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STRONG DUALITY PRINCIPLE FOR FOUR-DIMENSIONAL OSSERMAN MANIFOLDS

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Abstract. In this paper we deal with pseudo-Riemannian Osserman curvature tensors. We define a concept of strong duality principle, and we prove that strong duality principle holds for four-dimensional Osserman curvature tensor.

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1. INTRODUCTION

Let us introduce the basic notation and terminology which are used throughout this work. Let R be an algebraic curvature tensor on a vector space $\mathcal{V}$ equipped with indefinite metric g of signature $(\nu, n-\nu)$. The norm of $X \in \mathcal{V}$ is $\varepsilon_X = g(X,X)$ and it determines various types of vectors. We say that $X \in \mathcal{V}$ is timelike (if $\varepsilon_X < 0$), spacelike ($\varepsilon_X > 0$), null ($\varepsilon_X = 0$), nonnull ($\varepsilon_X \neq 0$), or unit ($\varepsilon_X \in \{-1, 1\}$). The link between R and curvature operator $\mathcal{R}$ is given by $R(X,Y,Z,W) = g(\mathcal{R}(X,Y)Z,W)$. If $(E_1, E_2, ..., E_n)$ is an orthonormal basis of $\mathcal{V}$, we shall use short notation $\varepsilon_i = \varepsilon_{E_i}$

and $R_{ijkl} = R(E_i, E_j, E_k, E_l)$ for $1 \leq i, j, k, l \leq n$. For the initial definitions and deeper explanations of this topic, the reader can consult Gilkey's book [7].
The Polarized Jacobi operator $\mathcal{J}(X,Y) : \mathcal{V} \to \mathcal{V}$ is given by

$$\mathcal{J}(X,Y)(Z) = \frac{1}{2}\{\mathcal{R}(Z,X)Y + \mathcal{R}(Z,Y)X\}.$$

The Jacobi operator $\mathcal{J}_X : \mathcal{V} \to \mathcal{V}$ is defined by $\mathcal{J}_X = \mathcal{J}(X,X)$, which means that $\mathcal{J}_X(Z) = \mathcal{R}(Z,X)(X)$, for every $Z \in \mathcal{V}$. For nonnull $X \in \mathcal{V}$, $\mathcal{J}_X$ preserves nondegenerate hyperspace $\{X\}^\perp = \{Y \in \mathcal{V} : X \perp Y\}$, and we have reduced Jacobi operator $\tilde{\mathcal{J}}_X : \{X\}^\perp \to \{X\}^\perp$, given by $\tilde{\mathcal{J}}_X = \mathcal{J}_X|_{\{X\}^\perp}$.

We say that R is Osserman curvature tensor if the characteristic polynomial of $\mathcal{J}_X$ is constant on both unit pseudo-spheres, in particular on positive ($\varepsilon_X = 1$) and negative ($\varepsilon_X = -1$) one. R is Jordan Osserman if the Jordan normal form of $\mathcal{J}_X$ is constant on both unit pseudo-spheres. R is diagonalizable if the related Jacobi operator is diagonalizable. We say that R is k-stein if there exist constants C_j for $1 \leq j \leq k$, such that equations $\mathrm{Tr}((\mathcal{J}_X)^j) = (\varepsilon_X)^j C_j$ hold for any $X \in \mathcal{V}$. Special cases are Einstein ($k = 1$) and zwei-stein ($k = 2$), and it is well known that every Osserman R is k-stein for every k.

In our previous work [3] we defined the duality principle on the following way.

