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ON THE EXISTENCE OF (196,91,42) HADAMARD DIFFERENCE SETS

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Abstract. We use group representations and factorization in the cyclotomic rings to show that $(196, 91, 42)$ Hadamard difference sets exist only in group $(C_7 \times C_7) \rtimes C_4$ with Gap location number $[196, 8]$. We also show that $(980, 89, 8)$ difference sets may only exist in four groups of order 980.

1. Introduction

Suppose that G is a multiplicative group of order v and D is a subset of G consisting k elements with $1 < k < v - 1$. If every non-identity element of G can be recovered λ times by the multi-set $\{d_1 d_2^{-1} : d_1, d_2 \in D, d_1 \neq d_2\}$, then D is called a non-trivial (v, k, λ) difference set. The natural number $n := k - \lambda$ is usually called the order of the difference set. If G is abelian (resp. non abelian or cyclic), then D is known as abelian (resp. non abelian or cyclic) difference set. Difference sets were introduced in the study of projective planes in cyclic groups by Singer [16] and since that time, there has been tremendous progress in the study of difference sets in abelian groups. Our interest in this paper is the class of difference sets with parameters $(4u^2, 2u^2 - u, u^2 - u)$, where u is any natural number. These difference sets are known as Menon-Hadamard difference sets and they exist in abelian groups of the form $(C_3)^{2s} \times C_4^a \times (C_2)^{2b}$. McFarland [14] ruled out the existence of abelian $(4p^2, 2p^2 - p, p^2 - p)$ difference sets for primes $p > 3$ and Smith [17] constructed

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an infinite family of non-abelian difference sets with parameters $(4t^2, 2t^2 - t, t^2 - t)$, where $t = 2^q \cdot 3^5 \cdot 5 \cdot 10^s, q, r, s \geq 0$ and $r > 0 \Rightarrow q > 0$. Part of Iiams's [6] work on difference sets with parameters $(4p^2, 2p^2 - p, p^2 - p)$ produced Theorem 2.4, in which he listed groups that could possibly admit difference sets with these parameters while AbuGhneim [1] showed that two of such groups cannot admit these difference sets. Other authors have also investigated difference sets with parameters $(980, 89, 8)$ and $(196, 91, 42)$. For instance, Lander [10] proved there are no $(980, 89, 8)$ difference sets in C_{980} while using Turyn's test [18], there are no $(196, 91, 42)$ difference sets in C_{196}. Also, Kopilovich [9] showed that groups $(C_2)^2 \times C_5 \times (C_7)^2$, $(C_2)^2 \times C_5 \times C_{49}$, $C_4 \times C_5 \times (C_7)^2$, $(C_2)^2 \times C_{49}$, $(C_2)^2 \times (C_7)^2$, and $C_4 \times (C_7)^2$ do not admit the respective difference sets. This paper is therefore, an extension of the efforts of the above authors and we state the main results.

Theorem 1.1. *There is no Hadamard* $(196, 91, 42)$*-difference set in any group* G *of order* 196 *with a normal subgroup* N *such that* $G/N \cong K$*, where* K *is a group of order* 28*,* C_{98} *or* D_{49}*.*

Corollary 1.1. *There is Hadamard* $(196, 91, 42)$*-difference set in group* $G = (C_7)^2 \rtimes C_4 = \langle x, y, z : x^7 = y^7 = z^4 = 1, xy = yx, zyz^{-1} = x, zxz^{-1} = y^{-1} \rangle$*, with Gap library location number* $[196, 8]$*.*

Theorem 1.2. *There is no* $(980, 89, 8)$*-difference set in any group* G *of order* 980 *with a normal subgroup* N *such that* $G/N \cong C_{14}$ *or* D_7*.*

Section two gives a brief background information required for this work while sections three and four establish our main results.

2. Preliminaries

Let G be any group of order v, it is more convenient to view elements of a difference set D as a member of a group ring $K[G]$, where K is a commutative ring with identity and G is a finite group. Without loss of generality, we take $K = \mathbb{Z}$ and view difference set $D = \{d_1, d_2, \ldots, d_k\}$ as a member of the group ring $\mathbb{Z}[G]$. In $\mathbb{Z}[G]$, $D = d_1 + d_2 + \ldots + d_k$ and the set of inverses of elements of D is $D^{(-1)} = d_1^{-1} + d_2^{-1} + \ldots + d_k^{-1}$. Thus, D satisfies the group ring equation

$$DD^{(-1)} = n \cdot 1_G + \lambda G \text{ and } DG = kG, \tag{2.1}$$

where 1_G is the identity element of G. If g is a non identity element of G, then the left and right translates of D denoted by gD and Dg respectively are also difference sets. Furthermore, if α is an automorphism of G, then $D^\alpha := \{\alpha(d) : d \in D\}$ is also a difference set. For each $g \in G$, if we take the left translates of D as blocks, then the resulting structure is called the development of D, $Dev(D)$ and G is the automorphism group of $Dev(D)$. In fact, ([10], Theorem 4.2),

Theorem 2.1. *Suppose that D is a (v, k, λ) difference set in a group G. Then the $Dev(D)$ is a (v, k, λ) symmetric design and G acts as a regular automorphism group of this design.*

A $\mathbb{C}$-representation of G is a homomorphism, $\chi : G \to GL(d, \mathbb{C})$, where $GL(d, \mathbb{C})$ is the group of invertible $d \times d$ matrices over $\mathbb{C}$. The positive integer d is the degree of χ. A linear representation (character) is a representation of degree one. We denote the set of all linear representations of G by G^*. In fact, G^* is an abelian group under multiplication and if G' is the derived group of G, then G^* is isomorphic to G/G'. Furthermore, the positive integer m is the exponent of the group G if $g^m = 1$ for all $g \in G$ and m is the smallest number with such property. If $\zeta_m := e^{2\frac{\pi i}{m}}$ is a primitive m-th root of unity, then $K_m := \mathbb{Q}(\zeta_m)$ is the cyclotomic extension of the set of rational numbers, $\mathbb{Q}$. Without loss of generality, we may replace $\mathbb{C}$ with the field K_m also known as the splitting field of G. This field is a Galois extension of degree $\phi(m)$, (ϕ is the Euler function) and a basis for K_m over $\mathbb{Q}$ is $S = \{1, \zeta_m, \zeta_m^2, \ldots, \zeta_m^{\phi(m)-1}\}$. S is also the integral basis for $\mathbb{Z}[\zeta_m]$. If G is an abelian group then the element

$$e_{\chi_i} = \frac{\chi_i(1)}{|G|} \sum_{g \in G} \chi_i(g) g^{-1} = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_i(g)} g \tag{2.2}$$

is the central primitive idempotents in $\mathbb{C}[G]$, where χ_i is an irreducible character of G. The set $\{e_{\chi_i} : \chi_i \in G^*\}$ is a basis for $K_m[G]$.

