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APPLICATION OF FIXED POINT THEOREM TO BEST SIMULTANEOUS APPROXIMATION IN CONVEX METRIC SPACES

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Abstract. We present existence of common fixed point results as best simultaneous approximation for uniformly $\mathcal{R}$-subweakly mappings on non-starshaped domains in convex spaces. This work provides extension as well as substantial improvement of some results in the existing literature.

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1. INTRODUCTION AND PRELIMINARIES

For the sake of convenience, we gather some basic definitions and set out our terminology needed in the sequel.

Definition 1.1. [27] *Let* $(\mathcal{X},d)$ *be a metric space. A continuous mapping* $\mathcal{W} : \mathcal{X} \times \mathcal{X} \times [0,1] \to \mathcal{X}$ *is said to be a convex structure on* $\mathcal{X}$*, if for all* $x, y \in \mathcal{X}$ *and* $\lambda \in [0,1]$ *the following condition is satisfied:*

$$d(u, \mathcal{W}(x,y,\lambda)) \leq \lambda d(u,x) + (1-\lambda)d(u,y), \ \ for \ \ all \ \ u \in \mathcal{X}.$$

A metric space $\mathcal{X}$ with convex structure is called a convex metric space. Banach space and each of its convex subsets are simple examples of convex metric spaces with $\mathcal{W}(x,y,\lambda) = \lambda x + (1-\lambda)y$. But a *Fréchet* space is not necessary a convex metric space. There are many examples of convex metric spaces which are not imbedded in any Banach space.

The two preliminary examples are given below:

Example 1.2. Let I be the unit interval $[0,1]$ and $\mathcal{X}$ be the family of closed intervals $[a_i, b_i]$ such that $0 \leq a_i \leq b_i \leq 1$. For $I_i = [a_i, b_i]$, $I_j = [a_j, b_j]$ and λ $(0 \leq \lambda \leq 1)$, we define a mapping $\mathcal{W}$ by $\mathcal{W}(I_i, I_j; \lambda) = [\lambda a_i + (1-\lambda)a_j, \lambda b_i + (1-\lambda)b_j]$ and define a metric d in $\mathcal{X}$ by the Hausdorff distance, i.e.

$$d(I_i, I_j) = \sup_{a \in I}\{|\inf_{b \in I_i}\{|a-b|\} - \inf_{c \in I_j}\{|a-c|\}|\}.$$

Example 1.3. Consider a linear space $\mathcal{L}$ which is also a metric space with the following properties:

(1) For $x, y \in \mathcal{L}$, $d(x,y) = d(x-y, 0)$;

(2) For $x, y \in \mathcal{L}$ and λ $(0 \leq \lambda \leq 1)$,

$$d(\lambda x + (1-\lambda)y, 0) \leq \lambda d(x,0) + (1-\lambda)d(y,0).$$

Definition 1.4. [27] *A subset* $\mathcal{K}$ *of a convex metric space* $\mathcal{X}$ *is said to be convex, if* $\mathcal{W}(x,y,\lambda) \in \mathcal{K}$ *for all* $x, y \in \mathcal{K}$ *and* $\lambda \in [0,1]$*. The set* $\mathcal{K}$ *is said to p-starshaped* *if there exists* $p \in \mathcal{K}$ *such that* $\mathcal{W}(p, x, \lambda) \in \mathcal{K}$ *for all* $x \in \mathcal{K}$ *and* $\lambda \in [0, 1]$. *Clearly p-starshaped subsets of* $\mathcal{X}$ *contain all convex subsets of* $\mathcal{X}$ *as a proper subclass.*

Definition 1.5 [27] *A convex metric space* $\mathcal{X}$ *is said to satisfy property* (I), *if for all* $x, y \in \mathcal{X}$ *and* $\lambda \in [0, 1]$,

$$d(\mathcal{W}(p, x, \lambda), \mathcal{W}(p, y, \lambda)) \leq \lambda d(x, y).$$

Definition 1.6. [27] *A continuous function* $\mathcal{S}$ *from a closed convex subset* $\mathcal{K}$ *of a convex metric space* $\mathcal{X}$, *into itself is said to be affine if* $\mathcal{S}(\mathcal{W}(x, y, \lambda)) = \mathcal{W}(\mathcal{S}x, \mathcal{S}y, \lambda)$ *whenever* $\lambda \in [0, 1] \cap \mathcal{Q}$ *and* $x, y \in \mathcal{K}$, *where* $\mathcal{Q}$ *denotes, the set of rational numbers.*