Definition 1. *Let R be an algebraic curvature tensor on vector space $\mathcal{V}$. We say that duality principle holds for R if for every $\lambda \in \mathbb{R}$, and for all mutually orthogonal units $X, Y \in \mathcal{V}$ holds*

$$\mathcal{J}_X(Y) = \varepsilon_X \lambda Y \Rightarrow \mathcal{J}_Y(X) = \varepsilon_Y \lambda X. \tag{1}$$

Our natural aim is to prove duality principle for Osserman curvature tensor in general or construction of eventual counterexamples. Unfortunately, this problem is too hard, and we are able to give just results under some specific conditions. The first restriction can be a low index ν of the metric g. In the case of $\nu = 0$ (Riemannian settings) Rakić proved that duality principle holds [10]. In the case of $\nu = 1$ (Lorentzian settings) R has constant sectional curvature [5] and duality principle holds. The second restriction can be a small number of eigenvalues of reduced Jacobi operator. In the case of diagonalizable R with two eigenvalues [2], the present author recently gave some results. The last restriction can be a low dimension n. The case of four-dimensional R is solved in our previous work [1, 3], but this proof was complicated and based on the discussion of the possibly Jordan normal forms. In this article we give the much easier proof. Moreover, in the four-dimensional case we prove that strong duality principle holds.

2. PRELIMINARIES

The following theorem contains the well known equations for Einstein and zwei-stein curvature tensor.

Theorem 1. *For a zwei-stein curvature tensor R in any orthonormal basis hold*

$$\sum_{1\leq p\leq n} \varepsilon_i\varepsilon_p R_{piip} = C_1, \tag{2}$$

$$\sum_{1\leq p\leq n} \varepsilon_p R_{pijp} = 0, \tag{3}$$

$$\sum_{1\leq p,q\leq n} \varepsilon_p\varepsilon_q (R_{piiq})^2 = C_2, \tag{4}$$

$$\sum_{1\leq p,q\leq n} \varepsilon_p\varepsilon_q R_{piiq}R_{pijq} = 0, \tag{5}$$

$$2\sum_{1\leq p,q\leq n} \varepsilon_p\varepsilon_q R_{piiq}R_{pjjq} - 2\varepsilon_i\varepsilon_j C_2 + \sum_{1\leq p,q\leq n} \varepsilon_p\varepsilon_q (R_{pijq}+R_{pjiq})^2 = 0, \tag{6}$$

for every $1 \leq i \neq j \leq n$.

We left the theorem without a proof, but reader can consult [1] and [3] for the details. Let us remark that equations (2) and (3) represent Einstein condition. Let us finish the section with lemma from [2], which we need later.

Lemma 1. *Every null $N \neq 0$ from a nondegenerate space $\mathcal{V}$ can be decomposed as $N = P + Q$, where $P, Q \in \mathcal{V}$ and $\varepsilon_P = -\varepsilon_Q = 1$.*

Proof. A nondegenerate vector space $\mathcal{V}$ can be decomposed as a direct sum $\mathcal{V} = \mathcal{V}^+ \oplus \mathcal{V}^-$ of complementary orthogonal subspaces, where $\mathcal{V}^+$ is a maximal spacelike subspace and $\mathcal{V}^-$ is a maximal timelike subspace. This is why we can decompose $N = S + T$, with $S \in \mathcal{V}^+, T \in \mathcal{V}^-$. $S \perp T$ gives $\varepsilon_S = -\varepsilon_T > 0$, and if we set

$$P = \left(\frac{\varepsilon_S + 1}{2\varepsilon_S} S + \frac{\varepsilon_S - 1}{2\varepsilon_S} T \right), \; Q = \left(\frac{\varepsilon_S - 1}{2\varepsilon_S} S + \frac{\varepsilon_S + 1}{2\varepsilon_S} T \right),$$

it is easy to check that $P + Q = S + T = N$, and $\varepsilon_P = -\varepsilon_Q = 1$. □

