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FINITE DIFFERENCE METHOD FOR THE PARABOLIC PROBLEM WITH DELTA FUNCTION

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Abstract. We investigate the convergence of difference schemes for the one-dimensional heat equation with the coefficient of the time derivative containing a Dirac delta distribution. An abstract operator method is applied for analyzing this equation. The convergence rate estimate of the order $\mathcal{O}(h)$ in a special discrete $\widetilde{W}_2^{2,1}$ Sobolev norm, compatible with the smoothness of the solution, is obtained.

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1. INTRODUCTION

The finite-difference method is one of the basic tools for the numerical solution of partial differential equations. In the case of problems with discontinuous coefficients and concentrated factors (Dirac delta functions, free boundaries, etc.) the solution has weak global regularity and it is impossible to establish convergence of the finite difference schemes using the classical Taylor expansion. Often, the Bramble-Hilbert lemma takes the role of the Taylor formula for functions from the Sobolev spaces [4], [5], [10].

Following Lazarov et al. [10], a convergence rate estimate of the form

$$\|u - v\|_{W^k_{2,h}} \le C h^{s-k} \|u\|_{W^s_2}, \quad s > k,$$

is called **compatible** with the smoothness (regularity) of the solution u of the boundary value problem. Here v is the solution of the discrete problem, h is the mesh step, W^s_2 and $W^k_{2,h}$ are Sobolev spaces of functions with continuous and discrete argument, respectively, C is a constant which doesn't depend on u and h. For the parabolic case typical estimates are of the form

$$\|u - v\|_{W^{k,k/2}_{2,h\tau}} \le C(h + \sqrt{\tau})^{s-k} \|u\|_{W^{s,s/2}_2}, \quad s > k,$$

where τ is the time step. In the case of equations with variable coefficients the constant C in the error bounds depends on norms of the coefficients (see, for example, [5], [15], [1]).

One interesting class of parabolic problems model processes in heat-conducting media with concentrated capacity in which the heat capacity coefficient contains a Dirac delta function, or equivalently, the jump of the heat flow in the singular point is proportional to the time derivative of the temperature [12]. Such, the current problems are nonstandard and the classical tools of the theory of finite difference schemes are difficult to apply to their convergence analysis.

In the present paper a finite-difference scheme, approximating the one-dimensional initial-boundary value problem for the heat equation with concentrated capacity, is derived. Special Sobolev type norm (corresponding to the norm $W^{2,1}_2$ for a classical heat-conduction problem) is used for estimation of error.

Note that the convergence to classical solutions is studied in [3] and [17]. The parabolic problem with concentrated capacity on weak solution is studied in [7], [9] and [2]. The convergence of a difference schemes for hyperbolic problem with concentrated mass has been studied in [8].

2. PRELIMINARY RESULTS

Let H be a real separable Hilbert space endowed with inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$ and S an unbounded selfadjoint positive definite linear operator, with domain $D(S)$ dense in H. It is easy to see that the product $(u,v)_S = (Su,v)$ $(u,v \in D(S))$ satisfies the axioms of inner product. The closure of $D(S)$ in the norm $\|u\|_S = (u,u)_S^{1/2}$ is a Hilbert space $H_S \subset H$. The inner product (u,v) continuously extends to $H_S^* \times H_S$, where $H_S^* = H_{S^{-1}}$ is the dual space for H_S. The spaces H_S, H and $H_{S^{-1}}$ form a Gelfand triple $H_S \subset H \subset H_{S^{-1}}$, with continuous imbeddings. The operator S extends to the map $S : H_S \to H_S^*$. There exists an unbounded selfadjoint positive definite linear operator $S^{1/2}$, such that $D(S^{1/2}) = H_S$ and $(u,v)_S = (Su,v) = (S^{1/2}u, S^{1/2}v)$. We also define the Sobolev spaces $W_2^s(a,b;H)$, $W_2^0(a,b;H) = L_2(a,b;H)$, of the functions $u = u(t)$ mapping interval $(a,b) \subset R$ into H (see [11], [18]).

