Kragujevac Journal of Mathematics
Volume 34 (2010), Pages 91–102.

ON THE CONTINUOUS DEPENDENCE OF THE FIXED POINTS FOR $(\varphi,\ \psi)$–CONTRACTIVE-TYPE OPERATORS

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Abstract. In this paper, we establish some results pertaining to the continuous dependence of the fixed points in a Banach space setting for three iterative processes by using some general contractive conditions.

1. Introduction

In this paper, we establish some results pertaining to the continuous dependence of the fixed points in a Banach space setting for three iterative processes by using some general contractive conditions. Two of these iterative processes have been recently introduced in [14] and [15]. Our results are generalizations and extensions of some of the results of [3], [4], [19] and [26]. See also the recent results of [5], [16], [20], [21] and [22].

Let (E,d) be a complete metric space and $T : E \to E$ a selfmap of E. Suppose that $F_T = \{p \in E \mid Tp = p\}$ is the set of fixed points of T.

There are several iterative processes in the literature for which the fixed points of operators have been approximated over the years by various authors. In a complete metric space, the Picard iterative process $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, \ldots, \tag{1.1}$$

Key words and phrases. Continuous dependence, General contractive conditions, Banach space setting.

2010 *Mathematics Subject Classification.* 47H06, 47H10.

Received: February 09, 2010.

Revised: April 27, 2010.

has been employed to approximate the fixed points of mappings satisfying the inequality relation

$$d(Tx,Ty) \leq \alpha d(x,y), \text{ for all } x,y \in E \text{ and } \alpha \in [0,1). \tag{1.2}$$

Condition (1.2) is called the *Banach's contraction condition.* Any operator satisfying (1.2) is called *strict contraction.* Also, condition (1.2) is significant in the celebrated Banach's fixed point theorem [1].

In a normed linear space or a Banach space setting, we shall state some of the iterative processes generalizing (1.1) as follows:
For $x_0 \in E$, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \quad n = 0,1,\ldots, \tag{1.3}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$, is called the Mann iterative process (see [12]).
The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_i \, T^i x_n, \quad x_0 \in E, \quad n = 0,1,2,\ldots, \quad \sum_{i=0}^{k} \alpha_i = 1, \tag{1.4}$$

$\alpha_i \geq 0$, $\alpha_0 \neq 0$, $\alpha_i \in [0,1]$, where k is a fixed integer is called the Kirk iterative process (see [10]).
The following two iterative processes in Olatinwo [14] and [15] generalize several well-known iterative processes in the literature including those defined in (1.1), (1.3) and (1.4):
(I) For $x_0 \in E$, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,\,i} \, T^i x_n, \quad n = 0,1,2,\ldots, \quad \sum_{i=0}^{k} \alpha_{n,\,i} = 1, \tag{1.5}$$

$\alpha_{n,\,i} \geq 0$, $\alpha_{n,\,0} \neq 0$, $\alpha_{n,\,i} \in [0,1]$, where k is a fixed integer.
(II) Let $T_i : E \to E$ $(i = 0,1,\ldots,k)$ be selfmaps of E and $x_0 \in E$. Then, define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$x_{n+1} = \sum_{i=0}^{k} \alpha_{n,\,i} \, T_i x_n, \quad \sum_{i=0}^{k} \alpha_{n,\,i} = 1, \quad n = 0,1,2,\ldots, \tag{1.6}$$

$\alpha_{n,\,i} \geq 0$, $\alpha_{n,\,0} \neq 0$, $\alpha_{n,\,i} \in [0,1]$, where k is a fixed integer and T_0 is an identity operator.