3. STRONG DUALITY PRINCIPLE

In Definition 1. we can demand nonnull X and Y, because such X and Y can be painlessly scaled to the units $X/\sqrt{|\varepsilon_X|}$ and $Y/\sqrt{|\varepsilon_Y|}$ with the same properties. Which are the optimal conditions for vectors X and Y from the formula (1)? For null X we feel some problems, because assumption become $\mathcal{J}_X(Y) = 0$, which is independent of λ, but λ is relevant in the case of nonnull Y. Moreover, case $\varepsilon_X = 0$ fall already in the dimension 4. For example, pseudo-Riemannian manifold $(\mathbb{R}^4, g)$ with the metric $g = x_2x_3dx_1 \otimes dx_1 - x_1x_4dx_2 \otimes dx_2 + dx_1 \otimes dx_2 + dx_2 \otimes dx_1 + dx_1 \otimes dx_3 + dx_3 \otimes dx_1 + dx_2 \otimes dx_4 + dx_4 \otimes dx_2$ is globally Jordan-Osserman [6]. For this manifold we have $\mathcal{J}_{\frac{\partial}{\partial x_3}}(\frac{\partial}{\partial x_1}) = 0$ and $\mathcal{J}_{\frac{\partial}{\partial x_1}}(\frac{\partial}{\partial x_3}) = -\frac{1}{2}\frac{\partial}{\partial x_4}$, and for $X = \frac{\partial}{\partial x_3}$ and $Y = \frac{\partial}{\partial x_1}$ formula (1) does not work at any point. However, there is no visible reason for other restrictions, which is good ground for the introduction of strong duality principle.

Definition 2. *Let R be an algebraic curvature tensor on vector space $\mathcal{V}$. We say that strong duality principle holds for R if for every $\lambda \in \mathbb{R}$, and for all $X, Y \in \mathcal{V}$ with $\varepsilon_X \neq 0$ holds $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y \Rightarrow \mathcal{J}_Y(X) = \varepsilon_Y \lambda X$.*

The rest of this section contains the results which follow the arguments from our previous work [1, 3].

Lemma 2. *If formula (1) holds for all mutually orthogonal X and Y with $\varepsilon_X \neq 0$, then it holds with only restriction $\varepsilon_X \neq 0$.*

Proof. Let us suppose $\mathcal{J}_X(Y) = \varepsilon_X\lambda Y$ for $\varepsilon_X \neq 0$ with $g(X,Y) \neq 0$. The decomposition $Y = \alpha X + Z$ with $g(Z,X) = 0$ gives $\alpha \neq 0$. Image of $\mathcal{J}_X$ is orthogonal to X, so

$$0 = g(\mathcal{J}_X(\alpha X + Z), X) = g(\varepsilon_X\lambda(\alpha X + Z), X) = g(\varepsilon_X\lambda\alpha X, X) = \varepsilon_X^2\alpha\lambda,$$

and because of $\varepsilon_X \neq 0$ and $\alpha \neq 0$, we have $\lambda = 0$. From $\mathcal{J}_X(X) = 0$ follows $\mathcal{J}_X(Z) = \mathcal{J}_X(\alpha X + Z) = \varepsilon_X\lambda(\alpha X + Z) = 0$, and because of $g(X,Z) = 0$ and $\varepsilon_X \neq 0$, the lemma assumption gives $\mathcal{J}_Z(X) = 0$. Thus arose

$$\mathcal{J}_Y(X) = \mathcal{J}_{\alpha X+Z}(X) = \alpha\mathcal{R}(X,Z)X + \mathcal{R}(X,Z)Z = -\alpha\mathcal{J}_X(Z) + \mathcal{J}_Z(X) = 0,$$

which proves $\mathcal{J}_Y(X) = 0 = \varepsilon_Y\lambda X$. $\square$

Theorem 2. *If R is a diagonalizable algebraic curvature tensor, then duality principle and strong duality principle are equivalent.*

Proof. Let us suppose $\mathcal{J}_X(Y) = \varepsilon_X\lambda Y$, $g(X,Y) = 0$, $\varepsilon_X \neq 0$, and $\varepsilon_Y = 0$. Diagonalizable R implies nondegenerate eigensubspace $\ker(\tilde{\mathcal{J}}_X - \varepsilon_X\lambda\mathrm{Id}) \ni Y$. The decomposition $Y = S + T$ from Lemma 1 with $S, T \in \ker(\tilde{\mathcal{J}}_X - \varepsilon_X\lambda\mathrm{Id})$ and $\varepsilon_S = -\varepsilon_T = 1$ gives $g(S,T) = g(S,X) = g(T,X) = 0$ with

$$\mathcal{J}_X(S) = \varepsilon_X\lambda S,\ \mathcal{J}_X(T) = \varepsilon_X\lambda T,\ Y = S + T.$$

