*Kragujevac J. Math.* 33 (2010) 101–106.

# A GENERALIZATION OF QI'S INEQUALITY FOR SUMS

**Huan-nan Shi**

*Department of Electronic Information, Teacher's College,*
*Beijing Union University,*
*Beijing City, 100011, China*
(e-mails: shihuannan@yahoo.com.cn, sfthuannan@buu.edu.cn)

*(Received February 02, 2009)*

**Abstract.** By a majorization method, a pair of inequalities for sums of nonnegative sequences are established, and so an open problem posed by F. Qi is resolved.

**2010 Mathematics Subject Classification**: 26D15.
**Key words**: Generalization, Qi's inequality for sums, Majorization.

## 1. INTRODUCTION

In [4], the following inequality between the sum of squares and the exponential of sum of a nonnegative sequence was obtained: For $(x_1, x_2, \ldots, x_n) \in R_+^n$ and $n \geq 2$, the inequality

$$\frac{e^2}{4}\sum_{i=1}^{n} x_i^2 \leq \exp\left(\sum_{i=1}^{n} x_i\right) \tag{1}$$

is valid, where $R_+^n = \{(x_1, \ldots, x_n) \in R^n : x_i \geq 0, i = 1, \ldots, n\}$. The equality in (1) holds if $x_i = 2$ and $x_j = 0$ for some given $1 \leq i \leq n$ and all $1 \leq j \leq n$ with $j \neq i$. The constant $\frac{e^2}{4}$ in the inequality(1) is the best possible.

The first open problem in [4] may be quoted as follows: For $(x_1, x_2, \ldots, x_n) \in R_+^n$ and $n \geq 2$, determine the best possible constants $\alpha_n, \lambda_n \in R$ and $0 < \beta_n, \mu_n < \infty$ such that

$$\beta_n \sum_{i=1}^{n} x_i^{\alpha_n} \leq \exp\left(\sum_{i=1}^{n} x_i\right) \leq \mu_n \sum_{i=1}^{n} x_i^{\lambda_n}. \tag{2}$$

First of all, we claim that the right-hand side inequality in (2) is generally untenable. In fact, when $n = 2$, the right-hand side inequality in (2) becomes

$$e^{x_1+x_2} \leq \mu_2 \left(x_1^{\lambda_2} + x_2^{\lambda_2}\right). \tag{3}$$

Further taking $x_2 = 0$ in the inequality (3) reduces

$$\frac{e^{x_1}}{x_1^{\lambda}} \leq \mu. \tag{4}$$

For any given $\lambda > 0$, the function $\dfrac{e^{x_1}}{x_1^{\lambda}}$ tends to $\infty$ as $x_1 \to \infty$. Hence, the inequality (4) does not hold if $x_1$ is large enough.

In this short note, by using a method in the theory of majorization, we give an affirmative solution to the left-hand side inequality in (2), which is also a generalization of the inequality (1), as follows.

**Theorem 1.** *Let $(x_1, x_2, \ldots, x_n) \in R_+^n$ and $n \geq 2$. If $\alpha \geq 1$, then the inequality*

$$\frac{e^{\alpha}}{\alpha^{\alpha}} \left(\sum_{i=1}^{n} x_i^{\alpha}\right) \leq \exp\left(\sum_{i=1}^{n} x_i\right) \tag{5}$$

*is valid. The equality in (5) holds if and only if $x_i = \alpha$ and $x_j = 0$ for some given $1 \leq i \leq n$ and all $1 \leq j \leq n$ with $j \neq i$.*

**Theorem 2.** *Let $\{x_i\}_{i=1}^{\infty}$ be a nonnegative sequence such that $\sum_{i=1}^{\infty} x_i < \infty$. For $\alpha \geq 1$, the inequality*

$$\frac{e^{\alpha}}{\alpha^{\alpha}} \sum_{i=1}^{\infty} x_i^{\alpha} \leq \exp\left(\sum_{i=1}^{\infty} x_i\right) \tag{6}$$

*is valid.*

## 2. DEFINITIONS AND LEMMAS

In order to prove our theorems, the following definitions and lemmas are needed.

**Definition 1.** [1, 6] *Let* $x = (x_1, \ldots, x_n) \in R^n$ *and* $y = (y_1, \ldots, y_n) \in R^n$.

1. *The sequence* $x$ *is said to be majorized by* $y$ (*in symbols* $x \preceq y$) *if* $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ *for* $k = 1, 2, \ldots, n-1$ *and* $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, *where* $x_{[1]} \geq \cdots \geq x_{[n]}$ *and* $y_{[1]} \geq \cdots \geq y_{[n]}$ *are rearrangements of* $x$ *and* $y$ *in a descending order, and* $x$ *is said to strictly majorized by* $y$ (*in symbols* $x \prec y$) *if* $x$ *is not a permutation of* $y$.

2. *A function* $f : \Omega \to R$ *is said to be a strictly Schur-convex on* $\Omega \subset R^n$ *if the relation* $x \prec y$ *on* $\Omega$ *implies* $f(x) < f(y)$. *A function* $f$ *is said to be strictly Schur-concave on* $\Omega$ *if and only if* $-f$ *is strictly Schur-convex on* $\Omega$.