Let $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ be two mappings. A point $x \in \mathcal{K}$ is a common fixed point of $\mathcal{S}$ and $\mathcal{T}$ if $x = \mathcal{S}x = \mathcal{T}x$. The set of fixed points of $\mathcal{T}$ is denoted by $Fix(\mathcal{T})$. The pair $(\mathcal{S}, \mathcal{T})$ is called

(1) commuting if $\mathcal{S}\mathcal{T}x = \mathcal{T}\mathcal{S}x$ for all $x \in \mathcal{K}$;

(2) compatible [12] if $\lim_n d(\mathcal{T}\mathcal{S}x_n, \mathcal{S}\mathcal{T}x_n) = 0$ when $\{x_n\}$ is a sequence such that $\lim_n \mathcal{T}x_n = \lim_n \mathcal{S}x_n = t$ for some t in $\mathcal{K}$. Every commuting pair of mappings is compatible but the converse is not true in general [12];

(3) weakly compatible, if they commute at there coincidence points, i.e., if $\mathcal{T}u = \mathcal{S}u$ for some $u \in \mathcal{X}$, then $\mathcal{T}\mathcal{S}u = \mathcal{S}\mathcal{T}u$.

Definition 1.7. *Let* $\mathcal{K} \subset \mathcal{X}$ *be a metric space. A map* $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ *is said to be*

(1) a uniformly asymptotically regular on $\mathcal{K}$ *if, for each* $\eta > 0$, *there exists* $N(\eta) = N$ *such that* $d(\mathcal{T}^n x, \mathcal{T}^{n+1} x) < \eta$ *for all* $\eta \geq N$ *and all* $x \in \mathcal{K}$.

(2) $\mathcal{S}$*-nonexpansive, if there exists a self-map* $\mathcal{S}$ *on* $\mathcal{K}$ *such that*

$$d(\mathcal{T}x, \mathcal{T}y) \leq d(\mathcal{S}x, \mathcal{S}y) \ \ for \ \ all \ \ x, y \in \mathcal{K}.$$

(3) asymptotically $(\mathcal{S}, \mathcal{J})$*-nonexpansive, if there exists a sequence* $\{k_n\}$ *of real numbers with* $k_n \geq 1$ *and* $\lim_{n\to\infty} k_n = 1$ *such that* $d(\mathcal{T}^n x, \mathcal{T}^n y) \leq k_n d(\mathcal{S}x, \mathcal{J}y)$ *for all* $x, y \in \mathcal{K}$ *and* $n = 1, 2, 3, ..\infty$. *If* $\mathcal{J} = \mathcal{S}$, *then* $\mathcal{T}$ *is called asymptotically* $\mathcal{S}$*-nonexpansive.*

Definition 1.8. *Let $\mathcal{K}$ be a subset of a convex space $\mathcal{X}$. The set*

$$\mathcal{P}_{\mathcal{K}}(x_0) = \{z \in \mathcal{K} : d(x_0, z) = dist(x_0, \mathcal{K})\}$$

is called the set of best approximants to $x_0 \in \mathcal{X}$ out of $\mathcal{K}$, where

$$dist(x_0, \mathcal{K}) = \inf\{d(x_0, z) : z \in \mathcal{K}\}.$$

Suppose that $\mathcal{A}$ and $\mathcal{G}$ are bounded subsets of $\mathcal{X}$. Then we write

$$r_{\mathcal{G}}(\mathcal{A}) = \inf_{g \in \mathcal{G}} \sup_{a \in \mathcal{A}} d(a, g)$$

$$cent_{\mathcal{G}}(\mathcal{A}) = \{g_0 \in \mathcal{G} : \sup_{a \in \mathcal{A}} d(a, g_0) = r_{\mathcal{G}}(A)\}.$$

The number $r_{\mathcal{G}}(\mathcal{A})$ is called the Chebyshev radius of $\mathcal{A}$ w.r.t. $\mathcal{G}$ and an element $y_0 \in cent_{\mathcal{G}(\mathcal{A})}$ is called a best simultaneous approximation of $\mathcal{A}$ w.r.t. $\mathcal{G}$. If $\mathcal{A} = \{u\}$, then $r_{\mathcal{G}(\mathcal{A})} = dist(u, \mathcal{G})$ and $cent_{\mathcal{G}(\mathcal{A})}$ is the set of all best approximations, $\mathcal{P}_{\mathcal{G}}(u)$, of u out of $\mathcal{G}$.