Let A and B be unbounded selfadjoint positive definite linear operators, $A \neq A(t)$, $B \neq B(t)$, in the Hilbert space H, in general noncommutative, with $D(A)$ – dense in H and $H_A \subset H_B$. We consider an abstract Cauchy problem (comp. [13], [18])

$$B\frac{du}{dt} + Au = f(t), \quad 0 < t < T; \quad u(0) = u_0, \tag{1}$$

where $f(t)$ and u_0 are given and $u(t)$ is un unknown function with values in H. The following proposition holds (see [7]).

Lemma 1. *The solution of the problem* (1) *satisfies the priori estimate:*

$$\int_0^T \left(\|Au(t)\|_{B^{-1}}^2 + \left\| \frac{du(t)}{dt} \right\|_B^2 \right) dt \leq C \left(\|u_0\|_A^2 + \int_0^T \|f(t)\|_{B^{-1}}^2 \, dt \right), \tag{2}$$

if $u_0 \in H_A$ *and* $f \in L_2(0,T;H_{B^{-1}})$.

Analogous results hold for operator-difference schemes. Let H_h be finite-dimensional real Hilbert space with inner product $(\cdot,\cdot)_h$ and norm $\|\cdot\|_h$. Let $A_h \neq A_h(t)$ and $B_h \neq B_h(t)$ be selfadjoint positive linear operators defined on H_h, in the general case – noncommutative. By H_{S_h}, where $S_h = S_h^* > 0$, we denote the space with inner product $(y,v)_{S_h} = (S_h y, v)_h$ and norm $\|y\|_{S_h} = (S_h y, y)_h^{1/2}$.

Let ω_τ be a uniform mesh on $(0,T)$ with the step $\tau = T/m$, $\omega_\tau^- = \omega_\tau \cup \{0\}$, $\omega_\tau^+ = \omega_\tau \cup \{T\}$ and $\overline{\omega}_\tau = \omega_\tau \cup \{0,T\}$. Further we shall use standard notation from the theory of the difference schemes [14], [15]. In particular we set

$$v_{\bar{t}} = v_{\bar{t}}(t) = \frac{v(t) - v(t-\tau)}{\tau}, \quad v_t = v_t(t) = \frac{v(t+\tau) - v(t)}{\tau} = v_{\bar{t}}(t+\tau).$$

We will consider the simplest implicit operator-difference scheme

$$B_h v_{\bar{t}} + A_h v = \varphi(t), \ \ t \in \omega_\tau^+; \quad v(0) = v_0, \tag{3}$$

where v_0 is a given element of H_h, $\varphi(t)$ is known and $v(t)$ is an unknown mesh function with values in H_h. The following analog of Lemma 1 is true (comp. [5], [6]).

Lemma 2. *For the solution of the problem* (3) *the following estimate holds:*

$$\tau {\sum_{t\in\overline{\omega}_\tau}}' \|A_h v(t)\|^2_{B_h^{-1}} + \tau \sum_{t\in\omega_\tau^+} \|v_{\bar{t}}(t)\|^2_{B_h} \le C\left(\|v_0\|^2_{A_h} + \tau\|A_h v_0\|^2_{B_h^{-1}} + \tau \sum_{t\in\omega_\tau^+} \|\varphi(t)\|^2_{B_h^{-1}}\right)$$

where we denoted:

$${\sum_{t\in\overline{\omega}_\tau}}' w(t) = \frac{w(0)}{2} + \sum_{t\in\omega_\tau} w(t) + \frac{w(T)}{2}.$$

3. HEAT EQUATION WITH CONCENTRATED CAPACITY

Let us consider the initial-boundary-value problem for the heat equation in the presence of a concentrated capacity at the interior point $x = \xi$:

$$\begin{gathered}[1 + K\,\delta(x-\xi)]\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad \text{on } \ Q = (0,1)\times(0,T), \\ u(0,t) = 0, \ \ u(1,t) = 0\,, \ t \in (0,T) \\ u(x,0) = u_0(x), \ \ x \in (0,1).\end{gathered} \tag{4}$$

where K is a positive constant, $\delta(x)$ is the Dirac delta generalized function and equality is considered in the sense of distributions [16]. Our aim is to investigate the singularity of the solution of the problem (4) caused by the presence of singular coefficient $K\delta(x-\xi)$, therefore we restrict ourselves to the simplest Dirichlet boundary conditions.