If G is an abelian group, then every element $A \in K_m[G]$ can be expressed uniquely by its image under the character $\chi \in G^*$. That is, $A = \sum_{\chi \in G^*} \chi(A) e_\chi$ and consequently, $\chi(A) e_\chi = A e_\chi$. It then follows that if $A \in K_m[G]$, then $A = A \sum_{\chi \in G^*} e_\chi = \sum_{\chi \in G^*} A e_\chi = \sum_{\chi \in G^*} \chi(A) e_\chi$. This brings us to an instrument, called an alias that is an interface between the values of group rings and combinatorial analysis. Aliases are members of group ring. Aliases allow us to transfer information from $K_m[G]$ to the group algebra $\mathbb{Q}[G]$ and then to $\mathbb{Z}[G]$. Suppose that G is an abelian group and $\Omega = \{\chi_1, \chi_2, \ldots, \chi_h\}$ is a set of characters of G. The element $\beta \in \mathbb{Z}[G]$ is known as

Ω-alias if for $A \in \mathbb{Z}[G]$ and all $\chi_i \in \Omega$, $\chi_i(A) = \chi_i(\beta)$. Suppose that $g \in G$, ϕ is a representation of G and σ is a Galois automorphism of K_m fixing $\mathbb{Q}$. Then $\sigma(\phi)$ is also a representation. In this case, σ and $\sigma(\phi)$ are algebraically conjugates. In particular, two linear representations of G are algebraically conjugates if they have the same kernel. Algebraically conjugacy is an equivalence relation.

With this information, we describe the rational idempotents of G as follows: If K_m is the Galois field over Q, then **central rational idempotents** in $\mathbb{Q}[G]$ are obtained by summing over the equivalence classes X_i on the e_χ's under the action of the Galois group of K_m over $\mathbb{Q}$. That is, $[e_{\chi_i}] = \sum_{e_{\chi_j} \in X_i} e_{\chi_j}, i = 1, \ldots, s$. For example, if G is a cyclic group of the form $C_{p^m} = \langle x : x^{p^m} = 1 \rangle$ (p is prime) whose characters are of the form $\chi_i = \zeta^i, i = 0, \ldots, m-1$, then the rational idempotents are

$$[e_{\chi_0}] = \frac{1}{p^m}\langle x \rangle, \tag{2.3}$$

and $0 \leq j \leq m-1$

$$[e_{\chi_{p^j}}] = \frac{1}{p^{j+1}}\left(p\langle x^{p^{m-j}}\rangle - \langle x^{p^{m-j-1}}\rangle\right). \tag{2.4}$$

The following is the basic formula employed in the search of difference set [7].

Theorem 2.2. *Let G be an abelian group and K_m be a field. Suppose that $G^*/\sim$ is the set of equivalence classes of characters with $\{\chi_o, \chi_1, \ldots, \chi_s\}$ a system of distinct representatives for the equivalence classes.*

Then for $A \in K[G]$, we have

$$A = \sum_{i=o}^{s} \alpha_i [e_{\chi_i}], \tag{2.5}$$

where α_i is any χ_i-alias for A.

Equation (2.5) is known as **the rational idempotent decomposition** of A.

Given that G is a group of order v and D is a (v, k, λ) difference set in G. Suppose that N is a normal subgroup of G. Then $\psi : G \longrightarrow G/N$ is a homomorphism. We can also extend ψ, by linearity, to the corresponding group rings. The difference set image in G/N (also known as the contraction of D with respect to the kernel N) is the multi-set $D/N = \psi(D) = \{dN : d \in D\}$. If $T^* = \{1, t_1, \ldots, t_h\}$ is a left transversal of N in G, then we write $\psi(D) = \sum_{t_j \in G} d_j t_j N$, where the integer $d_j = |D \cap t_j N|$ is called the **intersection number** of $\psi(D)$ with respect to N. In this work, we shall always use the notation $\hat{D}$ for $\psi(D)$, the difference set image in a homomorphic image of G

and denote the number of times d_i equals i by $m_i \geq 0$. Group m' is the abbreviation for groups of order m'. The following lemma states the properties of $\hat{D}$.

Lemma 2.1. *Let D be a difference set in a group G and N be a normal subgroup of G. Suppose that $\psi : G \longrightarrow G/N$ is a natural epimorphism. Then*

(a) $\hat{D}\hat{D}^{(-1)} = n \cdot 1_{G/N} + |N|\lambda(G/N)$

(b) $\sum d_i^2 = n + |N|\lambda$

(c) $\chi(\hat{D})\overline{\chi(\hat{D})} = n \cdot 1_{G/N}$, *where χ is a non trivial representation of G/N.*

The following lemma gives information about the character value of $\chi(\hat{D})$.

Lemma 2.2. *Suppose that G is group of order v with normal subgroup N such that G/N is abelian. If $\hat{D} \in \mathbb{Z}[G/N]$ and $\chi \in (G/N)^*$, then*

$$|\chi(\hat{D})| = \begin{cases} k, & \text{if } \chi \text{ is a principal character of } G/N \\ \sqrt{k-\lambda}, & \text{otherwise.} \end{cases}$$

Dillon [3] proved the following results which will be used to obtain difference set images in dihedral group of a certain order if the difference images in the cyclic group of same order are known.

Theorem 2.3 (Dillon Dihedral Trick)**.** *Let H be an abelian group and let G be the generalized dihedral extension of H. That is, $G = \langle Q, H : Q^2 = 1, QhQ = h^{-1}, \forall h \in H\rangle$. If G contains a difference set, then so does every abelian group which contains H as a subgroup of index 2.*

Corollary 2.1. *If the cyclic group Z_{2m} does not contain a (nontrivial) difference set, then neither does the dihedral group of order $2m$.*

The method we intend to use in this paper is known as representation theoretic method made popular by Leibler ([13]). Smith and others also used this method in search of difference sets [17]. This approach entails getting information about the difference set D in G by first obtaining a comprehensive lists $\Omega_{G/N}$, of difference set image distribution scheme in factor groups of G. The reason for doing this is to enable us to garner information about D as we gradually increase the size of the factor group. If at a point the distribution list $\Omega_{G/N}$ is empty, then this signifies non-existence. In obtaining this list, $\Omega_{G/N}$, we use lemmas 2.1, 2.2 and the difference set equation (2.5). In the study of difference sets in groups of order 196, we look at the difference set

images in the factor groups of order $7, 14, 28, 49$ and 98. We shall use the following result to show that some of the images of difference set in groups of order 14 cannot be lifted to those of group 28.

