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COMPLETENESS THEOREM FOR PROBABILITY MODELS WITH FINITELY MANY VALUED MEASURE IN LOGIC WITH INTEGRALS

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Abstract. The aim of the paper is to prove the completeness theorem for probability models, in logic with integrals, whose measures have any finite ranges.

1. Introduction

The probability logics are created as logics appropriate for study of structures arising in Probability Theory. They have been introduced by Jerome Keisler who studied first order structures in which probability is defined on the domen, i.e. structures $\langle \mathcal{M}, \mu \rangle$ where $\mathcal{M}$ is a first order structure and μ is probability. He began his study by probability logic $L_{\mathbb{A}P}$ in which ordinary quantifiers $\forall x$ and $\exists x$ are not allowed and instead of them, quantifier $(Px > r)$ is incorporated, where $\mathbb{A}$ is a countable admissible set such that $\mathbb{A} \subseteq \mathbb{HC}$ and $\omega \in \mathbb{A}$ (see [1]). The formula $(P\vec{x} \geq r)\varphi(\vec{x})$ means that the set $\{\vec{x} \mid \varphi(\vec{x})\}$ has a probability greater then or equal to r.

The development of probability model theory has engendered the need for the study of logics with a stronger expressive power than that of the logic $L_{\mathbb{A}P}$. Properties of random variables are usually easier to express using integrals rather than probability quantifiers. The logic $L_{\mathbb{A}\int}$, introduced by Keisler in [6] as an equivalent of the logic $L_{\mathbb{A}P}$, allows us to express many properties of random variables in an easier way. In this logic the quantifiers $\int \ldots dx$ are incorporated instead of the quantifiers $(Px \geq r)$. The

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logics $L_{\mathbb{A}P}$ and $L_{\mathbb{A}\int}$ correspond to two alternative approaches to integration theory: Lebesgue measure theory and the Daniel integral.

In [9], the quite similar investigation has been made for propositional probabilistic logic which are conceived as the logics for reasoning about uncertainity. The logic developed here is more expressive and more powerful.

2. Background

In order to prove the main results of the paper, we shall use the following theorem (see [2]).

Theorem 2.1. *Let $\mathcal{F}$ be a field of subset of set Ω. Then μ is a finitely many valued probability measure on $\mathcal{F}$ if and only if there is a real number $c > 0$ such that $\mu(A) > c$ whenever $A \in \mathcal{F}$ and $\mu(A) > 0$.*

Since, it is, in a way, measure in a probability models, in the logic $L_{\mathbb{A}P}$, expressed more explicitly, this feature of a measure can be obtained by axiom

$$\bigvee\nolimits_{c\in\mathbb{Q}^+} \bigwedge\nolimits_{\varphi\in\Phi_n} \Big((P\vec{x} > 0)\varphi(\vec{x}) \to (P\vec{x} > c)\varphi(\vec{x})\Big)\,,$$

where $\Phi_n \in \mathbb{A}$ and $\Phi_n = \{\varphi \mid \varphi \text{ has } n \text{ free variables}\}$ (see also [9]).

We will use the next result to create further interest in *axiom of finitely many valued measure* in the logic $L^{\text{fin}}_{\mathbb{A}\int}$. By induction on the complexity of terms we can prove that, for each term $\tau(\vec{x})$ of $L_{\mathbb{A}\int}$ and for each $r \in \mathbb{R} \cap \mathbb{A}$ there is a formula $\varphi_{\tau,r}$ of $L_{\mathbb{A}P}$ such that in every probability structure $\langle \mathfrak{A}, \mu\rangle$ for each $\vec{a} \in A^n$, the following is true: $\tau^{\mathfrak{A}}(\vec{a}) > r$ iff $\mathfrak{A} \models \varphi_{\tau,r}[\vec{a}]$ (see [8]).

3. $L^{\text{fin}}_{\mathbb{A}\int}$ logic and completeness theorem

Let $\mathbb{A}$ be a countable admissible set such that $\mathbb{A} \subseteq \mathbb{HC}$ and $\omega \in \mathbb{A}$ and let L be an Σ–definable set of finitary relation and constant symbols which has countably many variables $v_1, v_2, \ldots,$ n–ary connective $\mathbb{F}$ for each continuous function $F\colon \mathbb{R}^n \to \mathbb{R}$, $n \geq 0$ such that $F{\restriction}\mathbb{Q}^n \in \mathbb{A}$, and quantifiers $\int \ldots dx$ which build terms with bound variables. The logic $L^{\text{fin}}_{\mathbb{A}\int}$ is similar to the logic $L_{\mathbb{A}\int}$, the only difference is that the list of axioms of $L^{\text{fin}}_{\mathbb{A}\int}$ has the following *axiom of finitely many valued measure* added:

