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NONIMMERSION RESULTS FOR THE REAL FLAG MANIFOLDS $\mathbb{R}F(1,1,1,n-3)$

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Abstract. By computing non-vanishing dual Stiefel-Whitney classes of the incomplete real flag manifold of length 3, $\mathbb{R}F(1,1,1,n-3)$, $n>4$, we obtain nonimmersion and non-embedding results for the manifold and give solution to the immersion / embedding problem for $n=5,6$ and 7 by showing that Lam's estimate are best possible for these.

1. Introduction

Let $n = n_1 + n_2 + \ldots + n_q$, $q > 2$, be the partition of the positive integer n. The real flag manifold,

$$\mathbb{R}F(n_1, n_2, \ldots, n_q) = O(n)/O(n_1) \times \ldots \times O(n_q),$$

(where $O(n), O(n_1), \ldots, O(n_q)$ are the appropriate orthogonal groups) is a smooth compact connected homogeneous manifold of dimension $\frac{1}{2}(n^2 - \sum_{i=1}^{q} n_i^2)$. In particular when $n_1 = n_2 = \ldots = n_p = 1$, $p = q-1$, we have the incomplete flag manifold of length p, $\mathbb{R}F(\underbrace{1,1,\ldots,1,}_{p-\text{times}} n-p)$. The incomplete flag manifold of length 1, $\mathbb{R}F(1,n-1)$ is the real projective space $\mathbb{R}P^{n-1}$. For any smooth compact manifold M^m of dimension m, the problem of finding integers k and s such that M^m can be embedded in $\mathbb{R}^{m+k}$ (denoted by $M^m \subset \mathbb{R}^{m+k}$) and immersed in $\mathbb{R}^{m+s}$ (denoted by $M^m \subseteq \mathbb{R}^{m+s}$), has

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its starting point, for calculating upper bounds in the classical theorems of Whitney [16], [17], that $M^m \subset \mathbb{R}^{2m}$ and $M^n \subseteq \mathbb{R}^{2m-1}$. The problem of immersion/embedding of the projective spaces and other real flag manifolds has been studied very much by different methods (see for examples, [7], [6], [12], [13], [14]) and is still unsolved. A table of known embedding and immersion results for real projective spaces can be viewed at [5]. Lam [9] gave upper bounds for immersing any flag manifold in the Euclidean space which improved on the classical Whitney's result and Cohen's theorem (cf. [4]), $M^m \subseteq \mathbb{R}^{2m-\alpha(m)}$ (where $\alpha(m)$ is the number of 1's in the dyadic expansion of m) only for $n = 5$ to 10 in the case of $\mathbb{R}F(1,1,1,n-3)$, although Stong [15] showed that for many cases of real flag manifolds, Lam's immersion results are best possible. In [2], Ajayi and Ilori, obtained lower bounds for the embeddings and immersions of $\mathbb{R}F(1,1,n-2)$ in the Euclidean space by finding some non-vanishing dual Stiefel-Whitney classes and showed that for $\mathbb{R}F(1,1,n-2)$, Lam's immersions for $n = 4$ and 5 are best possible. In this paper, we obtain some lower bounds for immersion and embedding of $\mathbb{R}F(1,1,1,n-3)$ in the Euclidean space and show that Lam's estimate are best possible for $n = 5, 6$ and 7 thereby giving solution to the immersion/ embedding problem for these manifolds.

2. Statement of Results

Let $s = 2^r$ be the integer defined by $2^{r+1} < 3n < 2^{r+2}$, $n > 4$ we have

Theorem 2.1. *The following hold*

(i) $\mathbb{R}F(1,1,1,n-3) \not\subset \mathbb{R}^{3s-3}$
$\mathbb{R}F(1,1,1,n-3) \not\subseteq \mathbb{R}^{3s-4}$ *if* $\frac{2}{3}s < n \leq s-1$;

(ii) $\mathbb{R}F(1,1,1,n-3) \not\subset \mathbb{R}^{3(2s-1)}$
$\mathbb{R}F(1,1,1,n-3) \not\subseteq \mathbb{R}^{2(3s-2)}$ *if* $s+3 \leq n < \frac{4}{3}s$;

(iii) $\mathbb{R}F(1,1,1,n-3) \not\subset \mathbb{R}^{3n-3}$
$\mathbb{R}F(1,1,1,n-3) \not\subseteq \mathbb{R}^{3n-4}$ *if* $n = 2^r$;

(iv) $\mathbb{R}F(1,1,1,n-3) \not\subset \mathbb{R}^{3s-2}$
$\mathbb{R}F(1,1,1,n-3) \not\subseteq \mathbb{R}^{3s-3}$ *if* $n = s+1$;

(v) $\mathbb{R}F(1,1,1,n-3) \not\subset \mathbb{R}^{3(s+1)}$
$\mathbb{R}F(1,1,1,n-3) \not\subseteq \mathbb{R}^{3s+2}$ *if* $n = s+2$.