Lemma 2.3 (The Variance Technique)**.** *Suppose that G is a group and D is a (v, k, λ) difference set in G. Suppose that N is a normal subgroup of G and $\psi : G \longrightarrow G/N$ is a homomorphism. Let $\hat{D}$ be the difference set image in G/N and T^* is a left transversal of N in G such that $\{d_i\}$ is a sequence of intersection numbers and $\{m_i\}$, where m_i the number of times d_i equals i. Then*

$$\sum_{i=0}^{|N|} m_i = |G/N|, \tag{2.6}$$

$$\sum_{i=0}^{|N|} i m_i = k, \tag{2.7}$$

$$\sum_{i=0}^{|N|} i(i-1)m_i = \lambda(|N| - 1). \tag{2.8}$$

The determination of difference set images in cyclic factor groups requires aliases. The aliases require the knowledge of how the ideal generated by $\chi(\hat{D})$ factors in the cyclotomic ring $\mathbb{Z}[\zeta_{m'}]$, $\zeta_{m'}$ is the m'^{th} root of unity and m' is the exponent of G/N. For the purpose of this paper, if χ is not a principal character then $|\chi(\hat{D})| = m$, where $m = 3^2, 7$ and we require how the ideals generated by 3 or 7 factors in $\mathbb{Z}[\zeta'_m]$, $m' = 7, 14, 28$. We need the following results:

Suppose p is any prime and m' is an integer such that $\gcd(p, m') = 1$. Suppose that d is the order of p in the multiplicative group $\mathbb{Z}^*_{m'}$ of the modular number ring $\mathbb{Z}_{m'}$. Then the number of prime ideal factors of the principal ideal (p) in the cyclotomic integer ring $\mathbb{Z}[\zeta_{m'}]$ is $\frac{\phi(m')}{d}$, where ϕ is the Euler ϕ-function, i.e. $\phi(m') = |\mathbb{Z}^*_{m'}|$ [11]. Using this information, the ideal generated by 3 is prime in $\mathbb{Z}[\zeta_{m'}]$, $m' = 7, 14$ while the ideal generated by 7 ramifies in $\mathbb{Z}[\zeta_{m'}]$, $m' = 7, 14, 28$.

According to Turyn [18], an integer n is said to be semi-primitive modulo m' if for every prime factor p of n, there is an integer i such that $p^i \equiv -1 \mod m'$. In this case, -1 belongs to the multiplicative group generated by p. Furthermore, n is self conjugate modulo m' if every prime divisor of n is semi primitive modulo m'_p, m'_p is the largest divisor of m' relatively prime to p. This means that every prime ideals over n in $\mathbb{Z}[\zeta_{m'}]$ are fixed by complex conjugation. For instance, $3^3 \equiv -1 \pmod{28}$. In this paper, we shall use the phase $\underline{m \text{ factors trivially}}$ in $\mathbb{Z}[\zeta_{m'}]$ if the ideal generated by m

is prime (or ramifies) in $\mathbb{Z}[\zeta_{m'}]$ or m is self conjugate modulo m'. In this situation, if G/N is a group with exponent m', $\hat{D}$ is the difference set image of order $n = m^2$ in G/N and χ is a non trivial representation of G/N, then $\chi(\hat{D}) = m\zeta_{m'}^i$.

Finally, we look at subgroup properties of a group that can aid the construction of difference set image. For the convenience of the reader, we reproduce the idea of Gjoneski, Osifodunrin and Smith [5] with some additions. Suppose that H is a group of order $2h$ with a central involution z. We take $T = \{t_i : i = 1, \ldots, h\}$ to be the transversal of $\langle z\rangle$ in H so that every element in H is viewed as $t_i z^j$, $0 \leq i \leq h$, $j = 0, 1$. Denote the set of all integral combinations, $\sum_{i=1}^{h} a_i t_i$ of elements of T, $a_i \in \mathbb{Z}$ by $\mathbb{Z}[T]$. The subgroup $\langle z\rangle$ has two irreducible representations: $z \mapsto 1$ or $z \mapsto -1$. Let φ_0 be the representation induced on H by the trivial representation $z \mapsto 1$ and φ_1 be the representation induced on H by the non trivial representation $z \mapsto -1$. Using the Frobenius reciprocity theorem [12], every irreducible representation of H is a constituent of φ_0 or φ_1. Thus, we may write any element X of the group ring $\mathbb{Z}[H]$ in the form

$$X = X\left(\frac{1+z}{2}\right) + X\left(\frac{1-z}{2}\right). \tag{2.9}$$

Let A be the group ring element created by replacing every occurrence of z in X by 1. Also, let B be the group ring element created by replacing every occurrence of z in H by -1. Then

$$X = A\left(\frac{\langle z\rangle}{2}\right) + B\left(\frac{2-\langle z\rangle}{2}\right), \tag{2.10}$$

where $A = \sum_{i=1}^{h} a_i t_i$ and $B = \sum_{j=1}^{h} b_j t_j, a_i, b_j \in \mathbb{Z}$. As $X \in \mathbb{Z}[H]$, A and B are both in $\mathbb{Z}[T]$ and $A \equiv B \mod 2$. We may equate A with the homomorphic image of X in $G/\langle z\rangle$. Consequently, if X is a difference set, then the coefficients of t_i in the expression for A will be intersection number of X in the coset $\langle z\rangle$. In particular, if K is a subgroup of H such that

$$H \cong K \times \langle z\rangle, \tag{2.11}$$

then we may assume that A and B are in the group ring $\mathbb{Z}[K]$ and $BB^{(-1)} = (k-\lambda)\cdot 1$. The search for the homomorphic image A in K gives considerable information about the element B. We describe B in terms of A as follows: If the structure of a group H is like (2.11), then the characters of the group are induced by those of K and $\langle z\rangle$. Let $\varphi_{0,0}$ be the characters of H induced by both trivial characters of K and $\langle z\rangle$; $\varphi_{1,s}$,

induced by non-trivial characters of K and $\langle z\rangle$; $\varphi_{1,0}$, induced by trivial character of K and non-trivial character of $\langle z\rangle$ while $\varphi_{0,s}$, is the character induced by non-trivial characters of K and trivial character of $\langle z\rangle$. Suppose that A is a difference set image in K. Then by Lemma 2.2,

$$\varphi_{0,0}(A) = k, |\varphi_{0,s}(A)| = \sqrt{n}, |\varphi_{1,0}(B)| = \sqrt{n}, |\varphi_{1,s}(B)| = \sqrt{n}. \tag{2.12}$$

