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AFFINE INVARIANT L-SYSTEMS

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Abstract. The purpose of this article is to investigate if the L-systems introduced and developed mainly by Lindenmayer, and Prusinkiewicz [8]–[12] possess affine invariance property. The main result given in Theorem 1 establishes a negative answer.

1. Introduction

The first fractal objects that appeared in mathematics were described by recursive constructions. The examples are: *Cantor set* (Henry J. S. Smith, 1875; Paul du Bois-Reymond, c. 1880; Vito Volterra, 1881; Georg Cantor, 1883); *Peano space-filling curve* (Giuseppe Peano, 1890); *Hilbert space-filling curve* (David Hilbert, 1891); *Koch snowflake* (Helge von Koch, 1904); *Lévy C-curve* (Ernesto Cesàro, 1906; G. Farber, 1910; Paul Pierre Lévy, 1938); *Sierpinski triangle* (Waclaw Sierpinski, 1915); *Sierpinski carpet* (Waclaw Sierpinski, 1916), etc. These recursive constructions are later called "deterministic algorithms" or *initiator-generator* type algorithms. *Initiator* is a simple geometric object, usually a simplex, or simplicial sequence. *Generator* is an union of initiator's images upon certain collection of contractive mappings $\{w_1, \ldots, w_k\}$, usually defined on a real metric space of dimension that is rarely higher than 3. In fact, in all above examples, known as "classic fractals", the w_i's are affine automorphisms of $\mathbf{R}$ or $\mathbf{R}^2$. Such constructions were generalized in the concept of Iterated Function Systems (IFS), $\{\mathbf{X}; w_1, \ldots, w_k\}$, i.e. the set of contractive mappings that act in the metric space $(\mathbf{X}, d)$, where d is a suitable metric (see [1]).

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Definition 1.1. The mapping w is called contractive if there exists a real number $\lambda \in (0,1)$, such that the inequality $\|w(x) - w(y)\| \le \lambda \|x - y\|$ is exact for some norm $\|\cdot\|$ in $\mathbf{R}^n$. The minimal value $c = \min\{\lambda\}$ is called Lipschitz factor of w.

Definition 1.2. Given the set $\{w_1, w_2, \ldots, w_k\}$ of contractive mappings in the metric space $(\mathbf{R}^n, d)$ with Lipschitz factors $0 < c_i < 1$, $i = 1, \ldots, k$ respectively. Then the Iterated Function Systems (IFS)

$$\Sigma = \{\mathbf{R}^n; w_1, w_2, \ldots, w_k\}, \tag{1.1}$$

is called hyperbolic IFS.

Definition 1.3. Let H be the set of nonempty compact subsets of $\mathbf{R}^n$. The operator W, known as Hutchinson operator, defined on H by

$$W(\cdot) = \bigcup_i w_i(\cdot), \tag{1.2}$$

is associated with the hyperbolic IFS (1.1).

Definition 1.4. The Hausdorff distance $h(A, B)$ between any two sets $A, B \in H$, is given by

$$h(A,B) = \max\left\{ \max_{a\in A} \min_{b\in B} d_E(a,b),\ \max_{b\in B} \min_{a\in A} d_E(b,a) \right\} \quad (d_E \text{ is Euclid metric}). \tag{1.3}$$

Theorem 1.1. *Let $\Sigma = \{\mathbf{R}^n; w_1, w_2, \ldots, w_k\}$ be a hyperbolic IFS. Then, the associated Hutchinson operator* (1.2) *is contractive in Hausdorff metric space (H, h), with Lipschitz factor $c = \max_i \{c_i\}$, where c_i is Lipschitz factor of the contraction w_i. The operator equation*

$$X = W(X) = \bigcup_i w_i(X), \quad X \in H, \tag{1.4}$$

has the unique solution X_0 on H, called attractor of the IFS Σ, $X_0 = att(\Sigma)$.