In many applications, the operator T in the Picard iteration of (1.1) depends on an additional parameter $\lambda \in Y$, where Y is a parameter space. Therefore, (1.1) is replaced by the equation

$$x_\lambda = S_\lambda\, x_\lambda, \quad x_\lambda \in E, \quad \lambda \in Y. \tag{1.7}$$

Condition (1.2) was employed in Zeidler [26] to prove a result on the stability of the fixed points (that is, continuous dependence of the fixed points on a parameter) for the Picard iteration. In Rus [19] and also Berinde [3] and Berinde [4], the continuous dependence of the fixed points on a parameter has been well formulated in the following general context in a metric space: Let $(E,\ d)$ be a complete metric space, (Y,τ) a topological space and $S_\lambda : E \times Y \to E$ a family of operators depending on the parameter $\lambda \in Y$. Suppose that $S_\lambda := S(\cdot,\lambda)$, $\lambda \in Y$, has a unique fixed point x^*_λ, for any $\lambda \in Y$.
Define the operator $U : Y \to E$ by

$$U(\lambda) = x^*_\lambda, \quad \text{for all } \lambda \in Y.$$

We are interested in finding sufficient conditions on S_λ that guarantee the continuity of U. In Rus [19], the following contractive condition was used: For a continuous mapping $S_\lambda : E \times Y \to E$, there exists a strict comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $x, y \in E$,

$$d(S(x,\lambda), S(y,\lambda)) \leq \varphi(d(x,y)), \quad \text{for all } x, y \in E, \lambda \in Y. \tag{1.8}$$

In Olatinwo [16], the concept of the continuous dependence of the fixed points on a parameter has been extended from the complete metric space to the normed linear space for the Schaefer and Mann iterative processes. Since metric is induced by norm, we have that $d(x,y) = ||x-y||$, for all $x, y \in E$, for the normed linear space or Banach space setting.

2. Preliminaries

We shall require the following definition and lemmas in the sequel:

Definition 2.1. (a) A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a *comparison function* if it satisfies the following conditions:

(i) ψ is monotone increasing;

(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0$, for all $t \geq 0$.

(b) A comparison function satisfying $t - \psi(t) \to \infty$ as $t \to \infty$ is called a *strict comparison function.*

See Berinde [2], [3], [4], Rus [19] and Rus et al [23] for the definition and examples of comparison function.

Remark 2.1. Every comparison function satisfies $\psi(0) = 0$.

We shall use the following contractive conditions:

(a) For a continuous mapping $S_\lambda : E \times Y \to E$, there exist:

(i) a real number $L \geq 0$ and a strict comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $x, y \in E$,

$$||S(x,\lambda) - S(y,\lambda)|| \leq L||x - S(x,\lambda)|| + \psi(||x - y||), \quad \lambda \in Y; \tag{2.1}$$

(ii) a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and a strict comparison function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that, for all $x, y \in E$,

$$||S(x,\lambda) - S(y,\lambda)|| \leq \varphi(||x - S(x,\lambda)||) + \psi(||x - y||), \quad \lambda \in Y. \tag{2.2}$$

(b) For continuous mappings $S_i : E \times Y \to E$, there exist a monotone increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi(0) = 0$ and strict comparison functions $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$, such that, for all $x, y \in E$, we have

$$||S_i(x,\lambda) - S_i(y,\lambda)|| \leq \varphi(||x - S_i(x,\lambda)||) + \psi_i(||x - y||), \; i = 0, 1, 2, \ldots, \; \lambda \in Y, \tag{2.3}$$

where $\psi_0(||x - y||) = I(||x - y||) = ||x - y||$=identity function and $S_0 = I$ (identity mapping).

We shall require the following Lemmas in the sequel:

Lemma 2.1. *Let $(E, ||\cdot||)$ be a Banach space and let $S_\lambda : E \times Y \to E$ be a mapping satisfying* (2.1)*, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a sublinear comparison function. Then, for all $i \in \mathbb{N}$, we have*

$$||S^i(x,\lambda) - S^i(y,\lambda)|| \leq \sum_{j=1}^{i} \binom{i}{j} L^j \psi^{i-j}(||x - S(x,\lambda)||) + \psi^i(||x - y||), \tag{2.4}$$

$\lambda \in Y$, for all $x, y \in E$.