The identity element of $\mathbb{Z}[K]$ is K and since A is a rational idempotent, it is of the form $\frac{Y}{|K|}$, $Y \in \mathbb{Z}[K]$. We subtract $k+\sqrt{n}$ or $k-\sqrt{n}$ multiples of $\frac{K}{|K|}$ from both sides of $\varphi_{0,0}(A) = k$ to get $|\varphi_{0,0}(A - (\frac{k+\sqrt{n}}{|K|})K)| = \sqrt{n}$ or $|\varphi_{0,0}(A - (\frac{k-\sqrt{n}}{|K|})K)| = \sqrt{n}$. Set $\alpha = \frac{k+\sqrt{n}}{|K|}$ or $\alpha = \frac{k-\sqrt{n}}{|K|}$ and $B = A - \alpha K$, k is the size of difference set. The entries of A are non-negative integers and if $|K|$ divides $k+\sqrt{n}$ or $k-\sqrt{n}$, then $BB^{(-1)} = (k-\lambda)\cdot 1$ and

$$\hat{D} = A\left(\frac{\langle z\rangle}{2}\right) + gB\left(\frac{2-\langle z\rangle}{2}\right), \tag{2.13}$$

$g \in H$. (2.13) can be used to determine the existence or otherwise of difference set image in H. However, this approach fails to yield a definite result if $|K| \nmid (k+\sqrt{n})$ and $|K| \nmid (k-\sqrt{n})$. To buttress the point being made here, consider the parameter set $(70, 24, 8)$ in the group $C_{70} \cong C_{35} \times C_2$. Take $K = C_{35}$. This shows that $|K| = 35$ and 35 does not divide $(24+4)$ or $(24-4)$. It is known that the group C_{70} does not admit this difference set ([10], Table 6-1). On the other hand, consider $(320, 88, 24)$ difference set in the group $H = (C_2)^6 \times C_5$. Take $K = (C_2)^5 \times C_5$ and $|K| = 160$. Also 160 does not divide $(88+8)$ or $(88-8)$. Davis and Jedwab [2] constructed $(320, 88, 24)$ difference set in H. The following theorem summarizes part of Iiam's [6] work on non-existence of difference sets with parameters $(4p^2, 2p^2-p, p^2-p)$.

Theorem 2.4. *If $p \geq 5$ is a prime and G is a group of order $4p^2$ containing a $(4p^2, 2p^2-p, p^2-p)$ difference set, then one of the following holds:*

(a) *G has an irreducible complex representation of degree 4. In particular, G is isomorphic to one of G_4, G_{13}, G_{14}, G_{15} or G_{16}.*
(b) *$G \cong G_{11}$ and $p \equiv 1 \pmod 4$, where*

$$\begin{aligned} G_4 &\cong \langle x, z | x^{p^2} = z^4 = 1, zxz^{-1} = x^f \rangle \\ &\cong \langle x | x^{p^2} = 1\rangle \rtimes \langle z | z^4 = 1\rangle; \end{aligned}$$

$$G_{11} \cong \langle x, y, z | x^p = y^p = z^4 = 1, xy = yx, zyz^{-1} = y^{-1}, zxz^{-1} = x \rangle$$
$$\cong \langle x, y | x^p = y^p = 1, xy = yx \rangle \rtimes \langle z | z^4 = 1 \rangle;$$

$$G_{13} \cong \langle x, y, z | x^p = y^p = z^4 = 1, xy = yx, zyz^{-1} = x, zxz^{-1} = y^{-1} \rangle$$
$$\cong \langle x, y | x^p = y^p = 1, xy = yx \rangle \rtimes \langle z | z^4 = 1 \rangle;$$

$$G_{14} \cong \langle x, y, z | x^p = y^p = z^4 = 1, xy = yx, zyz^{-1} = y^f, zxz^{-1} = x \rangle$$
$$\cong \langle x, y | x^p = y^p = 1, xy = yx \rangle \rtimes \langle z | z^4 = 1 \rangle;$$

$$G_{15} \cong \langle x, y, z | x^p = y^p = z^4 = 1, xy = yx, zyz^{-1} = y^f, zxz^{-1} = x^{-1} \rangle$$
$$\cong \langle x, y | x^p = y^p = 1, xy = yx \rangle \rtimes \langle z | z^4 = 1 \rangle;$$

$$G_{16} \cong \langle x, y, z | x^p = y^p = z^4 = 1, xy = yx, zyz^{-1} = y^f, zxz^{-1} = x^f \rangle$$
$$\cong \langle x, y | x^p = y^p = 1, xy = yx \rangle \rtimes \langle z | z^4 = 1 \rangle;$$

with $f^2 \equiv -1 \pmod{p^2}$.

AbuGhneim [1] proved in his dissertation that G_{11} and G_{14} do not admit a difference set. This conclusion is based on the fact if G is a group of order $4p^2$ and N a normal subgroup of G such that $G/N \cong C_{4p}$, $C_{2p} \times C_2$ or D_{2p}, then G does not admit $(4p^2, 2p^2 - p, p^2 - p)$ difference sets. In our case, $p = 7$ and we will extend this result by showing that the fourth group of order $4p$ do not admit this difference set.

3. There are no group 28 images in G, $|G| = 196$

We compute the possible intersection numbers of $\hat{D}$ with the cosets of normal subgroups N for which G/N is either isomorphic to a 2-group or a group of order 14. Thereafter, we show that the fourth group of order 28 does not admit $(196, 91, 42)$ difference sets. In this section, G is a group of order 196.

3.1. THE 2-GROUP IMAGES

We generate difference set image in factor groups of G whose orders are powers of 2.

3.1.1. The C_2 Image. Suppose that N is a normal subgroup of G such that $G/N \cong C_2 = l < x : x^2 = 1 \rangle$. Let the image of difference set in C_2 be $\hat{D} = d_0 + d_1 x$.