Since $\varepsilon_{S+\theta T} = \varepsilon_S + \theta^2\varepsilon_T = (1-\theta^2)$, for $\theta^2 \neq 1$, vectors $S + \theta T$, S and T are nonnull, orthogonal on X, eigenvectors of operator $\mathcal{J}_X$ for eigenvalue $\varepsilon_X\lambda$, and therefore if duality principle holds we have

$$\mathcal{J}_{S+\theta T}(X) = \varepsilon_{S+\theta T}\lambda X,\ \mathcal{J}_S(X) = \varepsilon_S\lambda X,\ \mathcal{J}_T(X) = \varepsilon_T\lambda X.$$

After the substitution in the standard calculation

$$\mathcal{J}_{S+\theta T}(X) = \mathcal{J}_S(X) + \theta^2\mathcal{J}_T(X) + 2\theta\mathcal{J}(S,T)X, \tag{7}$$

we get $\varepsilon_{S+\theta T}\lambda X = \varepsilon_S\lambda X + \theta^2\varepsilon_T\lambda X + 2\theta\mathcal{J}(S,T)X$. For $\theta \notin \{-1,0,1\}$ it gives $\mathcal{J}(S,T)X = 0$ and (7) becomes $\mathcal{J}_{S+\theta T}(X) = \mathcal{J}_S(X) + \theta^2\mathcal{J}_T(X)$. Especially for $\theta = 1$

holds

$$\mathcal{J}_Y(X) = \mathcal{J}_{S+T}(X) = \mathcal{J}_S(X) + \mathcal{J}_T(X) = \varepsilon_S \lambda X + \varepsilon_T \lambda X = 0 = \varepsilon_Y \lambda X,$$

which proves (1) for $X \perp Y$ and $\varepsilon_X \neq 0$. Lemma 2 eliminates the condition $X \perp Y$, which completes the proof. $\square$

Theorem 2 is very useful for the extension of duality principle. For example, in Riemannian settings duality principle holds [10], and such curvature tensor is diagonalizable, so by Theorem 2 strong duality principle holds. It is worth noting that the condition of diagonalizability is pretty natural, because every Jordan Osserman curvature tensor of non-balanced signature is necessarily diagonalizable [8].

4. FOUR-DIMENSIONAL OSSERMAN

Let R be four-dimensional zwei-stein curvature tensor. Theorem 1 gives useful information. From (2) for $1 \leq i \leq 4$ we have four equations

$$\begin{aligned}
\varepsilon_1\varepsilon_2 R_{2112} + \varepsilon_1\varepsilon_3 R_{3113} + \varepsilon_1\varepsilon_4 R_{4114} &= C_1,\\
\varepsilon_2\varepsilon_1 R_{1221} + \varepsilon_2\varepsilon_3 R_{3223} + \varepsilon_2\varepsilon_4 R_{4224} &= C_1,\\
\varepsilon_3\varepsilon_1 R_{1331} + \varepsilon_3\varepsilon_2 R_{2332} + \varepsilon_3\varepsilon_4 R_{4334} &= C_1,\\
\varepsilon_4\varepsilon_1 R_{1441} + \varepsilon_4\varepsilon_2 R_{2442} + \varepsilon_4\varepsilon_3 R_{3443} &= C_1,
\end{aligned}$$

and after solve the system of equations

$$\begin{aligned}
\varepsilon_2\varepsilon_3 R_{3223} &= \varepsilon_1\varepsilon_4 R_{4114},\\
\varepsilon_2\varepsilon_4 R_{4224} &= \varepsilon_1\varepsilon_3 R_{3113},\\
\varepsilon_3\varepsilon_4 R_{4334} &= \varepsilon_1\varepsilon_2 R_{2112}.
\end{aligned} \tag{8}$$