In a standard manner one obtains the following weak form of initial-boundary-value problem (4):

$$\int_0^T\!\!\int_0^1 \frac{\partial u}{\partial t} v\,dxdt + \int_0^T \frac{\partial u}{\partial t}(\xi,t)v(\xi,t)\,dt + \int_0^T\!\!\int_0^1 \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dxdt = \int_0^T\!\!\int_0^1 fv\,dxdt$$
$$\forall v \in \overset{\circ}{W}{}_2^{1,0}(Q) = \{v \in W_2^{1,0}(Q) \,:\, v = 0 \text{ on } \{0,1\} \times (0,T)\}. \tag{5}$$

The same weak form (5) corresponds to the following initial-boundary-value problem:

$$\begin{aligned} &\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), \quad (x,t) \in Q_1 \cup Q_2, \\ &[u]_{x=\xi} \equiv u(\xi+0,t) - u(\xi-0,t) = 0, \quad \left[\frac{\partial u}{\partial x}\right]_{x=\xi} = K\frac{\partial u}{\partial t}(\xi,t), \\ &u(0,t) = 0, \;\; u(1,t) = 0, \;\; 0 < t < T, \quad u(x,0) = u_0(x), \;\; x \in (0,1), \end{aligned} \tag{6}$$

where $Q_1 = (0,\xi) \times (0,T)$ and $Q_2 = (\xi,1) \times (0,T)$. In this sense, initial-boundary-value-problems (4) and (6) are equivalent.

Letting $H = L_2(0,1)$ it is easy to see that the initial boundary value problem (4) can be written in the form (1), where

$$Au = -\frac{\partial^2 u}{\partial x^2}, \quad Bu = [1 + K\delta(x-\xi)]\,u(x,t),$$

or

$$(Av,w) = \int_0^1 v'(x)w'(x)\,dx,$$

for $v, w \in \overset{\circ}{W}{}_2^1(0,1)$ and

$$(Bv,w) = \int_0^1 v(x)w(x)\,dx + Kv(\xi)w(\xi).$$

We immediately obtain

$$\|w\|_A^2 = \int_0^1 |w'(x)|^2\,dx \asymp \|w\|^2_{W_2^1(0,1)}, \quad w \in \overset{\circ}{W}{}_2^1(0,1),$$

so, we can put $H_A = \overset{\circ}{W}{}_2^1(0,1)$ and $H_{A^{-1}} = W_2^{-1}(0,1)$.

The operator B is defined on the subset H_B of functions in $L_2(0,1)$ with finite norm

$$\|w\|_B^2 = \|w\|^2_{L_2(0,1)} + Kw^2(\xi) \asymp \|w\|^2_{L_2(0,1)} + w^2(\xi) = \|w\|^2_{\widetilde{L}_2(0,1)},$$

so, we can put $H_B = \widetilde{L}_2(0,1) =$ closure of the set $C[0,1]$ in the norm $\|\cdot\|_{\widetilde{L}_2(0,1)}$. Obviously, $H_A = \overset{\circ}{W}{}_2^1(0,1) \subset C[0,1] \subset \widetilde{L}_2(0,1) = H_B$. The "negative" norm $\|w\|_{B^{-1}}$ satisfies the relation

$$\|w\|_{B^{-1}} = (B^{-1}w, w)^{1/2} = \sup_{0\neq v\in H_B} \frac{|(w,v)|}{\|v\|_B}.$$

4. DIFFERENCE PROBLEM

Let ω_h be a uniform mesh on $(0,1)$ with step $h = 1/n$, $\omega_h^- = \omega_h \cup \{0\}$ and $\overline{\omega}_h = \omega_h \cup \{0,1\}$. Suppose for simplicity that ξ is a rational number. Then one can choose the step h, so that $\xi \in \omega_h$. Also, we suppose that the condition $c_1 h^2 \leq \tau \leq c_2 h^2$ is satisfied. Define finite differences on the usual way:

$$v_{\bar{t}}(x,t) = \frac{v(x,t) - v(x,t-\tau)}{\tau} = v_t(x,t-\tau),$$
$$v_{\overline{x}}(x,t) = \frac{v(x,t) - v(x-h,t)}{h} = v_x(x-h,t),$$
$$v_{x\overline{x}}(x,t) = (v_x(x,t))_{\overline{x}} = \frac{v(x+h,t) - 2v(x,t) + v(x-h,t)}{h^2}.$$