Then by Lemma 2.1, $(d_0 + d_1x)^2 = 49 + 4116C_2$. Expand both sides, simplify and equate corresponding coefficients of the powers of x to get

$$d_0^2 + d_1^2 = 4165 \quad \text{and} \quad 2d_0d_1 = 4116.$$

The above equations imply $d_0 - d_1 = \pm 7$ and $d_0 + d_1 = \pm 91$. We translate if necessary to obtain $d_0 - d_1 = 7$ and $d_0 + d_1 = 91$. Thus, up to translation, the difference set image in C_2 is $7 + 42\langle x\rangle$.

3.1.2 The C_4 Image. Suppose that there is a normal subgroup N such that $G/N \cong C_4 = \langle x : x^4 = 1\rangle$. We perceive the difference set image in C_4 as $\hat{D} = \sum_{j=0} d_j x^j$, $j = 0, 1, 2, 3$. The characters of C_4 are of the form $\chi_j(x) = i^j$, $j = 0, 1, 2, 3$, $i := \exp(\frac{2\pi j}{4})$. Thus, the rational idempotents are

$$[e_{\chi_0}] = \frac{1}{4}\langle x\rangle; \quad [e_{\chi_1}] = \frac{1}{4}(2\langle x^2\rangle - \langle x\rangle); \quad [e_{\chi_2}] = \frac{1}{2}(2 - \langle x^2\rangle).$$

As $\chi_j(\hat{D})(\overline{\chi_j(\hat{D})}) = 7^2$, $j \neq 0$ and the fact that the ideal generated by 7 does not factor in the cyclotomic ring $\mathbb{Z}[i]$, we have $\chi_0(\hat{D}) = 91$, $\chi_1(\hat{D}) = \pm 7$, $\chi_2(\hat{D}) = \pm 7i^s$. Consequently, the aliases are $\alpha_{\chi_0} = 91$, $\alpha_{\chi_1} = \pm 7$ and $\alpha_{\chi_2} = \pm 7x^s$. Therefore, the difference set equation is

$$\hat{D} = \alpha_{\chi_0}[e_{\chi_0}] + \alpha_{\chi_1}[e_{\chi_1}] + \alpha_{\chi_2}[e_{\chi_2}].$$

However a solution exists if and only if

$$\hat{D} = \frac{91}{4}\langle x\rangle + \frac{7}{4}\langle x^2\rangle(1 - x) + x^s\frac{7}{2}(1 - x^2), \qquad s = 0, 2. \tag{3.1}$$

We translate, if necessary, to obtain the unique C_4 image as $7 + 21\langle x\rangle$.

3.2. THE GROUP 14 IMAGES

3.2.1. The C_7 images. Suppose G has a normal subgroup N such that $G/N \cong C_7 = \langle x : x^7 = 1\rangle$. Suppose also that the difference set image $\hat{D} = \sum_{i=0}^{6} d_i x^i$ exists in C_7. This image could also be viewed as a 1×7 matrix with the columns indexed by the powers of x. The characters of C_7 are of the form $\chi_i(x) = \zeta^i$, $i = 0, \ldots, 6$. Using (2.3) and (2.4), the two rational idempotents are:

$$[e_{\chi_0}] = \left(\frac{\langle x\rangle}{7}\right) \quad \text{and} \quad [e_{\chi_1}] = \left(\frac{7 - \langle x\rangle}{7}\right).$$

Thus, the difference set image is

$$\hat{D} = \sum_{j=0,1} \alpha_{e_{\chi_i}}[e_{\chi_i}], \tag{3.2}$$

where $\alpha_{e_{\chi_i}}$ is an alias. As the ideal generated by 7 does not factor in the cyclotomic field $\mathbb{Z}[\zeta]$ and the fact that $\alpha_{e_{\chi_0}} = 91$, (3.2) becomes

$$\hat{D} = 91[e_{\chi 0}] \pm 7x^i[e_{\chi 1}], \qquad i = 0, \ldots, 6.$$

We translate if necessary, to get $A_1 = 7 + 12\langle x\rangle$ and $A_2 = -7 + 14\langle x\rangle$. Next, we look at the factor group of order 14.

3.2.2. The C_{14} images. We assume that there is also a normal subgroup N such that G/N is isomorphic to a group of order 14.

First take $G/N = C_{14} \cong C_7 \times C_2 = \langle x, y : x^7 = y^2 = 1 = [x, y]\rangle$. Suppose that the difference set image in C_{14} is $\hat{D} = \sum_{s=0}^{6}\sum_{t=0}^{1} d_{s,t}x^s y^t$, viewed as a 2×7 matrix with the columns indexed by the powers of x and rows by powers of y. Since G/N is of the form (2.11), we can use (2.13), with $|K| = 7$, $n = 49$, $k = 91$ and $B_j = A_j - 12K$, where A_j is a difference set image in C_7. Thus, the difference set equation is

$$\hat{D} = A_i\Big(\frac{1+y}{2}\Big) + x^m y^l B_j\Big(\frac{1-y}{2}\Big), \tag{3.3}$$

$m = 0, \ldots, 6$, $l = 0, 1$; $i, j = 1, 2$. In this case, $A_1\Big(\frac{1+y}{2}\Big) = \frac{1}{2}\Big((7+12\langle x\rangle)+(7+12\langle x\rangle)y\Big)$, $A_2\Big(\frac{1+y}{2}\Big) = \frac{1}{2}\Big((-7+14\langle x\rangle)+(-7+14\langle x\rangle)y\Big)$, $B_1\Big(\frac{1-y}{2}\Big) = \frac{1}{2}\Big(7-7y\Big)$, $B_2\Big(\frac{1-y}{2}\Big) = \frac{1}{2}\Big((-7+2\langle x\rangle)+(7-2\langle x\rangle)y$. Notice that each of the matrices $A_i\Big(\frac{1+y}{2}\Big)$ and $B_j\Big(\frac{1+y}{2}\Big)$ has only one column of fractions and consequently, the value of m must be 0. Up to translation, the difference set images in C_{14} are $F_1 = (7+6\langle x\rangle)+(6\langle x\rangle)y$, $F_2 = (-7+7\langle x\rangle)+(7\langle x\rangle)y$, $F_3 = (7+5\langle x\rangle)+(7\langle x\rangle)y$ and $F_4 = (-7+8\langle x\rangle)+(6\langle x\rangle)y$. Secondly, take $G/N = D_7 = \langle x, y : x^7 = y^2 = 1 = yxy = x^{-1}\rangle$ and $\hat{D} = \sum_{s=0}^{6}\sum_{t=0}^{1} d_{s,t}x^s y^t$, the difference set image in G/N. We use Dillon dihedral trick with the difference set images in C_{14}. As the presentation of this group is similar to that of C_{14}, we just apply the representation:

$$\chi : x \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where ζ is the seventh root of unity, to each image set in C_{14} and verify $\chi(F_i)\overline{\chi(F_i)} = 49I_2$. Since this condition is satisfied for each i, then $F_i, i = 1, 2, 3, 4$ is also image set in D_7.