Lemma 2.2. *Let $(E, ||\cdot||)$ be a Banach space and $S_\lambda : E \times Y \to E$ a mapping satisfying* (2.2)*, where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a sublinear comparison function and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, a sublinear monotone increasing function such that $\varphi(0) = 0$ and $\psi^s(\varphi^r(x)) \leq \varphi^r(\psi^s(x))$, for all $x \in \mathbb{R}_+$, $r, s \in \mathbb{N}$. Then, for all $i \in \mathbb{N}$, we have*

$$||S^i(x,\lambda) - S^i(y,\lambda)|| \leq \sum_{j=1}^{i} \binom{i}{j} \varphi^j(\psi^{i-j}(||x - S(x,\lambda)||)) + \psi^i(||x - y||), \tag{2.5}$$

$\lambda \in Y$, *for all* $x, y \in E$.

Proof. The proof of sublinearity of both φ and ψ is as follows:
In order to show that ψ^i (i.e. iterate of ψ) is sublinear, we have to show that ψ^i is both subadditive and positively homogeneous.
We first establish that ψ subadditive implies that each iterate ψ^i of ψ is also subadditive: Since ψ is subadditive, we have

$$\psi(x+y) \leq \psi(x) + \psi(y), \quad \text{for all } x, y \in \mathbb{R}_+.$$

Therefore,using subadditivity of ψ in ψ^2 yields

$$\psi^2(x+y) = \psi(\psi(x+y)) \leq \psi(\psi(x) + \psi(y)) \leq \psi(\psi(x)) + \psi(\psi(y)) = \psi^2(x) + \psi^2(y),$$

which implies that ψ^2 is subadditive.
Similarly, applying subadditivity of ψ^2 in ψ^3, we get

$$\begin{aligned}\psi^3(x+y) = \psi(\psi^2(x+y)) \leq \psi(\psi^2(x) + \psi^2(y)) &\leq \psi(\psi^2(x)) + \psi(\psi^2(y)) \\ &= \psi^3(x) + \psi^3(y),\end{aligned}$$

which implies that ψ^3 is also subadditive.
Hence, in general, each ψ^n, $n = 1, 2, \ldots,$ is subadditive.
We now prove that ψ positively homogeneous implies that each iterate ψ^i of ψ is also positively homogeneous: Therefore, we have that

$$\psi(\alpha x) = \alpha\psi(x), \quad \text{for all } x \in \mathbb{R}_+, \quad \alpha > 0.$$

Using positive homogeneity of ψ in ψ^2, we have

$$\psi^2(\alpha x) = \psi(\psi(\alpha x)) = \psi(\alpha\psi(x)) = \alpha\psi(\psi(x)) = \alpha\psi^2(x), \text{ for all } x \in \mathbb{R}_+, \ \alpha > 0,$$

which implies that ψ^2 is positively homogeneous.
Hence, in general, each ψ^n, $n = 1, 2, \ldots,$ is positively homogeneous.
Thus, we have that ψ^n, $n = 1, 2, \ldots,$ is sublinear.
Similarly, we can prove the sublinearity of φ as that of ψ.
The second part of the proof of this lemma is by induction on i as follows: Now, throughout the proof, we shall use the fact that $S(x, \lambda) = S_\lambda x$, $x \in E$, $\lambda \in Y$.
If $i = 1$, then (2.5) becomes

$$||S_\lambda x - S_\lambda y|| \leq \sum_{j=1}^{1} \binom{1}{j} \varphi^j(\psi^{1-j}(||x - S_\lambda x||)) + \psi(||x-y||) = \varphi(||x - S_\lambda x||) + \psi(||x-y||),$$

that is, (2.5) reduces to (2.2) when $i = 1$ and hence the result holds.