Corollary 2.1. *Let* $imm(M)$ *be the immersion dimension of a manifold* M. *Then,*

$imm\mathbb{R}F(1,1,1,2) = 10$,

$imm\mathbb{R}F(1,1,1,3) = 15,$
$imm\mathbb{R}F(1,1,1,4) = 21.$

3. Proof of Results

Put $F = \mathbb{R}F(1,1,1,n-3)$ and let ν_1, ν_2, ν_3 be the canonical line bundles over F and $x = w_1(\nu_1)$, $y = w_1(\nu_2)$, $z = w_1(\nu_3)$ be the Stiefel-Whitney classes of ν_1, ν_2, ν_3 respectively. Let

$$\sigma_1 = x + y + z, \quad \sigma_2 = xy + yz + xz, \quad \sigma_3 = xyz.$$

To prove the results we need the following:

Lemma 3.1. [1]

$$w(F) = (1 + \sigma_1 + \sigma_2 + \sigma_3)^n \cdot (1 + \sigma_1^2 + \sigma_2 + \sigma_1\sigma_2 + \sigma_3)^{-1}.$$

Proof. Over F, $\nu_1 \oplus \nu_2 \oplus \nu_3 \oplus \xi$ is an n-plane trivial bundle, where ξ is an $(n-3)$-plane bundle. From [9]

$$\tau(F) \cong (\nu_1 \otimes \nu_2) \oplus (\nu_1 \otimes \nu_3) \oplus (\nu_2 \otimes \nu_3) \oplus (\nu_1 \otimes \xi) \oplus (\nu_3 \otimes \xi)$$

and

$$\tau(F) \oplus (\nu_1 \otimes \nu_1) \oplus n\xi \oplus (\nu_1 \otimes \nu_2) \oplus (\nu_2 \otimes \nu_2) \oplus (\nu_1 \otimes \nu_3) \oplus (\nu_2 \otimes \nu_3) \oplus (\nu_1 \otimes \nu_3)$$

is an n^2-plane trivial bundle. Therefore taking the total Stiefel-Whitney classes and using the Whitney product formula, we have

$$w(F) \cdot w(\nu_1 \otimes \nu_2) \cdot w(\nu_1 \otimes \nu_3) \cdot w(\nu_2 \otimes \nu_3) w(n\xi) = 1$$

i.e.

$$w(F) = \bar{w}(n\xi)\overline{w}(\nu_1 \otimes \nu_2) \cdot \bar{w}(\nu_1 \otimes \nu_3) \cdot \bar{w}(\nu_2 \otimes \nu_3)$$

where $\bar{w}$ is the total dual Stiefel-Whitney class of F, and

$$\begin{aligned} w(F) &= [w(\nu_1 \oplus \nu_2 \oplus \nu_3]^n \cdot (1+x+y)^{-1} \cdot (1+x+z)^{-1} \cdot (1+y+z)^{-1} \\ &= (1 + \sigma_1 + \sigma_2 + \sigma_3)^n \cdot (1 + \sigma_1^2 + \sigma_2 + \sigma_1\sigma_2 + \sigma_3)^{-1}. \end{aligned}$$

□

Proposition 3.1. *Let $\bar{w}$ be the total dual Stiefel-Whitney class of F then*

$$\bar{w}(F) = (1 + \sigma_1 + \sigma_2 + \sigma_3)^{2s-n} \cdot (1 + \sigma_1^2 + \sigma_2 + \sigma_1\sigma_2 + \sigma_3).$$

Proof. From the above lemma,

$$w(F) = (1+\sigma_1+\sigma_2+\sigma_3)^n \cdot (1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3)^{-1}.$$