$$\bigvee_{c\in\mathbb{Q}^+} \bigwedge_{\tau\in D} \bigwedge_{r\in\mathbb{Q}} \bigvee_{k} \left[\int \mathbb{F}_{k,r}\Big(\tau(\vec{x})\Big)\, d\vec{x} > 0 \to \int \mathbb{F}_{k,r}\Big(\tau(\vec{x})\Big)\, d\vec{x} > c\right],$$

where

$$F_{k,r}(a)=\begin{cases}1, & a\geq r,\\ 0, & a\leq r-\frac{1}{k},\\ \text{linear}, & \text{otherwise}\,,\end{cases}$$

and D is a set of terms with finitely many free variables and $D\in\mathbb{A}$.

Since the conjunction in the axiom above can be made only according to elements of the admissible set $\mathbb{A}$, that is $\bigwedge_{\tau\in D}$ where $D\in\mathbb{A}$, we will apply the construction of the middle model (see [7]).

The probability structure for $L^{\mathrm{fin}}_{\mathbb{A}\int}$ is a structure $\langle\mathfrak{A},\mu\rangle$ such that $\mathfrak{A}$ is a first–order structure and μ is probability measure on A such that each singleton is measurable, each $R_i^{\mathfrak{A}}$ is $\mu^{(n_i)}$–measurable and μ is a finitely many valued probability measure. (The measure $\mu^{(n)}$ is the restriction of the completion of μ^n to the σ–algebra generated by the measurable rectangles and the diagonal sets $\{x\in A^n\mid x_i=x_j\}$.) Graded model is defined as in the logic $L_{\mathbb{A}P}$ with the exception that the measures μ_n are with a finite range in our logic.

We shall prove that this axiomatization is complete for Σ_1–definable theories with respect to the class of probability models with finitely many valued measure, by combining a Henkin construction (see [6] and [8]), and a weak–middle–strong model construction, such as that of Rašković [7] (see also [3]).

We shall introduce two sorts of auxiliary structures.

Definition 3.1. (i) A weak structure for $L^{\mathrm{fin}}_{\mathbb{A}\int}$ is a structure $\langle\mathfrak{A},I\rangle$ where $\mathfrak{A}$ is a first–order structure for L and I is a positive linear real function defined on the set of terms with at most one free variable x and parameters from A (I may be called an $\mathbb{A}$–Daniell integral an $\mathfrak{A}$).

(ii) A middle structure for $L^{\mathrm{fin}}_{\mathbb{A}\int}$ is a weak structure $\langle\mathfrak{A},I\rangle$ such that the axiom of finitely many valued measure holds uniformly in any term τ; that is, there is a $c\in\mathbb{Q}^+$ such that for each term τ and $\vec{a}\in A^n$, and for each $r\in\mathbb{Q}$, there is k, if $I\Big(\mathbb{F}_{k,r}\Big(\tau(x,\vec{a})\Big)\Big)>0$ then $I\Big(\mathbb{F}_{k,r}\Big(\tau(x,\vec{a})\Big)\Big)>c$ where

$$F_{k,r}(a)=\begin{cases}1, & a\geq r,\\ 0, & a\leq r-\frac{1}{k},\\ \text{linear}, & \text{otherwise}\,.\end{cases}$$

In both cases, for a term τ, we define $\tau^{\mathfrak{A}}$ inductively as for probability models, except that at the integral step we define

$$\left(\int \tau\,(x,\vec{y})\ dx\right)^{\mathfrak{A}}(\vec{a}) = I\Big(\tau,(x,\vec{a})\Big)\,.$$

By means of a Henkin construction argument similarly as in [8] we prove that a Σ_1–definable theory of $L^{\mathrm{fin}}_{\mathbb{A}\int}$ is consistent if and only if it has a weak model in which each theorem of $L^{\mathrm{fin}}_{\mathbb{A}\int}$ is true. Let $C \in \mathbb{A}$ be a set of new constant symbols introduced in this Henkin construction and let $K = L \cup C$.

The construction of the middle model from the weak model is the key part of the proof of the main result and we present it with the following theorem.