Assume as an inductive hypothesis that (2.4) holds for $i = m$, $m \in \mathbb{N}$, that is,

$$||S_\lambda^m x - S_\lambda^m y|| \leq \sum_{j=1}^{m} \binom{m}{j} \varphi^j(\psi^{m-j}(||x - S_\lambda x||)) + \psi^m(||x - y||), \quad \text{for all } x, y \in E.$$

We then show that the statement is true for $i = m + 1$

$$\begin{aligned}
& ||S_\lambda^{m+1} x - S_\lambda^{m+1} y|| \\
&= ||S_\lambda^m(S_\lambda x) - S_\lambda^m(S_\lambda y)|| \\
&\leq \sum_{j=1}^{m} \binom{m}{j} \varphi^j(\psi^{m-j}(||S_\lambda x - S_\lambda^2 x||)) + \psi^m(||S_\lambda x - S_\lambda y||) \\
&\leq \sum_{j=1}^{m} \binom{m}{j} \varphi^j(\psi^{m-j}(\varphi(||x - S_\lambda x||) + \psi(||x - S_\lambda x||))) \\
&+ \psi^m(\varphi(||x - S_\lambda x||) + \psi(||x - y||)) \\
&\leq \sum_{j=1}^{m} \binom{m}{j} \varphi^j[\psi^{m-j}(\varphi(||x - S_\lambda x||)) + \psi^{m+1-j}(||x - S_\lambda x||)] \\
&+ \psi^m(\varphi(||x - S_\lambda x||)) + \psi^{m+1}(||x - y||) \\
&\leq \sum_{j=1}^{m} \binom{m}{j} \varphi^j[\varphi(\psi^{m-j}(||x - S_\lambda x||)) + \psi^{m+1-j}(||x - S_\lambda x||)] \\
&+ \varphi(\psi^m(||x - S_\lambda x||)) + \psi^{m+1}(||x - y||) \\
&\leq \sum_{j=1}^{m} \binom{m}{j} \varphi^{j+1}(\psi^{m-j}(||x - S_\lambda x||)) + \sum_{j=1}^{m} \binom{m}{j} \varphi^j(\psi^{m+1-j}(||x - S_\lambda x||)) \\
&+ \varphi(\psi^m(||x - S_\lambda x||)) + \psi^{m+1}(||x - y||) \\
&= \binom{m}{m} \varphi^{m+1}(||x - S_\lambda x||) + \left[\binom{m}{m-1} + \binom{m}{m}\right] \varphi^m(\psi(||x - S_\lambda x||)) \\
&+ \left[\binom{m}{m-2} + \binom{m}{m-1}\right] \varphi^{m-1}(\psi^2(||x - S_\lambda x||)) + \ldots \\
&+ \left[\binom{m}{1} + \binom{m}{0}\right] \varphi(\psi^m(||x - S_\lambda x||)) + \psi^{m+1}(||x - y||) \\
&= \binom{m+1}{m+1} \varphi^{m+1}(||x - S_\lambda x||) + \binom{m+1}{m} \varphi^m(\psi(||x - S_\lambda x||)) \\
&+ \binom{m+1}{m-1} \varphi^{m-1}(\psi^2(||x - S_\lambda x||)) + \ldots \\
&+ \binom{m+1}{2} \varphi^2(\psi^{m-1}(||x - S_\lambda x||)) + \binom{m+1}{1} \varphi(\psi^m(||x - S_\lambda x||)) \\
&+ \psi^{m+1}(||x - y||)
\end{aligned}$$

$$= \sum_{j=1}^{m+1} \binom{m+1}{j} \varphi^j(\psi^{m+1-j}(||x - S_\lambda x||)) + \psi^{m+1}(||x - y||).$$

□

Lemma 2.3. [13] *Let* $\{\psi^k(t)\}_{k=0}^n$ *be a finite set of comparison functions. Then, any linear combination* $\sum_{j=0}^n c_j\psi^j(t)$ *of the comparison functions is also a comparison function, where* $\sum_{j=0}^n c_j = 1$ *and* $c_o, c_1, \ldots, c_n$ *are positive constants.*