We also refer the reader to Milman [19] and Vijayaraju [28] for further details.

The interplay between the geometry of Banach spaces and fixed point theory has been very strong and fruitful. In particular, geometric properties play a key role in metric fixed point problems, see for example [7] and references mentioned therein. These results mainly rely on geometric properties of Banach spaces. These results were the starting point for a new mathematical field: the application of geometric theory of Banach spaces to fixed point theory. Fixed point theorems have been applied in the field of approximation theory.

In the realm of best approximation theory, it is viable, meaningful and potentially productive to know whether some useful properties of the function being approximated is inherited by the approximating function. In this perspective, Meinardus [18] observed the general principle that could be applied, while doing so the author has employed a fixed point theorem as a tool to establish it. The result of Meinardus was further generalized by Habiniak [8], Smoluk [25] and Subrahmanyam [26].

On the other hand, Beg and Sahazad [1], Hicks and Humphries [9], Singh [22, 23, 24] and many others have used fixed point theorems in approximation theory to prove existence of best approximation. The concept of best simultaneous approximation is a offshoot of the best approximation, an approximation technique exists in the theory of approximation. Several analysts have studied the problem of best simultaneous approximation under different conditions. Some applications of the fixed point theorems to best simultaneous approximation is given by Sahney and Singh [21]. Vijayaraju [28] also studied this type of results to the class of asymptotically $(\mathcal{S}, \mathcal{J})$-nonexpansive map $\mathcal{T}$. For the detail survey of the subject we refer the reader to Cheney [4].

It is not out of the course that the class of asymptotically nonexpansive mappings was introduced by Goeble and Kirk [7] and further studied by various authors. Recently, Chen and Li [3] introduced the notion of Banach operator pair as a new class of noncommuting maps.

The ordered pair $(\mathcal{T}, \mathcal{S})$ of two self-maps of a metric space $(\mathcal{X}, d)$ is called a Banach operator pair, if the set $Fix(\mathcal{S})$ is $\mathcal{T}$-invariant, namely $\mathcal{T}(Fix(\mathcal{S})) \subseteq Fix(\mathcal{S})$. Obviously commuting pair $(\mathcal{T}, \mathcal{S})$ is Banach operator pair but not conversely in general (see [3]). If $(\mathcal{T}, \mathcal{S})$ is Banach operator pair then $(\mathcal{S}, \mathcal{T})$ need not be Banach operator pair (see [3, Example 1]).

If the self-maps $\mathcal{T}$ and $\mathcal{S}$ of $\mathcal{X}$ satisfy

$$d(\mathcal{S}\mathcal{T}x, \mathcal{T}x) \leq kd(\mathcal{S}x, x) \tag{1}$$

for all $x \in \mathcal{X}$ and $k \geq 0$, then $(\mathcal{T}, \mathcal{S})$ is Banach operator pair. In particular, when $\mathcal{S} = \mathcal{T}$ and $\mathcal{X}$ is a normed space, (1) can be rewritten as

$$\|\mathcal{T}^2 x - \mathcal{T}x\| \leq k\|\mathcal{T}x - x\| \tag{2}$$

for all $x \in \mathcal{X}$. Such $\mathcal{T}$ is called Banach operator of type k in [26].

This new class of noncommuting mappings is different from the class of noncommuting mappings (viz. $\mathcal{R}$-weakly commuting, $\mathcal{R}$-subweakly commuting, compatible, weakly compatible, $\mathcal{C}_p$-commuting etc) existing in the literature so far. Hence the concept of Banach operator pair is of basic importance for study of common fixed points in best approximation. Using this concept, Chen and Li [3] obtained the results without assuming the linearity or affinity of the mapping $\mathcal{T}$ but the convexity or starshapedness of the domain is retained.

The result of Chen and Li [3] is further investigated by Hussain [11]. Recently, Khan and Akbar [17] improved and extended the invariant approximation results of Chen and Li [3] and Vijayaraju [28] to the class of asymptotically $(\mathcal{S},\mathcal{J})$-nonexpansive map $\mathcal{T}$ where $(\mathcal{T},\mathcal{S})$ and $(\mathcal{T},\mathcal{J})$ are Banach operator pairs.