Theorem 1.2. *Let $W^{om} = W(W^{o(m-1)})$ be the m-th auto-composition of the Hutchinson operator W associated with the IFS Σ, given by* (1.1)*. Then, the attractor $att(\Sigma)$ of Σ, is given as a limit of the sequence*

$$B, W(B), W^{o2}(B), \ldots, W^{om}(B), \ldots, \tag{1.5}$$

where B is an arbitrary point in H, from the open neighborhood of $att(\Sigma)$. In other words,

$$att(\Sigma) = \lim_{k\to\infty} W^{ok}(B), \quad B \in H. \tag{1.6}$$

By the rule, attractors are fractal sets ([1], [2]). Then, the sets (1.5) are called *pre-attractors* of orders $0, 1, 2, \ldots, m, \ldots$, respectively.

The simplest Iterated Function Systems having the widest applications anyway, consist out of affine mappings of the form

$$w(x) = Ax + b, \quad x \in \mathbf{R}^m, \tag{1.7}$$

where A is an $m \times m$ nonsingular real matrix and b is an m-dimensional real vector. Let $S = (s_{ij})_{i,j=1}^{m+1}$ be an $(m+1) \times (m+1)$ row-stochastic real matrix (i.e. real matrix which rows sum up to 1). Then the following theorem holds.

Theorem 1.3. [7] *For any affine mapping of the form* (1.7)*, there exists a nonsingular row-stochastic real matrix $S = (s_{ij})_{i,j=1}^{m+1}$, such that* (1.7) *can be represented by the linear mapping $L : \mathbf{R}^m \to \mathbf{R}^m$, given by*

$$L(r) = S^T r, \tag{1.8}$$

where $r = (r_1\ r_2\ \cdots\ r_{m+1})^T \in \mathbf{R}^{m+1}$, with restriction

$$\sum_{i=1}^{m+1} r_i = 1. \tag{1.9}$$

The mapping (1.8), $L : \mathbf{R}^m \to \mathbf{R}^m$ is said to be associated to the affine mapping $w : \mathbf{R}^m \to \mathbf{R}^m$, given by (1.7).

Note that $\{r_1, r_2, \ldots, r_{m+1}\}$ is the set of affine coordinates, so called *barycentric coordinates* with respect to an m-simplex.

Definition 1.5. A (non-degenerate) m-simplex $\hat{P}_m$ is the convex hull of a set of $m+1$ points (or vectors) $P_m^T = (p_1\ p_2 \cdots p_{m+1})$ such that their affine full coincide with $\mathbf{R}^m$.

Definition 1.6. [5, 6] Let $\hat{P}_m$ be a non-degenerate simplex and let $\{S_i\}_{i=1}^n$ be a set of real square nonsingular row-stochastic matrices of order m. If the linear mappings associated with S_i are contractions in $(\mathbf{R}^m, d)$, the system

$$\Omega(\hat{P}_m) = \{\hat{P}_m; S_1, S_2, \ldots, S_n\}, \tag{1.10}$$

is (hyperbolic) Affine invariant IFS (AIFS), with the unique attractor att(Ω).

Theorem 1.4. [7] *Given the IFS* $\Sigma = \{\mathbf{R}^n; w_1, w_2, \ldots, w_k\}$ *with affine mappings, and AIFS* $\Omega(\hat{P}_m) = \{\hat{P}_m; S_1, S_2, \ldots, S_n\}$ *with associated linear mappings. Then these systems are equihyperbolic, and share the same attractor* $att(\Sigma) = att(\Omega)$.

Theorem 1.5. [7] *The AIFS* (1.10) *has affine invariance property that is, for any affine transformation* $\varphi : \mathbf{R}^m \to \mathbf{R}^m$, $att\left(\Omega\Big(\varphi(\hat{P}_m)\Big)\right) \equiv \varphi\left(att\Big(\Omega(\hat{P}_m)\Big)\right)$ *is valid.*

Example 1.1. One of the classical fractals is the famous Lévy C-curve, which is the attractor of the IFS $\Sigma_C = \{\mathbf{R}^2; w_1, w_2\}$, where

$$w_1 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad w_2 : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}.$$

The AIFS with the same attractor is given by $C = \{\hat{T}; S_1, S_2\}$, where $\hat{T}$ is 2-simplex (proper triangle) $T^T = (a \;\; b \;\; c)$ with vertices $a = (0 \;\; 0)^T$, $b = (0.5 \;\; 0.5)^T$, $c = (1 \;\; 0)^T$, and two contractive mappings given by row-stochastic matrices

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & -0.5 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 0 & 1 & 0 \\ -0.5 & 1 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}.$$

The attractor, $C = \mathrm{att}(C)$ is displayed in Figure 1 (upper row) next to the simplex $\hat{T}$. Altering $\hat{T}$ into $\hat{T}'$ affinely causes altering the attractor's shape by exactly the same affine transform $C \mapsto C'$ (Figure 1, bottom row). This is the point of introducing the AIFS: ability of gaining control over the global shape of fractal attractors.