3. Main Results

Theorem 3.1. *Let* $(E, \|\cdot\|)$ *be a Banach space and* (Y, τ) *a topological space. Let* $S_i : E \times Y \to E$ $(i = 0, 1, \ldots, k)$ *be continuous mappings satisfying* (2.3). *Suppose that* $\psi_i : \mathbb{R}_+ \to \mathbb{R}_+$ *(for each* i*) is a strict comparison function and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a monotone increasing function such that* $\varphi(0) = 0$*. Let* x^*_λ *be the unique common fixed point of* $S_{i\lambda}$*, for each* i*, where* $S_{i\lambda}x = S_i(x, \lambda)$*,* $x \in E$*,* $\lambda \in Y$*. Suppose* $\{x_n\}_{n=0}^\infty$ *is the Kirk-Mann iterative process defined by* (1.6) *with* $\sum_{i=0}^k \alpha_{n,i} = 1$*. Then, the mapping* $U : Y \to E$*, given by* $U(\lambda) = x^*_\lambda$*,* $\lambda \in Y$*, is continuous.*

Proof. Let $\lambda_1, \lambda_2 \in Y$. Then, we shall apply the contractive condition and the triangle inequality in the sequel. Let $S_0 = I$ (identity mapping). Then, $I(x_\lambda, \lambda) = I_\lambda x_\lambda = x_\lambda$. Since

$$\begin{aligned}
&||x^*_{\lambda_1} - x^*_{\lambda_2}|| \\
&\leq \sum_{i=0}^k \alpha_{\lambda_1,i}||S_i(x^*_{\lambda_1}, \lambda_1) - S_i(x^*_{\lambda_2}, \lambda_2)|| + \sum_{i=0}^k |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}| \; ||S_i(x^*_{\lambda_2}, \lambda_2)|| \\
&\leq \sum_{i=0}^k \alpha_{\lambda_1,i}[\; ||S_i(x^*_{\lambda_1}, \lambda_1) - S_i(x^*_{\lambda_2}, \lambda_1)|| + ||S_i(x^*_{\lambda_2}, \lambda_1) - S_i(x^*_{\lambda_2}, \lambda_2)|| \;] \\
&+ \sum_{i=0}^k |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}| \; ||S_i(x^*_{\lambda_2}, \lambda_2)|| \\
&= \sum_{i=0}^k \alpha_{\lambda_1,i}||S_i(x^*_{\lambda_1}, \lambda_1) - S_i(x^*_{\lambda_2}, \lambda_1)|| \\
&+ \sum_{i=0}^k \alpha_{\lambda_1,i}||S_i(x^*_{\lambda_2}, \lambda_1) - S_i(x^*_{\lambda_2}, \lambda_2)|| + \sum_{i=0}^k |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}| \; ||S_i(x^*_{\lambda_2}, \lambda_2)|| \\
&\leq \sum_{i=0}^k \alpha_{\lambda_1,i} \left\{\varphi(||x^*_{\lambda_1} - S_i(x^*_{\lambda_1}, \lambda_1)||) + \psi_i(||x^*_{\lambda_1} - x^*_{\lambda_2}||)\right\} \\
&+ \sum_{i=0}^k \alpha_{\lambda_1,i}||S_i(x^*_{\lambda_2}, \lambda_1) - S_i(x^*_{\lambda_2}, \lambda_2)|| + \sum_{i=0}^k |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}| \; ||S_i(x^*_{\lambda_2}, \lambda_2)||
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{k} \alpha_{\lambda_1,i}\psi_i(||x^*_{\lambda_1} - x^*_{\lambda_2}||) + \sum_{i=0}^{k} \alpha_{\lambda_1,i}||S_i(x^*_{\lambda_2}, \lambda_1) - S_i(x^*_{\lambda_2}, \lambda_2)|| \\
&+ \sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}| \; ||S_i(x^*_{\lambda_2}, \lambda_2)||.
\end{aligned}
\tag{3.1}
$$