Theorem 3.1 (Middle Completeness Theorem)**.** *A Σ_1–definable theory $\mathbf{T}$ of $K^{\mathrm{fin}}_{\mathbb{A}\int}$ is consistent if and only if it has a middle model in which each theorem of $K^{\mathrm{fin}}_{\mathbb{A}\int}$ is true.*

Proof. In order to prove that consistent Σ_1–definable theory $\mathbf{T}$ of $K^{\mathrm{fin}}_{\mathbb{A}\int}$ has a middle model, we introduce language M with four kind of variables: $X, Y, Z, \ldots$ variables for sets, $x, y, z, \ldots$ variables for urelements, $r, s, t, \ldots$ for reals and $P, Q, R, S, T, \ldots$ variables for terms which represent functions $A^n \to \mathbb{R}, n \geq 0$. Predicates are $\leq$ for reals, $E^s_n(\vec{x}, X)$ ($n \geq 1$ and $\vec{x} = x_1, \ldots, x_n$) for sets, $E^t_{n+1}(\vec{x}, r, T)$ ($n \geq 0$) (with the meaning $T(x_1, \ldots, x_n) = r$), and $I(T, r)$ ($T\colon A^0 \to \mathbb{R}$ or $T\colon A^1 \to \mathbb{R}$ and $I(T) = r$ iff $I(T, r)$). Function symbols are $+, \cdot$ and set of function symbols $\mathbb{F}$ for terms and reals, such that $F \in \mathbb{C}_{\mathbb{A}}(\mathbb{R}^n)$. Constant symbols are X_φ for each formula φ of $K^{\mathrm{fin}}_{\mathbb{A}\int}$, T_τ for each term τ and $\bar{r}$ for each real number $r \in \mathbb{R} \cap \mathbb{A}$.

Let $\mathbf{S}$ be the following theory of $M_{\mathbb{A}}$ which has the following list of formulas:

1. Axioms of well–definedness
   (a) $(\forall X)\ \bigwedge_{n<m} \neg\,(\exists \vec{x}, \vec{y})\,\Big(E^s_m(\vec{x}, \vec{y}, X) \wedge E^s_n\,(\vec{x}, X)\Big)$ where $\{\vec{x}\} \cap \{\vec{y}\} = \varnothing$,
   (b) $(\forall T)\ \bigwedge_{n<m} \neg(\exists \vec{x}, \vec{y}, r, s)\Big(E^t_{m+1}(\vec{x}, \vec{y}, r, T) \wedge E^t_{n+1}(\vec{x}, s, T)\Big)$,
   (c) $(\forall T)\ (\forall \vec{x}, r, s)\ \Big((E^t_{n+1}(\vec{x}, r, T) \wedge E^t_{n+1}(\vec{x}, s, T)) \to r = s\Big)$.
2. Axioms of extensionality
   (a) $(\forall \vec{x})\ \Big(E^s_n\,(\vec{x}, X) \leftrightarrow E^s_n\,(\vec{x}, Y)\Big) \leftrightarrow X = Y\,,$
   (b) $(\forall \vec{x}, r)\ \Big((E^t_{n+1}(\vec{x}, r, T) \leftrightarrow E^t_{n+1}(\vec{x}, r, S)) \leftrightarrow T = S\,.$
3. Axioms of terms
   (a) $(\forall x, y)\ \Big[E^t_{2+1}\Big(x, y, 0, T_{\mathbf{1}(x=y)}\Big) \vee E^t_{2+1}\Big(x, y, 1, T_{\mathbf{1}(x=y)}\Big)\Big]\,,$
   (b) $(\forall \vec{x})\ \Big[E^t_{n+1}\Big(\vec{x}, 0, T_{\mathbf{1}(R(\vec{x}))}\Big) \vee E^t_{n+1}\Big(\vec{x}, 1, T_{\mathbf{1}(R(\vec{x}))}\Big)\Big]\,,$

(c) $E^t_{0+1}(\bar{r}, T_\tau)$ where τ has the form r,
(d) $(\forall \vec{x}) \Big[E^t_{n+1}(\vec{x}, r, T_\tau) \leftrightarrow (\exists P) \Big((\forall y, s) \ \Big(E^t_{1+1}(y, s, P)$
$\leftrightarrow E^t_{n+1+1}(\vec{x}, y, s, T_\sigma)\Big) \wedge I(P, r)\Big)\Big]$ where τ has the form $\int \sigma(\vec{x}, y)\, dy$,
(e) $(\forall \vec{x}) \Big[E^t_{n+1}\Big(\vec{x}, r, T_{\mathbb{F}(\tau_1, \ldots, \tau_k)}\Big) \leftrightarrow \Big(\bigwedge_{i=1}^{k} E^t_{n+1}(\vec{x}, r_i, T_{\tau_i}) \wedge r = \mathbb{F}(r_1, \ldots, r_k)\Big)\Big]$.

