof of Theorem 5 it was claimed that if $diam(G) \leq 2$ and if none of the four graphs of Figure 1 is an induced subgraph of G, then any five-edge subsets of the edge sets of G induces one of the 18 graphs depicted in Figure 2. Since none of the line graphs of the graphs depicted in Figure 2 has F_i, $i = 1, 2, 3, 4$, as an induced subgraph one arrived at Theorem 5.

Theorem 5 is erroneous. The error is caused by the fact that in the case $diam(G) = 2$, in addition to the 18 five-edge induced subgraphs displayed in Figure 2, there exist other induced subgraphs generated by five edges of G, but containing more than five edges. One of the several such induced subgraphs is H_{19}, depicted in Figure 3. It is "obtained" from the graph G^* in Figure 3, by selecting its five edges numbered 1,2,3,4,5 and indicated by thick lines. In this case, of course, $G^* \cong H_{19}$.

By direct checking it can be seen that H_{19} does not contain F_i, $i = 1, 2, 3, 4$, as induced subgraphs. Nevertheless, its line graph $L(H_{19})$, also shown in Figure 3, contains F_1 as an induced subgraph. In fact, F_1 is induced by the vertices 1,2,3,4,5 of $L(H_{19})$, see Figure 3. This, in particular, implies $diam(L^2(G^*)) > 2$.

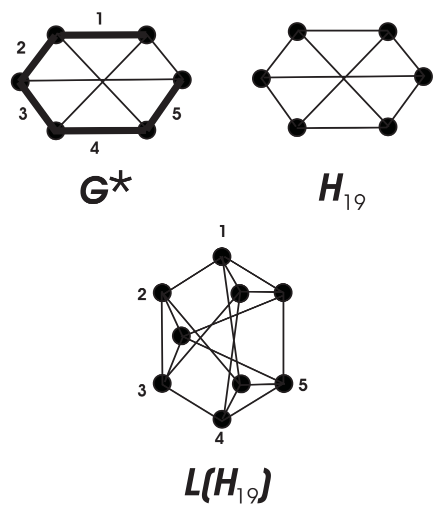

FIGURE 3. An overlooked induced subgraph, H_{19}, induced by the edges 1,2,3,4,5 of the graph G^* whose diameter is 2. The line graph $L(H_{19})$ of H_{19} possesses an induced subgraph isomorphic to F_1, induced by the vertices 1,2,3,4,5.

By this counterexample we see that Theorem 5 of [1] is not correct.

Since Theorem 6 is obtained by combining Theorems 4 and 5, also Theorem 6 is not generally valid. In fact, Theorem 6 holds for $k = 1$ and is violated for $k > 1$.

In connection with Theorem 6 it should be noted that the second line graph of the complete graph on four vertices, that is $L^2(K_4)$, also contains F_1 and F_2 as induced subgraphs. Consequently, $diam(L^3(K_4)) > 2$.

In view of the counterexamples outlined in this note, we are inclined to conjecture that a kind of inverse of Theorem 6 holds: If $diam(G) \leq 2$ and if none of the three graphs F_1, F_2, and F_3 from Figure 1 is an induced subgraph of G, then $diam(L^k(G)) > 2$ for some $k > 2$.

References

[1] H. S. Ramane, D. S. Revankar, I. Gutman, H. B. Walikar, *Distance spectra and distance energies of iterated line graphs of regular graphs*, Publ. Inst. Math. (Beograd) **85** (2009), 39–46.

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