Let $s = 2^r$, be the integer such that $2^{r+1} < 3n < 2^{r+2}$, we have

$$\begin{aligned}
(1+\sigma_1+\sigma_2+\sigma_3)^{2s} &= [(1+\sigma_1+\sigma_2+\sigma_3)^2]^s \\
&= 1+\sigma_1^{2s}+\sigma_2^{2s}+\sigma_3^{2s} \\
&= 1+(x+y+z)^{2s}+(xy+yz+xz)^{2s}+(xyz)^{2s} \\
&= 1+x^{2s}+y^{2s}+z^{2s}+x^{2s}y^{2s}+y^{2s}z^{2s}+x^{2s}z^{2s}+x^{2s}y^{2s}z^{2s} \\
&= 1
\end{aligned}$$

since $2s > n$ and the $\mathbb{Z}_2$-cohomology algebra $H^*(F,\mathbb{Z}_2)$ can be identified with $\mathbb{Z}_2[x,y,z]$ subject to the relations $\bar{\sigma}_{n-2} = \bar{\sigma}_{n-1} = \bar{\sigma}_n = 0$ where $\bar{\sigma_i} = \bar{\sigma_i}(x,y,z)$ is the i-th complete symmetric function in x, y and z so that $x^n = 0 = y^n = z^n$ [3]. An additive basis for $H^*(F,\mathbb{Z}_2)$ is the set $\{x^i y^j z^k | 0 \le i \le n-1, \;\; 0 \le j \le n-2, \;\; 0 \le k \le n-3\}$ and we have $\sigma_1^a \neq 0$, $\sigma_2^b \neq 0, \sigma_3^c \neq 0$, $1 \le a,b,c \le n-3$.
Hence since

$$w(F) = (1+\sigma_1+\sigma_2+\sigma_3)^n \cdot (1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3)^{-1}$$

and using

$$w(F)\bar{w}(F) = 1$$

where $\bar{w}$ is the total dual Stiefel-Whitney class of F we have

$$\bar{w}(F) = (1+\sigma_1+\sigma_2+\sigma_3)^{2s-n} \cdot (1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3).$$

□

From the Proposition above, if $n \ge s+3$, the Stiefel-Whitney class of maximal dimension is

$$\bar{w}_{6s-3n+3} = (\sigma_1\sigma_2+\sigma_3)\sigma_3^{2s-n}.$$

Since $\sigma_1\sigma_2+\sigma_3 \neq 0$ (cf. [8] and [1]) and $2s-n \le n-3$ for $s+3 \le n < \frac{4}{3}s$, then $\bar{w}_{6s-3n+3}$ is the non-zero class in the top dimension of $H^*(F,\mathbb{Z}_2)$ for $n \ge s+3$. Using the fact that if $\bar{w}_k(M) \neq 0$ then $M \not\subset \mathbb{R}^{m+k}$ and $M \not\subseteq \mathbb{R}^{m+k-1}$ where M is a smooth manifold of real dimension m [11], we have,

$$\mathbb{R}F(1,1,1,n-3) \not\subset \mathbb{R}^{6s-3}$$

and

$$\mathbb{R}F(1,1,1,n-3) \not\subseteq \mathbb{R}^{6s-4}$$

for $s+3 < n < \frac{4}{3}s$.

If $\frac{2}{3}s \leq n < s$, then $2s-n = s+q$, $0 < q < \frac{1}{3}s$. Therefore, from the Proposition, we have,

$$\begin{aligned}
\bar{w}(F) &= (1+\sigma_1+\sigma_2+\sigma_3)^{s+q} \cdot (1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3) \\
&= (1+\sigma_1^s+\sigma_2^s+\sigma_3^s) \cdot (1+\sigma_1+\sigma_2+\sigma_3)^q \cdot (1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3) \\
&= (1+\sigma_1+\sigma_2+\sigma_3)^q \cdot (1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3) \quad \text{since } s > n.
\end{aligned}$$

The maximal class

$$\bar{w}_{3(s-n)} = (\sigma_1\sigma_2+\sigma_3)\sigma_3^{s-n} \neq 0 \quad \text{for} \quad n < s.$$

And (i) follows.