4. Axioms of satisfaction
(a) $(\forall \vec{x}) \Big(E^s_n(\vec{x}, X_{\tau \geq 0}) \leftrightarrow E^t_{n+1}(\vec{x}, r, T_\tau) \wedge r \geq 0\Big)$,
(b) $(\forall \vec{x}) \Big(E^s_n(\vec{x}, X_{\neg\varphi}) \leftrightarrow \neg E^s_n(\vec{x}, X_\varphi)\Big)$,
(c) $(\forall \vec{x}) \Big(E^s_n(\vec{x}, X_{\wedge\varphi}) \leftrightarrow \bigwedge_\varphi E^s_n(\vec{x}, X_\varphi)\Big)$.

5. Axioms of integral operators I
(a) $(\forall T) \Big(\bigwedge_{n \geq 2} \neg (\exists \vec{x}, r) \ E^t_{n+1}(\vec{x}, r, T) \leftrightarrow (\exists_1 s) \ I(T, s)\Big)$,
(b) $(\forall r) \ I(T_r, r)$,
(c) $(\forall S, T, r, s) \Big(I(r \cdot T + s \cdot S) = r \cdot I(T) + s \cdot I(S)\Big)$,
(d) $(\forall T) \Big((\forall x) \ (\exists r \geq 0) \ E^t_{1+1}(x, r, T) \rightarrow (\exists s \geq 0) \ I(T, s)\Big)$.

6. Axiom of finitely many valued measure

$$(\exists c) \ (\forall T) \bigwedge_{r \in \mathbb{Q}} \bigvee_k \Big(I\Big(\mathbb{F}_{k,r}(T) > 0\Big) \rightarrow I\Big(\mathbb{F}_{k,r}(T) > c\Big)$$

$$\text{where } F_{k,r}(a) = \begin{cases} 1, & a \geq r \\ 0, & a \leq r - \dfrac{1}{k} \\ \text{linear}, & \text{otherwise} \end{cases}$$

7. Axiom for an Archimedean field and all theorems of the theory of real closed field

8. Axioms which are transformations of axiom of $K^{\text{fin}}_{\mathbb{A}\int}$
$(\forall \vec{x}) \ E^s_n(\vec{x}, X_\varphi)$.

9. Axioms of realizability of all sentences φ in $\mathbf{T}$
$(\forall x) \ E^s_1(x, X_\varphi)$.

A weak structure $\langle \mathfrak{A}, I \rangle$ can be transformed to a standard structure $\mathcal{B} = \langle \mathrm{B}, \mathrm{P}, \mathrm{Q}, \mathrm{F}, {E^s_n}^{\mathcal{B}}, {E^t_{m+1}}^{\mathcal{B}}, I^{\mathcal{B}}, \leq, +, \cdot, \mathbb{F}_n, X^{\mathcal{B}}_\varphi, T^{\mathcal{B}}_\tau, \bar{r}\rangle$ for $M_{\mathbb{A}}$ where $\mathrm{B} = \mathrm{A}$, $\mathrm{P} \subseteq \bigcup_{n \geq 1} \mathbb{P}(A^n)$, $\mathrm{Q} \subseteq \bigcup_{n \geq 0} \mathbb{P}(A^n \times \mathbb{R})$, $\mathrm{F} \subseteq \mathbb{R}$ a field, $X^{\mathcal{B}}_\varphi \in \mathrm{P}$ for a formula φ and $T^{\mathcal{B}}_\tau \in \mathrm{Q}$ for a term τ. We can take $X^{\mathcal{B}}_\varphi = \Big\{\vec{a} \in A^n \mid \mathfrak{A} \models \varphi[\vec{a}]\Big\}$, $\mathrm{P} = \Big\{X^{\mathcal{B}}_\varphi \mid \varphi \in K^{\text{fin}}_{\mathbb{A}\int}\Big\}$, $T^{\mathcal{B}}_\tau(\vec{a}) = \tau^{\mathfrak{A}}(\vec{a})$ for $\vec{a} \in A^n$, $\mathrm{Q} = \Big\{T^{\mathcal{B}}_\tau \mid \tau \text{ is term of } K^{\text{fin}}_{\mathbb{A}\int}\Big\}$ and $I^{\mathcal{B}}_k(T^{\mathcal{B}}_\tau) = I(\tau)$ for a term τ with at most one free variable.