Therefore, we have from (3.1) that

$$
\begin{aligned}
||x^*_{\lambda_1} - x^*_{\lambda_2}|| - \sum_{i=0}^{k} \alpha_{\lambda_1,i}\psi_i(||x^*_{\lambda_1} - x^*_{\lambda_2}||) &\le \sum_{i=0}^{k} \alpha_{\lambda_1,i}||S_i(x^*_{\lambda_2}, \lambda_1) - S_i(x^*_{\lambda_2}, \lambda_2)|| \\
&+ \sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}| \; ||S_i(x^*_{\lambda_2}, \lambda_2)||,
\end{aligned}
\tag{3.2}
$$

so that from (3.2) we have

$$
\begin{aligned}
&||x^*_{\lambda_1} - x^*_{\lambda_2}|| - \bar{\psi}(||x^*_{\lambda_1} - x^*_{\lambda_2}||) \\
&\le \sum_{i=0}^{k} \alpha_{\lambda_1,i}||S_i(x^*_{\lambda_2}, \lambda_1) - S_i(x^*_{\lambda_2}, \lambda_2)|| + \sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}| \; ||S_i(x^*_{\lambda_2}, \lambda_2)||
\end{aligned}
$$

$$
= \sum_{i=0}^{k} \alpha_{\lambda_1,i}||S_{i\lambda_1}x^*_{\lambda_2} - S_{i\lambda_2}x^*_{\lambda_2}|| + \sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}| \; ||S_{i\lambda_2}x^*_{\lambda_2}||, \tag{3.3}
$$

where $\bar{\psi}(||x^*_{\lambda_1} - x^*_{\lambda_2}||) = \sum_{i=0}^{k} \alpha_{\lambda_1,i}\psi_i(||x^*_{\lambda_1} - x^*_{\lambda_2}||)$.
By Lemma 2.3, we have that $\bar{\psi}$ is also a (strict) comparison function.
Since S_i is continuous for each i, we have

$$
||S_{i\lambda_1}x^*_{\lambda_2} - S_{i\lambda_2}x^*_{\lambda_2}|| \to 0 \text{ as } \lambda_2 \to \lambda_1,
$$

and also,

$$
\sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}| \; ||S_{i\lambda_2}x^*_{\lambda_2}|| \to 0 \text{ as } \lambda_2 \to \lambda_1,
$$

so that (3.3) leads to

$$
||x^*_{\lambda_1} - x^*_{\lambda_2}|| \to 0 \text{ as } \lambda_2 \to \lambda_1.
$$

That is,

$$
||U(\lambda_1) - U(\lambda_2)|| \to 0 \text{ as } \lambda_2 \to \lambda_1.
$$

Hence, the mapping $U : Y \to E$, defined by $U(\lambda) = x^*_\lambda$, $\lambda \in Y$, is continuous. □

Theorem 3.2. *Let* $(E, \|\cdot\|)$ *be a Banach space and* (Y, τ) *a topological space. Let* $S : E \times Y \to E$ *be a continuous mapping satisfying* (2.2). *Suppose* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a sublinear monotone increasing function such that* $\varphi(0) = 0$ *and* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *a sublinear comparison function such that* $\psi^s(\varphi^r(x)) \leq \varphi^r(\psi^s(x))$, *for all* $x \in \mathbb{R}_+$, $r,\ s \in \mathbb{N}$. *Let* x^*_λ *be the unique fixed point of* S_λ, *where* $S_\lambda x = S(x, \lambda)$, $x \in E$, $\lambda \in Y$. *Suppose* $\{x_n\}_{n=0}^\infty$ *is the Kirk-Mann iterative process defined by* (1.5) *with* $\sum_{i=0}^k \alpha_{n,\ i} = 1$. *Then, the mapping* $U : Y \to E$, *given by* $U(\lambda) = x^*_\lambda$, $\lambda \in Y$, *is continuous.*

Proof. Let $\lambda_1,\ \lambda_2 \in Y$. Then, we shall apply Lemma 2.2, Lemma 2.3 and the triangle inequality in the sequel. Let $T^0 = I$ (identity mapping). Then, $I(x_\lambda, \lambda) = I_\lambda x_\lambda = x_\lambda$.