The problem (4) can be approximated on the mesh $\overline{Q}_{h\tau} = \overline{\omega}_h \times \overline{\omega}_\tau$ by the following implicit difference scheme with averaged right-hand side:

$$\begin{aligned} [1 + K\,\delta_h(x-\xi)]\, v_{\bar{t}} - v_{x\bar{x}} &= T_x^2 T_t^- f, \quad \text{on } Q_{h\tau} = \omega_h \times \omega_\tau^+ \\ v(0,t) = 0, \quad v(1,t) &= 0, \quad t \in \omega_\tau^+, \\ v(x,0) &= u_0(x), \quad x \in \bar{\omega}_h. \end{aligned} \tag{7}$$

where

$$\delta_h(x-\xi) = \begin{cases} 0, & x \in \omega_h \setminus \{\xi\}, \\ 1/h, & x = \xi \end{cases}$$

is the mesh Dirac function, and T_x, T_t^- are Steklov averaging operators [16], defined as follows

$$T_x f(x,t) = T_x^- f(x+h/2,t) = T_x^+ f(x-h/2,t) = \frac{1}{h}\int_{x-h/2}^{x+h/2} f(x',t)\,dx',$$
$$T_t^- f(x,t) = T_t^+ f(x,t-\tau) = \frac{1}{\tau}\int_{t-\tau}^{t} f(x,t')\,dt'.$$

Notice that these operators commute and map derivatives into finite differences; for example,

$$T_x^2 \frac{\partial^2 u}{\partial x^2} = u_{x\bar{x}}, \quad T_t^- \frac{\partial u}{\partial t} = u_{\bar{t}}.$$

Let H_h denote the set of functions defined on the mesh $\overline{\omega}_h$ and equal to zero at $x=0$ and $x=1$. We define the following inner product and norms:

$$(v,u)_h = h\sum_{x\in\omega_h} v(x)u(x),$$
$$\|v\|_h = \|v\|_{L_{2,h}} = (v,v)_h^{1/2}, \quad |[v\|_h = \left(h\sum_{x\in\omega_h^-} v^2(x)\right)^{1/2}.$$

The difference scheme (7) can be reduced to the form (3) by setting $A_h v = -v_{x\bar{x}}$ and $B_h v = [1+K\delta_h(x-\xi)]\,v$. For each $v\in H_h$ we have

$$\|v\|_{A_h}^2 = (A_h v, v)_h = |[v_x\|_h^2, \quad \|v\|_{B_h}^2 = \|v\|_h^2 + Kv^2(\xi),$$
$$\|v\|_{B_h^{-1}}^2 = (B_h^{-1}v, v)_h = h\sum_{x\in\omega\setminus\{\xi\}} v^2(x) + \frac{h^2}{K+h}v^2(\xi),$$
$$\|v\|_{\widetilde{W}_{2,h}^2}^2 = \|v_{x\overline{x}}\|_{B_h^{-1}}^2 + |[v_x\|_h^2 + \|v\|_{B_h}^2.$$

Note that the norms $\|A_h v\|_{B_h^{-1}}$ and $\|v\|_{\widetilde{W}_{2,h}^2}$ are equivalent [9].

We also define the discrete $\widetilde{W}_2^{2,1}$ norm

$$\|v\|_{\widetilde{W}_2^{2,1}(Q_{h\tau})}^2 = \tau\sum_{t\in\overline{\omega}_\tau}{}' \|v(\cdot,t)\|_{\widetilde{W}_{2,h}^2}^2 + \tau\sum_{t\in\omega_\tau^+}\|v_{\bar{t}}(\cdot,t)\|_{B_h}^2.$$