On the other hand, Takahashi [27] introduced the notion of convex metric space and studied the fixed point theory for nonexpansive mappings in such a setting. Afterwards, many authors have discussed the existence of fixed point and the convergence of iterative processes for nonexpansive mappings in this kind of spaces (see [5, 6, 10]). The concept of convex metric space was also used by Beg et al. [2] to prove existence of common fixed point and then to apply it for proving existence of best approximant for relatively contractive commuting mappings. In this way, Beg et al. [2] generalized the result of Sahab et al. [20] and others related previous works.

The purpose of this paper is to improve and extend invariant approximation results of Beg et al. [2] to the class of asymptotically $(\mathcal{S},\mathcal{J})$-nonexpansive map $\mathcal{T}$ where $(\mathcal{T},\mathcal{S})$ and $(\mathcal{T},\mathcal{J})$ are Banach operator pairs without the condition of linearity or affinity of $\mathcal{S}$ and $\mathcal{J}$. Incidently, results of Khan and Akbar [17] are also extended to convex metric spaces.

The following recent result will also be needed which is a consequence of Hussain [11, Lemma 2.10]:

Lemma 1.9. *Let $\mathcal{K}$ be a nonempty subset of a metric space $(\mathcal{X},d)$, and $(\mathcal{T},\mathcal{S})$ and $(\mathcal{T},\mathcal{J})$ be Banach operator pair on $\mathcal{K}$. Assume that $cl\mathcal{T}(\mathcal{K})$ is complete, and $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{J}$ satisfy for all $x,y\in\mathcal{K}$ and $0\le h<1$,*

$$d(\mathcal{T}x,\mathcal{T}y)\le hd(\mathcal{S}x,\mathcal{J}y). \tag{3}$$

If $\mathcal{S}$ and $\mathcal{J}$ are continuous, $Fix(\mathcal{S})\cap Fix(\mathcal{J})$ is nonempty, then there is unique common fixed point of $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{J}$.

2. MAIN RESULTS

Theorem 2.1. *Let $\mathcal{K}$ be a nonempty subset of a convex metric space $\mathcal{X}$ satisfying property (I) and $y_1, y_2 \in \mathcal{X}$. Suppose that $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{J}$ are selfmaps of $\mathcal{K}$ such that $\mathcal{T}$ is asymptotically $(\mathcal{S},\mathcal{J})$-nonexpansive. Suppose that the set $Fix(\mathcal{S}) \cap Fix(\mathcal{J})$ is nonempty. Let the set $\mathcal{D}$, of best simultaneous $\mathcal{K}$-approximants to y_1 and y_2, is nonempty compact and starshaped with respect to an element $p \in Fix(\mathcal{S}) \cap Fix(\mathcal{J})$ and $\mathcal{D}$ is invariant under $\mathcal{T}$, $\mathcal{S}$ and $\mathcal{J}$. Assume further that $(\mathcal{T},\mathcal{S})$ and $(\mathcal{T},\mathcal{J})$ are Banach operator pairs on $\mathcal{D}$, $Fix(\mathcal{S})$ and $Fix(\mathcal{J})$ are p-starshaped with $p \in Fix(\mathcal{S}) \cap Fix(\mathcal{J})$, $\mathcal{S}$ and $\mathcal{J}$ are continuous and $\mathcal{T}$ is uniformly asymptotically regular on $\mathcal{D}$. Then $\mathcal{D}$ contains a $\mathcal{T}$-, $\mathcal{S}$- and $\mathcal{J}$-invariant point.*

Proof. For each $n \geq 1$, define $\mathcal{T}_n$ on $\mathcal{K}$ by $\mathcal{T}_n x = \mathcal{W}(p, \mathcal{T}^n x, \mu_n)$ for all $x \in \mathcal{K}$, $\mu_n = \frac{\lambda_n}{k_n}$ and $\{\lambda_n\}$ is a sequence of real number in $(0,1)$ such that $\lim_{n\to\infty} \lambda_n = 1$ and $\{k_n\}$ is defined as above.