$$
\begin{aligned}
& ||x^*_{\lambda_1} - x^*_{\lambda_2}|| \\
&= ||\sum_{i=0}^k \alpha_{\lambda_1,i}[\ S^i(x^*_{\lambda_1}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_2)\ ] + \sum_{i=0}^k (\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}) S^i(x^*_{\lambda_2}, \lambda_2)|| \\
&\leq \sum_{i=0}^k \alpha_{\lambda_1,i} ||S^i(x^*_{\lambda_1}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_2)|| + \sum_{i=0}^k |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}|\ ||S^i(x^*_{\lambda_2}, \lambda_2)|| \\
&\leq \sum_{i=0}^k \alpha_{\lambda_1,i}[\ ||S^i(x^*_{\lambda_1,i}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_1)|| + ||S^i(x^*_{\lambda_2}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_2)||\ ] \\
&+ \sum_{i=0}^k |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}|\ ||S^i(x^*_{\lambda_2}, \lambda_2)|| \\
&= \sum_{i=0}^k \alpha_{\lambda_1,i} ||S^i(x^*_{\lambda_1}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_1)|| \\
&+ \sum_{i=0}^k \alpha_{\lambda_1,i} ||S^i(x^*_{\lambda_2}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_2)|| \\
&+ \sum_{i=0}^k |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}|\ ||S^i(x^*_{\lambda_2}, \lambda_2)|| \\
&= \sum_{i=1}^k \alpha_{\lambda_1,i} ||S^i(x^*_{\lambda_1}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_1)|| + \alpha_{\lambda_1,0} ||I(x^*_{\lambda_1}, \lambda_1) - I(x^*_{\lambda_2}, \lambda_1)|| \\
&+ \sum_{i=0}^k \alpha_{\lambda_1,i} ||S^i(x^*_{\lambda_2}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_2)|| \\
&+ \sum_{i=0}^k |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}|\ ||S^i(x^*_{\lambda_2}, \lambda_2)||
\end{aligned}
$$

$$\leq \sum_{i=1}^{k} \alpha_{\lambda_1,i}[\ \sum_{j=1}^{i} \binom{i}{j} \varphi^j(\psi^{i-j}(||x^*_{\lambda_1,i} - S(x^*_{\lambda_1}, \lambda_1)||)) + \psi^i(||x^*_{\lambda_1} - x^*_{\lambda_2}||)\ ]$$
$$+ \alpha_{\lambda_1,0}||x^*_{\lambda_1} - x^*_{\lambda_2}|| + \sum_{i=0}^{k} \alpha_{\lambda_1,i}||S^i(x^*_{\lambda_2}, \lambda_1) - S^i(x^*_{\lambda_2}, \lambda_2)||$$
$$+ \sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}|\ ||S^i(x^*_{\lambda_2}, \lambda_2)||$$
$$= \sum_{i=1}^{k} \alpha_{\lambda_1,i}[\ \sum_{j=1}^{i} \binom{i}{j} \varphi^j(\psi^{i-j}(0)) + \psi^i(||x^*_{\lambda_1} - x^*_{\lambda_2}||)\ ] + \alpha_{\lambda_1,0}||x^*_{\lambda_1} - x^*_{\lambda_2}||$$
$$+ \sum_{i=0}^{k} \alpha_{\lambda_1,i}||S^i_{\lambda_1}x^*_{\lambda_2} - S^i_{\lambda_2}x^*_{\lambda_2}|| + \sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}|\ ||S^i_{\lambda_2}x^*_{\lambda_2}||,$$

i.e.

$$(3.4) \qquad \begin{aligned} ||x^*_{\lambda_1} - x^*_{\lambda_2}|| \leq \sum_{i=0}^{k} \alpha_{\lambda_1,i}\psi^i(||x^*_{\lambda_1} - x^*_{\lambda_2}||) + \sum_{i=0}^{k} \alpha_{\lambda_1,i}||S^i_{\lambda_1,i}x^*_{\lambda_2} - S^i_{\lambda_2}x^*_{\lambda_2}|| \\ + \sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}|\ ||S^i_{\lambda_2}x^*_{\lambda_2}||, \end{aligned}$$

since $\varphi^j(\psi^{i-j}(0)) = \varphi^j(0) = \psi^{i-j}(0) = \varphi(0) = \psi(0) = 0$.
It follows from (3.4) that