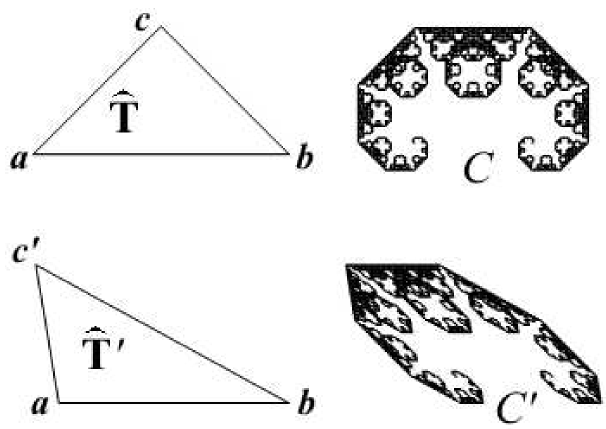

FIGURE 1. Affine invariance of the AIFS code.

2. L-systems

Trying to find a suitable mathematical model that will satisfactory describe growth and development of various biological forms, Lindenmayer, and later Prusinkiewicz (see [8]-[12]) elaborated a special iterated system based on iterated string rewriting, called *L-system.*

The structure $G = \{\Sigma, \Pi, \alpha\}$, calling *grammar* of L-systems contains the following elements:

1° A finite set of characters $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_n\}$, called *alphabet*, which is used to form words as strings of characters;

2° A finite set Π of *rules* for creating words;

3° The starting word, called *axiom*, $\alpha \in \Sigma^*$, the set of all possible words over Σ. Obviously, Π is a mapping

$$\Pi = \{\pi \mid \pi : \Sigma \to \Sigma^*\}, \tag{2.1}$$

of the alphabet into the set of all possible words Σ^*.

Definition 2.1. L-system over the grammar G is the set $L = \{\Pi^{ok}(\alpha) \mid k \in N_0\}$, where Π^{ok} denotes k-th auto-composition of the mapping (2.1).

Since the rule Π is applied repeatedly on the same characters simultaneously, it is sometimes called *parallel rewriting rule.*

There are a lot of types of L-systems, but in this paper, focus will be set on so called deterministic, context-free L-systems (shortly D0L-*systems*). *Deterministic* L-systems have all elements defined *a priori* (by the grammar G). *Context-free* means that the replacing rule applied to the i-th character of some word is independent of the neighbor characters. The acronym "D0L" stands for Deterministic 0-context L (-systems).

The concept of L-systems was accepted as a useful tool in the Theory of formal languages, developed by Chomsky as well as the model of growth of living cells and biological organisms. Originally, L-systems were invented by Lindenmayer [8] as an answer to the problem of formal description of plant growth. In this context, L-systems were necessarily connected with geometric configurations in the plane or space. These configurations are product of a particular geometry, known as *"turtle geometry"*. In one of the simplest setting, based on a simple L-system over the alphabet

$$\Sigma = \{F, f, +, -\}, \tag{2.2}$$

$2D$ turtle generates *"turtle interpretation map"* that give geometric meaning to the words from Σ^*.

Definition 2.2. The turtle interpretation map τ defined over the set of all words Σ^* is specified as follows:

- F -causes moving the turtle linearly forward for a fixed step of prescribed length (say 1) and tracing the line being passed over;
- f -the same as above, without marking the trace;
- $+$ -causes turning the turtle counterclockwise by a prescribed angle δ;
- $-$ -causes turning the turtle clockwise by a prescribed angle δ.

Definition 2.3. The turtle geometry is the set of geometric objects T obtained by the mapping $\tau : \Sigma^* \to T$, where τ is turtle interpretation map as given in Definition 2.2. The image of axiom α in T is called initiator, $\mathrm{In} = \tau(\alpha)$, and the image of $\Pi(\alpha)$ is called generator, $\mathrm{Gen} = \tau(\Pi(\alpha))$.