The theory $\mathbf{S}$ is Σ_1 definable over $\mathbb{A}$ and $\mathbf{S}_0 \subseteq \mathbf{S}$, $\mathbf{S}_0 \in \mathbb{A}$ has a standard model because the axiom of finitely many valued measure holds in the weak model. It follows by means of the Barwise Compactness Theorem (see [1]) that $\mathbf{S}$ has a standard model $\mathcal{B}$ which can be transformed to a middle model $\bar{\mathcal{B}}$ of $\mathbf{T}$ by taking:

$$\tau^{\bar{\mathcal{B}}}(\vec{x}) = r \quad \text{iff} \quad E^{t\,\mathcal{B}}_{m+1}(\vec{x}, 1, T_\tau)\left(R^{\bar{\mathcal{B}}}(\vec{a}) \quad \text{holds iff} \quad E^{t\,\mathcal{B}}_{m+1}\left(\vec{a}, r, T_{\mathbf{1}(R(\vec{x}))}\right)\right)$$
$$I^{\bar{\mathcal{B}}}(\tau(\vec{a}, y)) = I^{\mathcal{B}}(T_{\tau(\vec{a},y)})$$

for $\vec{a} \in B^n$.

This completes the proof. $\square$

Now we shall prove the completeness theorem for the logic $L^{\mathrm{fin}}_{\mathbb{A}\int}$. The method of proof is similar to the completeness proof of Keisler [6]. It uses Loeb's construction in [8] which corresponds to the Daniell integral. We first state a simple case of Loeb's results as a lemma.

Lemma 3.1 (Loeb). *In an ω_1–saturated nonstandard universe, let $\mathbf{L}$ be an internal vector lattice of functions from an internal set M into $^*\mathbb{R}$ (the set of hyper real numbers), and let I be an internal positive linear functional on $\mathbf{L}$, such that $1 \in \mathbf{L}$ and $I(1) = 1$. Then there is a complete probability measure μ on M such that for each finitely bounded $\phi \in \mathbf{L}$, the standard part of ϕ is integrable with respect to μ and its integral is equal to the standard part of $I(\phi)$.*

Theorem 3.2 (Completeness Theorem for $L^{\mathrm{fin}}_{\mathbb{A}\int}$)**.** *A Σ_1–definable theory $\mathbf{T}$ of $L^{\mathrm{fin}}_{\mathbb{A}\int}$ is consistent if and only if $\mathbf{T}$ has a probability model with finitely many valued measure.*

Proof. The soundness part is easy. To prove hard part let C be countable set of new constant symbols. We use the Henkin construction to obtain a maximal consistent set Φ of sentences of $K^{\mathrm{fin}}_{\mathbb{A}\int}$ where $K = L \cup C$, such that:

(1) if $\Gamma \subseteq \Phi$ and $\bigwedge \Gamma \in K^{\mathrm{fin}}_{\mathbb{A}\int}$, then $\bigwedge \Gamma \in \Phi$,

(2) if $(\int \tau(x)dx > 0) \in \Phi$, then $(\tau(c) > 0) \in \Phi$, for some $c \in C$.

Since Φ is complete and contains all the axioms of $K^{\mathrm{fin}}_{\mathbb{A}\int}$, the Henkin theory Φ induces a weak structure $\langle \mathfrak{A}, I\rangle$ with universe $A = C$, such that every sentence in Φ holds in $\langle \mathfrak{A}, I\rangle$.

The next step is the construction of the middle model for theory $\mathbf{T}$ of $L^{\mathrm{fin}}_{\mathbb{A}\int}$. Based on the Theorem 3.1. we get the middle model in which the axiom of finitely many valued measure holds uniformly in any term τ. Then we pass to an ω_1–saturated nonstandard universe and form internal middle integral structure $\langle ^*\mathfrak{A}, ^*I\rangle$. The Loeb construction of Daniell integral produces a probability measure μ on $^*A$ such that for each term $\tau(x)$ with parameters from ${}^*A$, the standard part of ${}^*I(\tau)$ is the integral of the standard part of $\tau^{{}^*\mathfrak{A}}$ with respect to μ. Similarly, we can construct a probability measure μ_n on ${}^*A^n$ using iterated integrals and in that way we get a graded model $\langle {}^*\mathfrak{A}, \mu_n\rangle$. From Loeb's construction, the axiom of finitely many valued measure and the Theorem 2.1. we conclude that the measures $\mu_n,\ n \in \mathbb{N}$ will have finite ranges. Finally, the axiom product measurability, which is essentially a translation of the axiom (B_4) for $L_{\mathbb{A}P}$ (see [6]), is satisfied almost every where in the graded model $\langle {}^*\mathfrak{A}, \mu_n\rangle$. This yields a probability model $\langle \mathfrak{B}, \mu\rangle$ of $\mathbf{T}$, where $\mathfrak{B} = {}^*\mathfrak{A}$ and $\mu = \mu_1$. $\square$

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