Since $\mathcal{T}(\mathcal{D}) \subset \mathcal{D}$ and $\mathcal{D}$ is p-starshaped, it follows that $\mathcal{T}_n$ maps $\mathcal{D}$ into $\mathcal{D}$. As $(\mathcal{T},\mathcal{S})$ is a Banach operator pair, $\mathcal{T}(Fix(\mathcal{S})) \subseteq Fix(\mathcal{S})$ implies that $\mathcal{T}^n(Fix(\mathcal{S})) \subseteq Fix(\mathcal{S})$ for each $n \geq 1$. On utilizing p-starshapedness of $Fix(\mathcal{S})$ we see that for each $x \in Fix(\mathcal{S})$, $\mathcal{T}_n x = \mathcal{W}(p, \mathcal{T}^n x, \mu_n) \in Fix(\mathcal{S})$, since $\mathcal{T}^n x \in Fix(\mathcal{S})$ for each $x \in Fix(\mathcal{S})$. Thus $(\mathcal{T}_n, \mathcal{S})$ is a Banach operator pair on $\mathcal{D}$ for each $n \geq 1$. Similarly, $(\mathcal{T}_n, \mathcal{J})$ is a Banach operator pair on $\mathcal{D}$ for each $n \geq 1$. For each $x, y \in \mathcal{D}$, we have

$$\begin{aligned} d(\mathcal{T}_n x, \mathcal{T}_n y) &= d(\mathcal{W}(p, \mathcal{T}^n x, \mu_n), \mathcal{W}(p, \mathcal{T}^n y, \mu_n)) \\ &\leq \mu_n d(\mathcal{T}^n x, \mathcal{T}^n y) \quad \text{by property (I)} \\ &\leq \lambda_n d(\mathcal{S}x, \mathcal{J}y). \end{aligned}$$

By Lemma 1.9, for each $n \geq 1$, there exists $x_n \in \mathcal{D}$ such that $x_n = \mathcal{S}x_n = \mathcal{J}x_n = \mathcal{T}_n x_n$. As $\mathcal{T}(\mathcal{D})$ is bounded, we have

$$\begin{aligned} d(x_n, \mathcal{T}^n x_n) = d(\mathcal{T}_n x_n, \mathcal{T}^n x_n) &= d(\mathcal{W}(p, \mathcal{T}^n x_n, \mu_n), \mathcal{T}^n x_n) \\ &\leq \mu_n d(\mathcal{T}^n x_n, \mathcal{T}^n x_n) + (1-\mu_n) d(p, \mathcal{T}^n x_n) \\ &= (1-\mu_n) d(p, \mathcal{T}^n x_n) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Since $(\mathcal{T},\mathcal{S})$ is a Banach operator pair and $\mathcal{S}x_n = x_n$, so $\mathcal{S}\mathcal{T}^n x_n = \mathcal{T}^n\mathcal{S}x_n = \mathcal{T}^n x_n$. Thus we have

$$\begin{aligned} d(x_n,\mathcal{T}x_n) &\le d(x_n,\mathcal{T}^n x_n) + d(\mathcal{T}^n x_n,\mathcal{T}^{n+1}x_n) + d(\mathcal{T}^{n+1}x_n,\mathcal{T}x_n)\\ &\le d(x_n,\mathcal{T}^n x_n) + d(\mathcal{T}^n x_n,\mathcal{T}^{n+1}x_n) + d(\mathcal{S}(\mathcal{T}^n x_n),\mathcal{J}x_n)\\ &= d(x_n,\mathcal{T}^n x_n) + d(\mathcal{T}^n x_n,\mathcal{T}^{n+1}x_n) + k_1 d(\mathcal{T}^n x_n,x_n). \end{aligned}$$

Since $\mathcal{T}$ is uniformly asymptotically regular on $\mathcal{D}$, it follows that

$$d(\mathcal{T}^n x_n,\mathcal{T}^{n+1}x_n) \to 0 \text{ as } n \ge 1.$$

Thus we have

$$d(x_n,\mathcal{T}x_n) \le d(x_n,\mathcal{T}^n x_n) + d(\mathcal{T}^n x_n,\mathcal{T}^{n+1}x_n) + k_1 d(\mathcal{T}^n x_n,x_n) \to 0$$

$n \to \infty$. Since $\mathcal{D}$ is compact, there exists a subsequence $\{x_m\}$ of $\{x_n\}$ such that $x_m \to y$ as $m \to \infty$. By the continuity of $\mathcal{I}-\mathcal{T}$, we have $(\mathcal{I}-\mathcal{T})x_m \to (\mathcal{I}-\mathcal{T})y$. But $(\mathcal{I}-\mathcal{T})x_m \to 0$, so we have $(\mathcal{I}-\mathcal{T})y = 0$. Since $\mathcal{S}$ and $\mathcal{J}$ are continuous, it follows that

$$\mathcal{S}y = \mathcal{S}(\lim_m x_m) = \lim_m \mathcal{S}x_m = \lim_m x_m = y$$

and

$$\mathcal{J}y = \mathcal{J}(\lim_m x_m) = \lim_m \mathcal{J}x_m = \lim_m x_m = y.$$

This completes the proof. □

An immediate consequence follows from Theorem 2.1 as condition (i) implies that $\mathcal{D}$ is $\mathcal{T}$-invariant.