Now if $n = 2^r$, $r > 2$, then

$$\bar{w}(F) = (1+\sigma_1+\sigma_2+\sigma_3)^{2s-n} \cdot (1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3)$$

and

$$\begin{aligned}
(1+\sigma_1+\sigma_2+\sigma_3)^n &= 1+(x^n+y^n+z^n)+(x^ny^n+y^nz^n+x^nz^n)+(x^ny^nz^n) \\
&= 1+\sigma_1^n+\sigma_2^n+\sigma_3^n \\
&= 1.
\end{aligned}$$

Thus,

$$\bar{w}(F) = 1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3.$$

and

$$\bar{w}_3(F) = \sigma_1\sigma_2+\sigma_3 \neq 0.$$

Therefore

$$\mathbb{R}F(1,1,1,n-3) \not\subset \mathbb{R}^{3n-3}$$

and

$$\mathbb{R}F(1,1,1,n-3) \not\subseteq \mathbb{R}^{3n-4}.$$

For $n = s+1$, we have from the Proposition,

$$\begin{aligned}
\bar{w}(F) &= (1+\sigma_1+\sigma_2+\sigma_3)^{s-1}\cdot(1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3)\\
&= (1+\sigma_1+\sigma_2+\sigma_3)^{s-1}\cdot((1+\sigma_1)(1+\sigma_1+\sigma_2)+\sigma_3)\\
&= ((1+\sigma_1)(1+\sigma_1+\sigma_2)+\sigma_3)\sum_{k=0}^{s-1}(1+\sigma_1+\sigma_2)^{s-k-1}\sigma_3^k\\
&= (1+\sigma_1)\sum_{k=0}^{s-1}(1+\sigma_1+\sigma_2)^{s-k}\sigma_3^k+\sum_{k=0}^{s-1}(1+\sigma_1+\sigma_2)^{s-k-1}\sigma_3^{k+1}\\
&= (1+\sigma_1)(1+\sigma_1+\sigma_2)^s+\sigma_1\sum_{k=0}^{s-1}(1+\sigma_1+\sigma_2)^{s-k}\sigma_3^k\\
&= 1+\sigma_1+\sigma_1^s+\sigma_1^{s+1}+\sigma_1\sigma_2^s+\sigma_2^s+\sigma_1\sum_{k=1}^{s-1}(1+\sigma_1+\sigma_2)^{s-k}\sigma_3^k
\end{aligned}$$

and

$$\bar{w}_1 = \sigma_1 \neq 0 \quad \text{for} \quad n = s+1.$$

This proves (iv).
For $n = s+2$, we have,

$$\bar{w}(F) = (1+\sigma_1+\sigma_2+\sigma_3)^{s-2}\cdot(1+\sigma_1^2+\sigma_2+\sigma_1\sigma_2+\sigma_3).$$

The only three dimensional terms appearing in $\bar{w}(F)$ for $n = s+2$, are $\sigma_1\sigma_2$ and σ_3, in the second factor, and it is non-zero in F, therefore

$$\bar{w}_3 = \sigma_1\sigma_2 + \sigma_3 \neq 0.$$

Hence the results. □

To prove the corollary note that in [9], K. Y. Lam proved, the following:
The real flag manifold $\mathbb{R}F(n_1, n_2, \ldots, n_s)$ can be immersed in Euclidean space with codimension $\frac{1}{2}\sum n_i(n_i-1)$ provided the codimension is non-zero.
From Lam's result we have, $\mathbb{R}F(1,1,1,2) \subseteq \mathbb{R}^{10}$, $\mathbb{R}F(1,1,1,3) \subseteq \mathbb{R}^{15}$, $\mathbb{R}F(1,1,1,4) \subseteq \mathbb{R}^{21}$. Combining these with result (iv) above for $n = 5$, $\mathbb{R}F(1,1,1,2) \nsubseteq \mathbb{R}^9$, result (v) for $n = 6$, $\mathbb{R}F(1,1,1,3) \nsubseteq \mathbb{R}^{14}$ and (ii) for $n = 7$, $\mathbb{R}F(1,1,1,4) \nsubseteq \mathbb{R}^{20}$; verify that Lam's estimates are best possible and give the immersion dimension in these three cases.

Remarks

(a) The non-embedding/ non-immersion results obtained in (i) (ii) and (iii) above are the best which could be obtained using Stiefel-Whitney classes, since we were able to obtain the non-zero dual Stiefel-Whitney class of maximal dimension. We have the strongest of the results when $n = s + 3$.

(b) In cases (iv) and (v), the results could be improved on but for $s = 4$, the results are best possible.

(c) Lam's immersions results, are not interesting for $n > 10$ since the estimates exceeds $2m$, Whitney's estimate.

(d) The results of the corollary coincides with the results in the table in [10] which was generated using the software Maple V Release 4.

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