3.3. THERE ARE NO GROUP 28 IMAGES

Suppose that there is a normal subgroup N of G such that G/N is isomorphic to a group of order 28. The work of AbuGhneim [1] showed G does not admit difference

sets if $G/N \cong C_{28}$, D_{14} and $C_{14} \times C_2$. Thus, we look at the generalized Quaternion group of order 28, with GAP location number $[28, 1]$.

3.3.1. There are no $C_7 \rtimes C_4$ images. Consider $G/N \cong C_7 \rtimes C_4 = \langle x, y : x^7 = y^4 = 1, yxy^{-1} = x^6 \rangle$. The derived of G/N is isomorphic to $\langle x \rangle$ and the center of G/N is $C(G/N) \cong \langle y^2 \rangle$. Suppose that the difference set image in G/N is $\hat{D} = \sum_{s=0}^{6} \sum_{t=0}^{3} d_{s,t} x^s y^t$, viewed as a 4×7 matrix with the columns indexed by the powers of x and rows by powers of y. Since $(G/N)/\langle y^2 \rangle \cong D_7$ and using the information about the difference set image in D_7, the map $y^2 \mapsto 1$ generates the system of equations

$$d_{s0} + d_{s2} = f_{s0}, \qquad d_{s1} + d_{s3} = f_{s1} \qquad s = 0, \ldots, 6 \tag{3.4}$$

where 2×7 matrix (f_{st}) is a difference set image set in D_7. Furthermore, $(G/N)/\langle x \rangle \cong C_4$ and the map $x \mapsto 1$ yields more linear equations

$$\sum_{s=0}^{6} d_{s0} = c_0, \qquad \sum_{s=0}^{6} d_{s1} = c_2, \qquad \sum_{s=0}^{6} d_{s2} = c_2, \qquad \sum_{s=0}^{6} d_{s3} = c_3, \tag{3.5}$$

where the 1×4 matrix (c_t), is the unique difference set image in C_4. We have used all the lifted representations of G/N from normal subgroups. The group G/N has three other equivalent degree two representations. One of them is

$$\chi : x \mapsto \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}, \quad y \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

where ζ and i are the seventh and fourth roots of unity respectively. By applying this representation to $\hat{D}$, we get $\chi(\hat{D}) = \begin{pmatrix} a & bi \\ \bar{b}i & \bar{a} \end{pmatrix}$, where $a = \sum_{s=0}^{6} (d_{s0} - d_{s2})\zeta^s$, $b = \sum_{s=0}^{6} (d_{s1} - d_{s3})\zeta^s$ and $a, b \in \mathbb{Z}[\zeta]$. Furthermore,

$$\chi(\hat{D})\overline{\chi(\hat{D})} = \begin{pmatrix} a\bar{a} + b\bar{b} & 0 \\ 0 & a\bar{a} + b\bar{b} \end{pmatrix}.$$

As we require $\chi(\hat{D})\overline{\chi(\hat{D})} = 49I_2$, where I_2 is a 2×2 matrix, it follows that

$$a\bar{a} + b\bar{b} = 49. \tag{3.6}$$

Our next step is to eliminate those difference set images in D_7 that cannot be extended to difference set images in H using the variance technique. We observe that since the coset bound is $|N| = 7$, every intersection number $d_{s,t}$ satisfies $0 \leq d_{s,t} \leq 7$. The distribution of the difference set images in D_7 are: $6^{13}13^1$, $0^1 7^{13}$, $5^6 7^7 12^1$ and $1^1 6^7 8^6$. The distribution $6^{13}13^1$ means that the intersection number 6 appears thirteen times

and the intersection number 13 appears once. The coset bound of 7 ensures that we look at the following five cases:

Case 1: The distribution $6^{13}13^1$, 13 split as $(7,6)$ and 6 split as $(6,0)$, $(5,1)$, $(4,2)$ or $(3,3)$;

Case 2: The distribution $5^67^712^1$, 12 split as $(7,5)$, 7 split as $(7,0)$, $(6,1)$, $(5,2)$ or $(4,3)$ and 5 split as $(5,0)$, $(4,1)$ or $(3,2)$;

Case 3: The distribution $5^67^712^1$, 12 split as $(6,6)$, 7 split as $(7,0)$, $(6,1)$, $(5,2)$ or $(4,3)$ and 5 split as $(5,0)$, $(4,1)$ or $(3,2)$;

Case 4: The distribution 0^17^{13}, 7 split as $(7,0)$, $(6,1)$, $(5,2)$ or $(4,3)$ while 0 split as $(0,0)$;

Case 5: The distribution $1^16^78^6$, 8 split as $(8,0)$, $(7,1)$, $(6,2)$, $(5,3)$ or $(4,4)$, 6 split as $(6,0)$, $(5,1)$, $(4,2)$ or $(3,3)$ and 1 can only split as $(1,0)$.

We claim that cases 3 and 4 are not feasible.

The distribution 0^17^{13} cannot be lifted to solution in G/N:

Let $0 \leq \alpha_i \leq 13$, $i = 0,1,2,3$ be the number of intersection number 7 that split as (7, 0), (6, 1), (5, 2) or (4, 3) respectively. Using the symbols of variance technique (Lemma 2.3), $m_0 = \alpha_0 + 2$, $m_1 = \alpha_2$, $m_2 = \alpha_2$, $m_3 = \alpha_3$, $m_4 = \alpha_3$, $m_5 = \alpha_2$, $m_6 = \alpha_1$ and $m_7 = \alpha_0$. The variance technique equations (2.6) - (2.8) are:

$$\sum_{i=0}^{3} \alpha_i = 13, \tag{3.7}$$

$$21\alpha_0 + 15\alpha_1 + 11\alpha_2 + 9\alpha_3 = 126. \tag{3.8}$$

In this case, (2.7) is redundant. From (3.7), the sum of four positive integers is odd. This implies that either one or three of the numbers are odd. Suppose that one of these numbers is odd. Then the remaining three numbers are even. Using this information in (3.8) with odd coefficients, we see that an odd number is on the left hand side of this equation while the right hand side is even. This is a contradiction. Also, if three of the numbers on the left hand side of (3.7) are odd and only one is even, then we reach the same conclusion. Thus, there is no feasible solution. A similar argument shows that the distribution $5^67^712^1$ cannot be lifted to solution in G/N.