$$(3.5) \qquad \begin{aligned} ||x^*_{\lambda_1} - x^*_{\lambda_2}|| - \bar{\psi}(||x^*_{\lambda_1} - x^*_{\lambda_2}||) \leq \sum_{i=0}^{k} \alpha_{\lambda_1,i}||S^i_{\lambda_1}x^*_{\lambda_2} - S^i_{\lambda_2}x^*_{\lambda_2}|| \\ + \sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}|\ ||S^i_{\lambda_2}x^*_{\lambda_2}||, \end{aligned}$$

where $\bar{\psi}(||x^*_{\lambda_1} - x^*_{\lambda_2}||) = \sum_{i=0}^{k} \alpha_{\lambda_1,i}\psi^i(||x^*_{\lambda_1} - x^*_{\lambda_2}||)$.
By Lemma 2.3, we have that $\bar{\psi}$ is also a (strict) comparison function.
Since S is continuous, we have

$$||S^i_{\lambda_1}x^*_{\lambda_2} - S^i_{\lambda_2}x^*_{\lambda_2}|| \to 0 \text{ as } \lambda_2 \to \lambda_1,$$

and also,

$$\sum_{i=0}^{k} |\alpha_{\lambda_1,i} - \alpha_{\lambda_2,i}|\ ||S^i_{\lambda_2}x^*_{\lambda_2}|| \to 0 \text{ as } \lambda_2 \to \lambda_1,$$

so that (3.5) leads to

$$||x^*_{\lambda_1} - x^*_{\lambda_2}|| \to 0 \text{ as } \lambda_2 \to \lambda_1.$$

That is, $||U(\lambda_1) - U(\lambda_2)|| \to 0$ as $\lambda_2 \to \lambda_1$.
Hence, the mapping $U : Y \to E$, defined by $U(\lambda) = x^*_\lambda,\ \lambda \in Y$, is continuous. □

Theorem 3.3. *Let $(E, \|\cdot\|)$ be a Banach space and (Y, τ) a topological space. Let $S : E \times Y \to E$ be a continuous mapping satisfying (2.1). Suppose that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a sublinear comparison function. Let x^*_λ be the unique fixed point of S_λ where $S_\lambda x = S(x, \lambda)$, $x \in E$, $\lambda \in Y$. Suppose $\{x_n\}_{n=0}^{\infty}$ is the Kirk-Mann iterative process defined by (1.5) with $\sum_{i=0}^{k} \alpha_{n,\ i} = 1$. Then, the mapping $U : Y \to E$, given by $U(\lambda) = x^*_\lambda$, $\lambda \in Y$, is continuous.*

Proof. The proof is similar to that of Theorem 3.2, except for the application of Lemma 2.1. □

Theorem 3.4. *Let $(E, \|\cdot\|)$ be a Banach space and (Y, τ) a topological space. Let $S : E \times Y \to E$ be a continuous mapping satisfying* (2.2). *Suppose $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a sublinear monotone increasing function such that $\varphi(0) = 0$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a sublinear comparison function. Let x^*_λ be the unique fixed point of S_λ where $S_\lambda x = S(x, \lambda)$, $x \in E$, $\lambda \in Y$. Suppose $\{x_n\}_{n=0}^{\infty}$ is the Kirk iterative process defined by* (1.4) *with $\sum_{i=0}^{k} \alpha_i = 1$. Then, the mapping $U : Y \to E$, given by $U(\lambda) = x^*_\lambda$, $\lambda \in Y$, is continuous.*

Proof. The proof is similar to that of Theorem 3.2. □

Remark 3.1. Our results generalize, extend and improve Proposition 1.2 of Zeidler [26]. Our results also extend Theorem 7.7 of Berinde [3] and [4] (which is Theorem 7.1.2 of Rus [19]) from the complete metric space to the Banach space setting. See also the results of [5], [16], [20], [21] and [22].

Acknowledgment: Some part of this research work was carried out while the author was on Research Visit to the Centre for Advanced Studies in Mathematics, Lahore University of Management Sciences, Lahore, Pakistan.

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