So, the "turtle" is a kind of "translator" that translates words (strings) into geometric objects. In $2D$ turtle geometry, these objects are made out of linear segments. Initiator and generator are pre-attractors of order 0 and 1.

What is amazing is that using some simple rules, the strings obtained by parallel rewriting rules may become highly complex. Since the process of rewriting is applied endlessly, the resulting limiting objects are often fractal sets [11] and [12].

Example 2.1. The Lévy C-curve defined in Example 1.1, can be coded by the single rule $\Pi = \{F \mapsto +F - -F+\}$, over the alphabet $\Sigma = \{F, +, -\}$, starting with the axiom $\alpha \equiv$"F" with angle $\delta = \pi/4$. Here, the initiator $\mathrm{In} = \tau(\alpha)$ is the linear segment (Figure 2, left, dashed line denoted by "0"), while the generator, $\mathrm{Gen} = \tau(" + F - -F + ")$ is the union of sides of isosceles triangle with the base angle of 45 degrees (Figure 2, left, solid line denoted by "1").

But, since the lengths of segments are one unit each, the distance between the end points of the turtle's path is $s = \sqrt{2}$, which is the Lipschitz factor indicating extension. The next step of iteration will be guided by the string "$++F--F+--+F--F++$", obtained by replacing of "F" in the generator string "$+F - -F+$". Consequently,

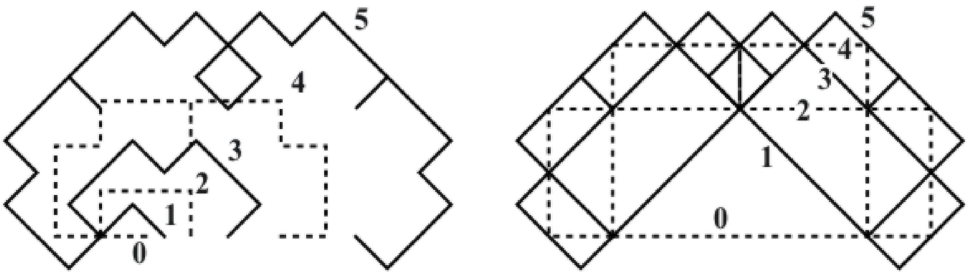

FIGURE 2. The pre-attractors of the Lévy C-curve of orders 0 to 5, generated with L-system without step reduction (left) and with step reduction (right).

the new geometric object will be $\tau('' + +F - -F + - - +F - -F + +'')$ (Figure 2, left, dashed line no. 2). End-points distance will now be $s^2 = 2$. The next iteration (Figure 2, left, solid line no. 3) increments the distance to $s^3 = 2\sqrt{2}$, and so on. In order to make this process equivalent to one produced by Iterated Function System, one must reduce the size of each step by s. Thus, the string $\Pi^{ok}(\alpha)$ of the L-system should be associated with the elementary step of length s^{-k}. Only in this case, "string rewriting" will cause proper "geometric replacement" which will result in getting the correct attractor.

In order to distinguish L-systems, which are algebraic structures from its graphical interpretation using "turtle graphics" endowed with turtle step reduction, the latter is suitable to call *Recursive L-systems* (RLS).

Definition 2.4. Let $L = \{\Pi^{ok}(\alpha) \mid k \in \mathbf{N}_0\}$ be the L-system with one replacement rule that allows $2D$ turtle geometry. Let $\mathrm{In} = \tau(\alpha)$, and $\mathrm{Gen} = \tau(\Pi(\alpha))$ are initiator and generator, respectively. The Recursive L-system (RLS) is the set

$$RL = \{\mathbf{R}^2; \mathrm{In} \to \mathrm{Gen}\}. \tag{2.3}$$

If exists, the attractor of RLS is given by

$$\mathrm{att}(RL) = \mathrm{att}\{\mathbf{R}^2; In \to Gen\} = \lim_{k\to\infty} \Big(In \to Gen\Big)^{ok}. \tag{2.4}$$

Note that, in some cases, transformation $In \to Gen$ is equivalent to Hutchinson operator of the IFS with affine mappings (Definition 1.3), and in these cases the Recursive L-systems and Iterated Function Systems can be identified (see [3] and [4]).