Next, we show that cases 1, 2 and 5 do not yield viable solutions. To achieve this, we garner information about the algebraic numbers a and b. But first, we rewrite

(3.4) as

$$d_{s2} = f_{s0} - d_{s0}, \qquad d_{s3} = f_{s1} - d_{s1} \qquad s = 0, \ldots, 6 \tag{3.9}$$

and substitute in a and b to get

$$A := 2\sum_{s=0}^{6} d_{s0}\zeta^s - \sum_{s=0}^{6} f_{s0}\zeta^s, \quad B := 2\sum_{s=0}^{6} d_{s1}\zeta^s - \sum_{s=0}^{6} f_{s1}\zeta^s$$

and $A, B \in \mathbb{Z}[\zeta]$. Since f_{s0} and f_{s1}, $s = 0, \ldots, 6$ are known, it turns out that for cases 1 and 2, $A = 2\sum_{s=0}^{6} d_{s0}\zeta^s - 7$ and $B = 2\sum_{s=0}^{6} d_{s1}\zeta^s - 7$. While for case 5, $A = 2\sum_{s=0}^{6} d_{s0}\zeta^s + 7$ and $B = 2\sum_{s=0}^{6} d_{s1}\zeta^s + 7$. Thus, (3.6) becomes

$$\frac{1}{7}\Big(A_1\bar{A}_1 + B_1\bar{B}_1\Big) = \frac{1}{2}\Big(A_1 + \bar{A}_1\Big) \qquad \text{for cases 1 and 2} \tag{3.10}$$

$$\frac{1}{7}\Big(A_1\bar{A}_1 + B_1\bar{B}_1\Big) = -\frac{1}{2}\Big(A_1 + \bar{A}_1\Big) \qquad \text{for case 5} \tag{3.11}$$

with $A_1 = \sum_{s=0}^{6} d_{s0}\zeta^s$ and $B_1 = \sum_{s=0}^{6} d_{s1}\zeta^s$. The right hand sides of equations (3.10) and (3.11) imply that

- d_{00} is any integer between 0 and 7,
- $d_{s0} + d_{7-s,0} \equiv 0 \mod 2$, $s = 1, \ldots, 6$,
- d_{s0} and $d_{7-s,0}$ are either both even integers or both odd integers,
- the sum $\sum_{s=1}^{6} d_{s0}$ is even.

With the above stipulations, we revisit each of the remaining three cases.

Case 1: The distribution $6^{13}13^1$

Without loss of generality we choose $d_{00} = 7$ and consequently, $d_{02} = 6$. The sum $\sum_{s=1}^{6} d_{s0}$ is even and consequently, $d_{00}+\sum_{s=1}^{6} d_{s0}$ is an odd integer. Also, (3.5) becomes

$$\sum_{s=0}^{6} d_{s0} = 21, \qquad \sum_{s=0}^{6} d_{s1} = 28, \qquad \sum_{s=0}^{6} d_{s2} = 21, \qquad \sum_{s=0}^{6} d_{s3} = 21. \tag{3.12}$$

Also, (3.4) yields the following thirteen equations

$$d_{s0} + d_{s2} = 6 \qquad s = 1, \ldots, 6; \qquad d_{s1} + d_{s3} = 6 \qquad s = 0, \ldots, 6 \tag{3.13}$$

The solutions to (3.6) are in quadratic subring of $\mathbb{Z}[\zeta]$ whose integral basis are $\{1, \zeta^2 + \zeta^5, \zeta^3 + \zeta^4\}$. Consequently, (3.6) yields three more equations

$$\sum_{s=0}^{6} a_s^2 + \sum_{s=0}^{6} b_s^2 - \sum_{s=0}^{6} a_s a_{s+1} - \sum_{s=0}^{6} b_s b_{s+1} = 49, \tag{3.14}$$

$$\sum_{s=0}^{6} a_{s+2} a_s + \sum_{s=0}^{6} b_{s+2} b_s - \sum_{s=0}^{6} a_s a_{s+1} - \sum_{s=0}^{6} b_s b_{s+1} = 0, \tag{3.15}$$

$$\sum_{s=0}^{6} a_{s+3}a_s + \sum_{s=0}^{6} b_{s+3}b_s - \sum_{s=0}^{6} a_s a_{s+1} - \sum_{s=0}^{6} b_s b_{s+1} = 0. \tag{3.16}$$

The subscripts of (3.14), (3.15), (3.16) are congruent to 0 modulo 7, $a_s = d_{s0} - d_{s2}$ and $b_s = d_{s1} - d_{s3}$, $s = 0, \ldots, 6$. Using (3.12) - (3.16), a computer search for feasible values of d_{st}, $t = 0, 1, 2, 3$ returns no results. For cases 2 and 5, we adjust (3.12) and (3.4) as appropriate and repeat the search. Thus, the generalized Quaternion group of order 28 does not admit $(196, 91, 42)$ difference sets.

3.4. THERE ARE NO C_{98} AND D_{49} IMAGES

We now look at two factor groups of G of order 98.