5. CONVERGENCE OF THE DIIFERENCE SCHEME

In this section we shall prove the convergence of the difference scheme (7) in the $\widetilde{W}_2^{2,1}$ norm. Let u be the solution of the boundary-value problem (4) and v the solution of the difference problem (7). The error $z = u - v$ satisfies the finite difference scheme

$$
\begin{aligned}
&[1 + K\,\delta_h(x-\xi)]\, z_{\bar{t}} - z_{x\bar{x}} = \psi_{\bar{t}} - \chi_{x\bar{x}} \quad \text{on } Q_{h\tau} \\
&z(0,t) = z(1,t) = 0, \ t \in \omega_\tau^+, \\
&z(x,0) = 0, \ x \in \bar{\omega}_h,
\end{aligned}
\tag{8}
$$

where

$$\chi = u - T_t^- u\,, \quad \psi = u - T_x^2 u.$$

In the sequel we shall assume that

$$u \in W_2^{3,3/2}(Q_1) \cap W_2^{3,3/2}(Q_2).$$

Using Lemma 2 we directly obtain the following a priori estimate for the solution of the difference scheme (8):

$$\|z\|_{\widetilde{W}_2^{2,1}(Q_{h\tau})} \le C \left(\tau \sum_{t\in\omega_\tau^+} \|\chi_{x\bar{x}}\|^2_{B_h^{-1}} + \tau \sum_{t\in\omega_\tau^+} \|\psi_{\bar{t}}\|^2_{B_h^{-1}} \right)^{1/2}. \tag{9}$$

Therefore, in order to estimate the rate of convergence of the difference scheme (7), it is sufficient to estimate the right-hand side of the inequality (9).

The following integral representation is true:

$$\chi(x,t) = \frac{1}{\tau} \int_{t-\tau}^{t} \int_{t'}^{t} \frac{\partial u(x,t'')}{\partial t}\, dt'' dt'.$$

Further,

$$
\begin{aligned}
\chi_{x\bar{x}}(x,t) = \ & \frac{1}{h^2\tau} \int_x^{x+h} \int_{t-\tau}^{t} \int_{t'}^{t} \frac{\partial^2 u(x',t'')}{\partial x \partial t}\, dt'' dt' dx' \\
& - \frac{1}{h^2\tau} \int_{x-h}^{x} \int_{t-\tau}^{t} \int_{t'}^{t} \frac{\partial^2 u(x',t'')}{\partial x \partial t}\, dt'' dt' dx'.
\end{aligned}
\tag{10}
$$

At the point $x \neq \xi$ we have

$$|\chi_{x\bar{x}}(x,t)| \le Ch^{-1/2}\left(\left\|\frac{\partial^2 u}{\partial x \partial t}\right\|_{L_2(g(x,t))} + \left\|\frac{\partial^2 u}{\partial x \partial t}\right\|_{L_2(g(x-h,t))}\right),$$

where $g(x,t) = (x, x+h) \times (t-\tau, t)$. Summing over the nodes of the meshes $\omega_h \setminus \{\xi\}$ and ω_τ^+ we get

$$\left(h\tau \sum_{x\in\omega_h\setminus\{\xi\}} \sum_{t\in\omega_\tau^+} |\chi_{x\bar{x}}(x,t)|^2\right)^{1/2} \le Ch(\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)}). \qquad (11)$$

At the point $x = \xi$, using representation (10), we have

$$\left|\frac{h^2}{K+h}\chi^2_{x\bar{x}}(\xi,t)\right| \le Ch|\chi^2_{x\bar{x}}(\xi,t)| \le C\left(\left\|\frac{\partial^2 u}{\partial x \partial t}\right\|^2_{L_2(g(\xi,t))} + \left\|\frac{\partial^2 u}{\partial x \partial t}\right\|^2_{L_2(g(\xi-h,t))}\right)$$

and

$$\left(\tau \sum_{t\in\omega_\tau^+} \frac{h^2}{K+h}\chi^2_{x\bar{x}}(\xi,t)\right)^{1/2} \le Ch(\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)}). \qquad (12)$$

From (11) and (12) we obtain

$$\left(\tau \sum_{t\in\omega_\tau^+} \|\chi_{x\bar{x}}\|^2_{B_h^{-1}}\right)^{1/2} \le Ch(\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)}). \qquad (13)$$