3. Lack of L-system's affine invariance

The main advantage of the concept of L-system is its simplicity and similarity to biological growth structures. The most successful tree and plant models are created doubtless by the help of L-systems ([3], [4]). Another curious feature of L-systems is their intrinsically algebraic structure based on simple string operations. For instant, the Lévy C-curve production rule $\Pi = \{F \;\mapsto\; +F - -F+\}$, transformed by the algebraic operator $X \mapsto FXF$ into $\Pi_1 = \{F \;\mapsto\; F + F - -F + F\}$ with the new angle $\delta = \pi/3$, generates another famous fractal set known as *Koch curve*. Besides, the initiator can be chosen differently. Instead the linear segment, some other piecewise linear configuration can be used. So, if the axiom will be changed to "$F - F - F$", with the same angle $\delta = \pi/3$, the initiator will be a triangle. The attractor becomes the *Koch snowflake*. On the other hand, the axiom "$F + F + F$" gives the "inward" snowflake. By extending the alphabet (2.2) with new characters "[" and "]", some branching structures will be created. Also, several rules combined with associated probabilities result in random fractal patterns.

From the point of view of modeling, the most desirable feature of some fractal generating system is predictability of the attractor's shape. Predictability is the main ingredient of what designers called free-form property of some modeling scheme. Similarly to IFS, L-systems, regarding the attractor's form, are unpredictable. This means that the form of attractor is very difficult to predict by using L-systems or IFS. The reasons partly lie in the fact that both systems suffer from the lack of affine invariance which is the condition *sine qua non* free-form property.

Definition 3.1. The RL-system (2.3) is invariant with respect to the mapping φ if $\mathrm{att}\Big\{\mathbf{R}^2; \varphi(\mathrm{In}) \to \varphi(\mathrm{Gen})\Big\} = \varphi\Big(\mathrm{att}\Big\{\mathbf{R}^2; \mathrm{In} \to \mathrm{Gen}\Big\}\Big)$.

Concerning, the following theorem holds.

Theorem 3.1. *The RL-system is not an affine invariant structure.*

Proof. A counterexample will be constructed based on the Lévy C-curve (Example 2.1), defined by the L-system is $\Pi = \{F \;\mapsto\; +F - -F+\}$, with $\delta = \pi/4$. As it is mentioned in Example 1.1, $\Sigma_C = \{\mathbf{R}^2; w_1, w_2\}$ is the associated IFS. Since the spectral norms of matrices $A_1 = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}$, are

$\|A_1\| = \|A_2\| = \frac{1}{\sqrt{2}} < 1$, both mappings w_1 and w_2 are contractions ([7]). Accordingly, the attractor, $\mathrm{att}(\Sigma_c)$, exists, and it is Lévy C-curve (Figure 1, upper row). Now, as in Example 2.1, denote

$$In = \tau(\text{"}F\text{"}) \equiv \Big\{(x,y) \mid 0 \le x \le 1,\ y = 0\Big\},$$

and

$$Gen = \tau(\text{"}+F--F+\text{"}) \equiv \Big\{(x,y) \mid \{0 \le x \le 1/2,\ y = x\} \wedge \{1/2 \le x \le 1,\ y = 1-x\}\Big\},$$

and apply on In and Gen the following affine mapping

$$\varphi : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & \tan\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad 0 < \beta < \frac{\pi}{2}. \tag{3.1}$$

It yields $\varphi(In) \equiv In$ (the initiator is not changed), while the generator transforms into an isosceles triangle with base angles β

$$\varphi(Gen) \equiv \Big\{(x,y) \mid \{0 \le x \le 1/2,\ y = (\tan\beta)x\} \wedge \{1/2 \le x \le 1,\ y = 1-(\tan\beta)x\}\Big\}.$$

The associated Hutchinson operator continuously depends of the angle β as a parameter, $W^\beta(\cdot) = w_1^\beta(\cdot) \cup w_2^\beta(\cdot)$, where

$$w_1^\beta : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} 1 & -\tan^2\beta \\ \tan\beta & \tan\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

$$w_2^\beta : \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \frac{1}{2} \begin{bmatrix} 1 & \tan^2\beta \\ -\tan\beta & \tan\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ \tan\beta \end{bmatrix}.$$