Corollary 2.2. *Let $\mathcal{X}$, $\mathcal{K}$, y_1, y_2, $\mathcal{S}$, $\mathcal{J}$ and $\mathcal{T}$ be as in Theorem 2.1. Assume that $\mathcal{T}$ satisfies the following condition:*
(i) $d(\mathcal{T}x,y_i) \le d(x,y_i)$ for all $x \in \mathcal{X}$ and $i = 1,2$.
Suppose that the set $\mathcal{D}$, of best simultaneous $\mathcal{K}$-approximants to y_1 and y_2, is nonempty compact and starshaped with respect to an element p in $Fix(\mathcal{S}) \cap Fix(\mathcal{J})$. Then $\mathcal{D}$ contains a $\mathcal{T}$-, $\mathcal{S}$- and $\mathcal{J}$-invariant point.

Take $\mathcal{J} = \mathcal{S}$ in Theorem 2.1 to get:

Corollary 2.3. *Let $\mathcal{K}$ be a nonempty subset of a convex metric space $\mathcal{X}$ satisfying property (I) and $y_1, y_2 \in \mathcal{X}$. Suppose that $\mathcal{T}$ and $\mathcal{S}$ are selfmaps of $\mathcal{K}$ such that $\mathcal{T}$ is asymptotically $\mathcal{S}$-nonexpansive. Suppose that the set $Fix(\mathcal{S})$ is nonempty. Let the set $\mathcal{D}$, of best simultaneous $\mathcal{K}$-approximants to y_1 and y_2, is nonempty compact and starshaped with respect to an element p in $Fix(\mathcal{S})$ and $\mathcal{D}$ is invariant under $\mathcal{T}$ and $\mathcal{S}$. Assume further that $(\mathcal{T}, \mathcal{S})$ is a Banach operator pair on $\mathcal{D}$, $Fix(\mathcal{S})$ is p-starshaped with $p \in Fix(\mathcal{S})$, $\mathcal{S}$ is continuous and $\mathcal{T}$ is uniformly asymptotically regular on $\mathcal{D}$. Then $\mathcal{D}$ contains a $\mathcal{T}$- and $\mathcal{S}$-invariant point.*

A commuting pair $(\mathcal{T}, \mathcal{S})$ is a Banach operator pair and affineness of $\mathcal{S}$ implies that $Fix(\mathcal{S})$ is p-starshaped, hence we get the following from Corollary 2.3.

Corollary 2.4. *Let $\mathcal{K}$ be a nonempty subset of a convex metric space $\mathcal{X}$ satisfying property (I) and $y_1, y_2 \in \mathcal{X}$. Suppose that $\mathcal{T}$ and $\mathcal{S}$ are selfmaps of $\mathcal{K}$ such that $\mathcal{T}$ is asymptotically $\mathcal{S}$-nonexpansive. Suppose that the set $Fix(\mathcal{S})$ is nonempty. Let the set $\mathcal{D}$, of best simultaneous $\mathcal{K}$-approximants to y_1 and y_2, is nonempty compact and starshaped with respect to an element p in $Fix(\mathcal{S})$ and $\mathcal{D}$ is invariant under $\mathcal{T}$ and $\mathcal{S}$. Assume further that $\mathcal{T}$ and $\mathcal{S}$ are commuting, $\mathcal{T}$ is uniformly asymptotically regular on $\mathcal{D}$ and $\mathcal{S}$ is affine. Then $\mathcal{D}$ contains a $\mathcal{T}$- and $\mathcal{S}$-invariant point.*

Remark 2.5. Notice that the condition $\mathcal{S}(\mathcal{D}) = \mathcal{D}$ in Theorem 2.3 of Vijayaraju [28] is not needed in our work.

Theorem 2.1 extends and improves the results due to Jungck and Sessa [15], Sahab et al. [20], Sahney and Singh [21], Singh [22, 23, 24] and Vijayaraju [28].

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