3.4.1. The C_{49} image. Suppose G has a normal subgroup N such that $G/N \cong C_{49} = \langle x : x^{49} = 1\rangle$. Let $\hat{D} = \sum_{i=0}^{48} d_i x^i$ be the difference set image in this factor group. We view this group ring element as a 1×49 matrix with columns indexed by powers of x. The characters of this group are $\chi(x) = \zeta^i, i = 0, \ldots, 48, \zeta$ is the forty ninth root of unity. Using the (2.3) and (2.4), the rational idempotents of C_{49} are

$$[e_{\chi_0}] = \frac{1}{49}\langle x\rangle, [e_{\chi_7}] = \frac{1}{49}(7\langle x^7\rangle - \langle x\rangle) \quad \text{and} \quad [e_{\chi_1}] = \frac{1}{7}(7 - \langle x^7\rangle).$$

Thus, the difference set equation is

$$\hat{D} = \alpha_{\chi_0}[e_{\chi_0}] + \alpha_{\chi_7}[e_{\chi_7}] + \alpha_{\chi_1}[e_{\chi_1}] \tag{3.17}$$

with $\alpha_{\chi_0} \in \mathbb{Z}$, $\alpha_{\chi_1}, \alpha_{\chi_7} \in \mathbb{Z}[\zeta]$. The linear combination of the rational idempotents having $\langle x^7\rangle$ in their kernel can be written as $\frac{A_i}{7}\langle x^7\rangle$, where A_i is a difference set image in C_7. Thus, (3.17) becomes

$$\hat{D} = \frac{A_i}{7}\langle x^7\rangle \pm 7x^k[e_{\chi_1}], \tag{3.18}$$

with $\frac{A_1}{7}\langle x^7\rangle = \frac{1}{7}\Big(12\langle x\rangle + 7\langle x^7\rangle\Big)$ and $\frac{A_2}{7}\langle x^7\rangle = \Big(2\langle x\rangle - \langle x^7\rangle\Big)$. Notice that the entries of matrix $49\Big(7[e_{\chi_1}]\Big) \equiv 0 \pmod{49}$, $49(\frac{A_1}{7}\langle x^7\rangle) \not\equiv 0 \pmod{49}$ and $49(\frac{A_2}{7}\langle x^7\rangle) \equiv 0 \pmod{49}$. As intersection numbers are integers, (3.18) becomes $\hat{D} = \frac{A_i}{7}\langle x^7\rangle \pm 7x^k[e_{\chi_1}]$. Up to translation, the solutions difference images in C_{49} are $A_3 = 7 + 2\langle x\rangle - 2\langle x^7\rangle$ and $A_4 = 7x + 2\langle x\rangle - \langle x^7\rangle - x\langle x^7\rangle$.

3.4.2. No C_{98} image. Suppose that $G/N \cong C_{98}$. Observe that $C_{98} \cong C_{49} \times C_2 = \langle x, y : x^7 = y^2 = 1 = [x, y]\rangle$. This group is of the form (2.11). We can use (2.13), with $|K| = 49$, $n = 49$, $k = 91$ and $B_j = A_j - 2K$, where $A_j, j = 3, 4$ is a difference set image in C_{49}. It is easy to see that the only solutions to (2.13) are

$A_5 = \Big(7 + \langle x\rangle - 2\langle x^7\rangle\Big) + \langle x\rangle y$ and $A_6 = \Big(7x + \langle x\rangle - \langle x^7\rangle - x\langle x^7\rangle\Big) + \langle x\rangle y$. However, A_5 consists of integers 6 and -1. The number 6 exceeds coset bound of 2 while -1 is not permissible. Also, A_6 contains number 7, which exceeds coset bound of 2. Thus, there are no viable solutions and there is no difference set image in C_{98} and consequently, by Dillon Dihedral trick, D_{49} does not admit this difference set. We generalize this results as follows: If $C_{2p^2} = \langle x, y : x^p = y^2 = 1 = [x, y]\rangle$ is a homomorphic image of G, a group of order $4p^2$ then the following hold:

- the coset bound for the intersection numbers of the difference set image $\hat{D}$ in C_{2p^2} is 2
- $B_j = A_j - 2K$ in (2.13), where A_j, $j = 1, 2$ is the difference set image in C_{p^2}
- the difference set images in C_{p^2} are $A_1 = p + 2\langle x\rangle - 2\langle x^p\rangle$ and $A_2 = px + 2\langle x\rangle - \langle x^p\rangle - x\langle x^p\rangle$
- as $C_{2p^2} \cong C_{p^2} \times C_2$, the solutions to (2.13) are
  (1) $\Big(p + \langle x\rangle - 2\langle x^p\rangle\Big) + \langle x\rangle y$, which consists of a number, $p - 1$, which exceeds coset bound and $(p-1)$-negative 1 (not admissible)
  (2) $\Big(px + \langle x\rangle - \langle x^9\rangle - x\langle x^p\rangle\Big) + \langle x\rangle y$, which consists of a number p, which exceeds coset bound.
- consequently, there is no difference set image in C_{2p^2} and D_{p^2}.

The above results eliminate all groups of order 196 except $(C_7 \times C_7) \rtimes C_4$ with gap location number $[196, 8]$. This is the only group, according to Theorem 2.4 admitting $(196, 91, 42)$ difference sets.

4. On the Existence of $(980, 89, 8)$ Difference Sets

In this section, G is a group of order 980 and N an appropriate normal subgroup of G.

4.1. THE C_7 IMAGES

Suppose that $G/N \cong C_7 = \langle x : x^7 = 1\rangle$. Using the same approach as in the case $(196, 91, 42)$, the unique difference set image in C_7 is $A = -9 + 14\langle x\rangle$.

4.2. THERE ARE NO C_{14} AND D_7 IMAGE

We assume that difference set image exist in $G/N \cong C_{14} = \langle x, y : x^7 = y^2 = 1 = [x, y]\rangle$ and we use the same approach as in the Section 2. Choose $|K| = 7$, $n = 81$,

$k = 89$ and $B = A - 14K$ in (2.13). Thus, the difference set image is $\hat{D} = A\Big(\frac{1+y}{2}\Big) + Bx^i y^j\Big(\frac{1-y}{2}\Big)$, $i = 0, \ldots, 6$; $j = 0, 1$. In this case, $A\Big(\frac{1+y}{2}\Big) = (-\frac{9}{2} + 7\langle x\rangle) + (-\frac{9}{2} + 7\langle x\rangle)y$ and $B\Big(\frac{1-y}{2}\Big) = -\frac{9}{2} + \frac{9}{2}y$. The fractions in matrix $A\Big(\frac{1+y}{2}\Big)$ forced $i = 0$ and up to translation, $\hat{D} = \Big((-9 + 7\langle x\rangle) + 7\langle x\rangle)y\Big)$. However, the intersection numbers are non-negative. Hence C_{14} does not admit difference set image and consequently, D_7 does not (by Dillon trick). These results imply that out of the 34 groups of order 980, only the groups with gap location numbers $[980, j]$, $j = 18, 22, 23$ could possibly admit this difference sets. Interestingly, all these three surviving groups of order 980 have $(C_7 \times C_7) \rtimes C_4$ with gap location number $[196, 8]$ as factor group. This is the same group of order 196 that admits $(196, 91, 42)$ difference set. The vital question is: Does this group also admit $(980, 89, 8)$ difference sets?

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