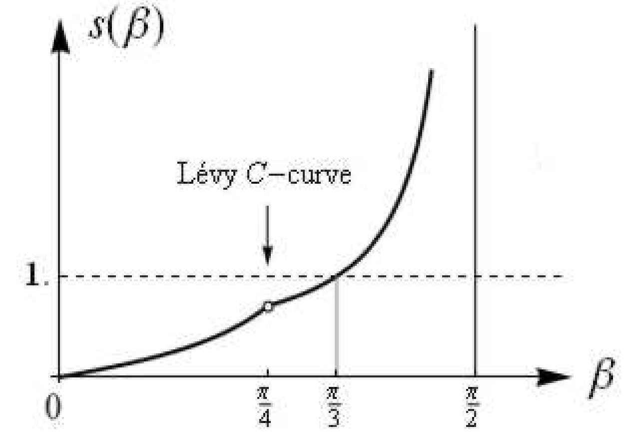

FIGURE 3. Lipschitz factor of Hutchinson operator W^β as function of β.

The matrices have the same norm, which is in the same time the Lipschitz factor $s(\beta)$ of the Hutchinson operator W^{β} which is clearly function of β

$$s(\beta) = \frac{\sqrt{1 - |\cos(2\beta)|}}{2\sqrt{2}\cos^2\beta}, \quad 0 < \beta < \frac{\pi}{2}. \tag{3.2}$$

As expected, the Lipschitz factor function is an increasing function (Figure 3). It is evident that $0 < s(\beta) < 1$ for $0 < \beta < \pi/3$, i.e. the IFS is hyperbolic, with a unique attractor that is Lévy C-curve-like (and is exact C-curve for $\beta = \pi/4$); The IFS is isometry (Lipschitz factor $s(\beta) = 1$) for $\beta = \pi/3$; And, finally, for all $\beta > \pi/3$, $s(\beta) > 1$, and the iterative process diverges, so, the attractor does not exist.

So, every RL-system $\{\mathbf{R}^2; F \mapsto +F - -F+\}$ with $\pi/3 < \delta < \pi/2$ is not affine invariant system. Generally, there exists affine mappings φ such that

$$\text{att}\{\mathbf{R}^2; \varphi(In) \to \varphi(Gen)\} \neq \varphi(\text{att}\{\mathbf{R}^2; In \to Gen\}),$$

i.e. the affine invariance property fails to exist. □

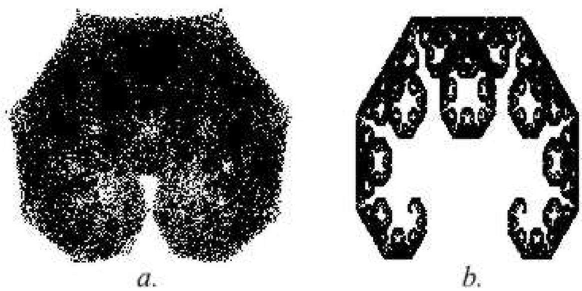

FIGURE 4. a) RL-system $\{\mathbf{R}^2;\ F \mapsto +F - -F+\}$ for $\delta = 11\pi/36$;
b) Affinely transformed C-curve

Example 3.1. The attractor of the RL-system $\{\mathbf{R}^2; F \mapsto +F - -F+\}$ for $\delta = 11\pi/36$ is approximated by the set of 6×10^4 points, using Random algorithm [1] (Figure 4 **a**). It differs grossly from the attractor of the Lévy C-curve, transformed affinely, by the mapping (3.1) for $\beta = \delta$ (Figure 4 **b).**

4. Conclusion

In this article, three different types of system for generating fractal sets are presented: The Iterated Function System (IFS) [1], the Affine invariant Iterated Function System (AIFS) [5]-[7], and RL-systems, the Recursive L-systems [8]–[12]. For the IFS is already known that it is not affine invariant system. On the other hand, the main feature of AIFS is its affine invariance. The main result presented in Theorem 3.1 is that L-systems also fail to have affine invariance property. The importance of this property for some modeling system reflects in the fact that affine invariance enables some elements of free-form modeling, which is precious feature especially when fractals are to be modeled.

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