Further, at the point $x \neq \xi$ we have

$$\psi(x,t) = \frac{1}{h}\int_{x-h}^{x+h}\int_{x'}^{x}\left(1 - \frac{|x'-x|}{h}\right)\frac{\partial u(x'',t)}{\partial x}dx''dx'$$

and

$$\psi_{\bar{t}}(x,t) = \frac{1}{h\tau}\int_{x-h}^{x+h}\int_{x'}^{x}\int_{t-\tau}^{t}\left(1 - \frac{|x'-x|}{h}\right)\frac{\partial^2 u(x'',t')}{\partial x \partial t}\,dt'dx''dx'.$$

From this representation we get

$$|\psi_{\bar{t}}(x,t)| \le Ch^{-1/2}\left\|\frac{\partial^2 u}{\partial x \partial t}\right\|_{L_2(g_1(x,t))}, \quad g_1(x,t) = (x-h, x+h) \times (t-\tau, t).$$

Summing over the nodes of the meshes $\omega_h \setminus \{\xi\}$ and ω_τ^+ we get

$$\left(h\tau \sum_{x\in\omega_h\setminus\{\xi\}} \sum_{t\in\omega_\tau^+} |\psi_{\bar{t}}(x,t)|^2\right)^{1/2} \le Ch(\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)}). \qquad (14)$$

At the point $x=\xi$ we have

$$\psi(\xi,t)= \frac{1}{h}\int_{\xi-h}^{\xi}\int_{x'}^{\xi}\left(1+\frac{x'-\xi}{h}\right)\frac{\partial u(x'',t)}{\partial x}\,dx''dx' + \frac{1}{h}\int_{\xi}^{\xi+h}\int_{x'}^{\xi}\left(1-\frac{x'-\xi}{h}\right)\frac{\partial u(x'',t)}{\partial x}\,dx''dx'$$

and

$$\psi_{\bar t}(\xi,t) = \frac{1}{h\tau}\int_{\xi-h}^{\xi}\int_{x'}^{\xi}\int_{t-\tau}^{t}\left(1+\frac{x'-\xi}{h}\right)\frac{\partial^2 u(x'',t')}{\partial x\partial t}\,dt'dx''dx' + \frac{1}{h\tau}\int_{\xi}^{\xi+h}\int_{x'}^{\xi}\int_{t-\tau}^{t}\left(1-\frac{x'-\xi}{h}\right)\frac{\partial^2 u(x'',t')}{\partial x\partial t}\,dt'dx''dx'.$$

Implies that

$$\left|\frac{h^2}{K+h}\psi_{\bar t}^2(\xi,t)\right| \le Ch|\psi_{\bar t}^2(\xi,t)| \le C\left(\left\|\frac{\partial^2 u}{\partial x\partial t}\right\|_{L_2(g(\xi,t))}^2 + \left\|\frac{\partial^2 u}{\partial x\partial t}\right\|_{L_2(g(\xi-h,t))}^2\right)$$

and

$$\left(\tau\sum_{t\in\omega_\tau^+}\frac{h^2}{K+h}\psi_{\bar t}^2(\xi,t)\right)^{1/2} \le Ch(\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)}). \tag{15}$$

From (14) and (15) we get

$$\left(\tau\sum_{t\in\omega_\tau^+}\|\psi_{\bar t}\|_{B_h^{-1}}^2\right)^{1/2} \le Ch(\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)}). \tag{16}$$

Finally, from (9), (13) and (16) we get the following result.

Theorem 1. *The solution of the difference scheme (7) converges to the solution of the differential problem (4), and the following estimate is valid:*

$$\|u-v\|_{\widetilde W_2^{2,1}(Q_{h\tau})} \le Ch(\|u\|_{W_2^{3,3/2}(Q_1)} + \|u\|_{W_2^{3,3/2}(Q_2)}). \tag{17}$$

This estimate is compatible with the smoothness of the solution of the differential problem (4).

Remark. The convergence of the order $\mathcal{O}(h^2)$ for the solution $u\in W_2^{4,2}(Q_1)\cap W_2^{4,2}(Q_2)$, $u(\xi,\cdot)\in W_2^2(0,T)$ is proved in [7].

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