**Definition 2.** [6] *Let set* $\Omega \subseteq R^n$. $\Omega$ *is said to be a convex set if* $x, y \in \Omega$, $0 \leq \alpha \leq 1$ *implies* $\alpha x + (1-\alpha)y = (\alpha x_1 + (1-\alpha)y_1, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

**Lemma 1.** [6, p. 5] *Let* $\Omega \subset R^n$ *be symmetric and have a nonempty interior convex set* $\Omega^0$, *and let* $f : \Omega \to R$ *be continuous on* $\Omega$ *and differentiable on* $\Omega^0$. *Then the function* $f$ *is strictly Schur-convex* (*or Schur-concave respectively*) *on* $\Omega$ *if and only if* $f$ *is symmetric on* $\Omega$ *and satisfies*

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) > 0 \quad (or \ < 0, \ respectively) \tag{7}$$

*for* $x = (x_1, x_2) \in \Omega^0$ *with* $x_1 \neq x_2$.

**Lemma 2.** *For any given positive real number* $s$ *and* $\alpha$, *we have*

$$\frac{e^\alpha}{\alpha^\alpha} \leq \frac{e^s}{s^\alpha}. \tag{8}$$

*The equality in (8) holds if and only if* $s = \alpha$.

**Proof.** Let $\varphi(s) = \alpha \ln s - s$. Then $\varphi'(s) = \frac{\alpha}{s} - 1 \leq 0$ for $s \geq \alpha > 0$, which means that $\varphi(s)$ is increasing, and $\varphi'(s) \geq 0$ for $0 < s \leq \alpha$, which means that $\varphi(s)$ is decreasing. Hence, for any $s > 0$, we have

$$\varphi(s) = \alpha \ln s - s \leq \varphi(\alpha) = \alpha \ln \alpha - \alpha,$$

i.e., the inequality (8) is valid and the equality in (8) holds if and only if $s = \alpha$. $\square$

## 3. PROOFS OF THEOREMS

Now we are in a position to prove our theorems.

**Proof of Theorem 1**

Let

$$f(x) = f(x_1, \ldots, x_n) = \ln\left(\sum_{i=1}^{n} x_i^\alpha\right) - s, \tag{9}$$

where $s = \sum_{i=1}^n x_i$. Simple calculation gives

$$\Delta := (x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) = \frac{\alpha(x_1 - x_2)(x_1^{\alpha-1} - x_2^{\alpha-1})}{\sum_{i=1}^n x_i^\alpha}.$$

When $\alpha > 1$, since $x^{\alpha-1}$ is strictly increasing on $(0, \infty)$, it follows easily that $(x_1 - x_2)(x_1^{\alpha-1} - x_2^{\alpha-1}) > 0$ for $x_1 \neq x_2$, and then $\Delta > 0$, so, by Lemma 1, $f(x)$ is strictly Schur-convex on $R_+^n$. It is easy to see that

$$x = (x_1, \ldots, x_n) \preceq \left(s, \underbrace{0, \ldots, 0}_{n-1}\right) = y \tag{10}$$

and $x \prec y$ unless $x_i = s$ and $x_j = 0$ for some given $1 \leq i \leq n$ and all $1 \leq j \leq n$ with $j \neq i$. Hence,

$$f(x_1, \ldots, x_n) = \ln\left(\sum_{i=1}^{n} x_i^\alpha\right) - s \leq f\left(s, \underbrace{0, \ldots, 0}_{n-1}\right) = \alpha \ln s - s, \tag{11}$$

that is,

$$\frac{e^s}{s^\alpha}\left(\sum_{i=1}^{n} x_i^\alpha\right) \leq \exp\left(\sum_{i=1}^{n} x_i\right), \tag{12}$$

and the equality in (12) holds if and only if $x_i = s$ and $x_j = 0$ for some given $1 \leq i \leq n$ and all $1 \leq j \leq n$ with $j \neq i$. Combining the inequality (12) with the inequality

(8) yields that the inequality (5) is valid and the equality in (5) holds if and only if $x_i = \alpha$ and $x_j = 0$ for some given $1 \le i \le n$ and all $1 \le j \le n$ with $j \ne i$.

The proof of Theorem 1 is complete. □

**Proof of Theorem 2**

Letting $n \to \infty$ in Theorem 1 yields Theorem 2 readily. □

**Remark.** *After the preprint [5] of this paper was announced, there have been several papers such as [2, 3] dedicated to discuss the open problems posed by F. Qi in [4].*

# References


[1] A. M. Marshall and I. Olkin. *Inequalities: Theory of Majorization and its Application*, Academic Press, New York, 1979.

[2] Y. Miao, L.-M. Liu and F. Qi, *Refinements of inequalities between the sum of squares and the exponential of sum of a nonnegative sequence*, J. Inequal. Pure Appl. Math. **9** (2008), no. 2, Art. 53; Available online at http://jipam.vu.edu.au/article.php?sid=985.

[3] T. F. Móri, *On an inequality of Feng Qi*, J. Inequal. Pure Appl. Math. **9** (2008), no. 3, Art. 87; Available online at http://jipam.vu.edu.au/article.php?sid=1024.

[4] F. Qi, *Inequalities between the sum of squares and the exponential of sum of a nonnegative sequence*, J. Inequal. Pure Appl. Math. **8** (2007), no. 3, Art. 78; Available online at http://jipam.vu.edu.au/article.php?sid=895.

[5] H.-N. Shi, *Solution of an open problem proposed by Feng Qi*, RGMIA Res. Rep. Coll. **10** (2007), no. 4, Art. 9; Available online at http://www.staff.vu.edu.au/rgmia/v10n4.asp.

[6] B.-Y. Wang, *Foundations of Majorization Inequalities*, Beijing Normal Univ. Press, Beijing, China, 1990. (Chinese)