From (3) for $1 \leq i \neq j \leq 4$ we get six equations

$$\begin{aligned}
R_{2443} &= -\varepsilon_1\varepsilon_4 R_{2113}, \ R_{1442} = -\varepsilon_3\varepsilon_4 R_{1332},\\
R_{2334} &= -\varepsilon_1\varepsilon_3 R_{2114}, \ R_{1443} = -\varepsilon_2\varepsilon_4 R_{1223},\\
R_{3224} &= -\varepsilon_1\varepsilon_2 R_{3114}, \ R_{1334} = -\varepsilon_2\varepsilon_3 R_{1224}.
\end{aligned} \tag{9}$$

From (5) for $1 \leq i \leq 4$ we have four equations

$$\begin{aligned}
R_{2112}^2 + R_{3113}^2 + R_{4114}^2 + 2\varepsilon_2\varepsilon_3 R_{2113}^2 + 2\varepsilon_2\varepsilon_4 R_{2114}^2 + 2\varepsilon_3\varepsilon_4 R_{3114}^2 = C_2,\\
R_{1221}^2 + R_{3223}^2 + R_{4224}^2 + 2\varepsilon_1\varepsilon_3 R_{1223}^2 + 2\varepsilon_1\varepsilon_4 R_{1224}^2 + 2\varepsilon_3\varepsilon_4 R_{3224}^2 = C_2,\\
R_{1331}^2 + R_{2332}^2 + R_{4334}^2 + 2\varepsilon_1\varepsilon_2 R_{1332}^2 + 2\varepsilon_1\varepsilon_4 R_{1334}^2 + 2\varepsilon_2\varepsilon_4 R_{2334}^2 = C_2,\\
R_{1441}^2 + R_{2442}^2 + R_{3443}^2 + 2\varepsilon_1\varepsilon_2 R_{1442}^2 + 2\varepsilon_1\varepsilon_3 R_{1443}^2 + 2\varepsilon_2\varepsilon_3 R_{2443}^2 = C_2,
\end{aligned}$$

and after use (8) and (9) hold

$$\begin{aligned}
\varepsilon_2\varepsilon_3 R_{2113}^2 + \varepsilon_2\varepsilon_4 R_{2114}^2 + \varepsilon_3\varepsilon_4 R_{3114}^2 &= \varepsilon_1\varepsilon_3 R_{1223}^2 + \varepsilon_1\varepsilon_4 R_{1224}^2 + \varepsilon_3\varepsilon_4 R_{3114}^2\\
&= \varepsilon_1\varepsilon_2 R_{1332}^2 + \varepsilon_1\varepsilon_4 R_{1224}^2 + \varepsilon_2\varepsilon_4 R_{2114}^2\\
&= \varepsilon_1\varepsilon_2 R_{1332}^2 + \varepsilon_1\varepsilon_3 R_{1223}^2 + \varepsilon_2\varepsilon_3 R_{2113}^2.
\end{aligned}$$

The solution of the previous system of equations is

$$\begin{aligned}
\varepsilon_2\varepsilon_3 R_{2113}^2 &= \varepsilon_1\varepsilon_4 R_{1224}^2,\\
\varepsilon_2\varepsilon_4 R_{2114}^2 &= \varepsilon_1\varepsilon_3 R_{1223}^2, \qquad (10)\\
\varepsilon_3\varepsilon_4 R_{3114}^2 &= \varepsilon_1\varepsilon_2 R_{1332}^2.
\end{aligned}$$

The derived equations are enough to prove duality principle.

Theorem 3. *For four-dimensional zwei-stein curvature tensor duality principle holds.*

Proof. Let $\mathcal{J}_X(Y) = \varepsilon_X \lambda Y$ holds for mutually orthogonal units X and Y from $\mathcal{V}$. Let us set $E_1 = X$, $E_2 = Y$ and extend them to an orthonormal basis (E_1, E_2, E_3, E_4) of $\mathcal{V}$. Initial $\mathcal{J}_{E_1}(E_2) = \varepsilon_1 \lambda E_2$, gives $R_{2112} = \varepsilon_1\varepsilon_2\lambda$, $R_{2113} = 0$, and $R_{2114} = 0$. By (10) it gives $R_{1224} = 0$ and $R_{1223} = 0$. Finally $\mathcal{J}_{E_2}(E_1) = \varepsilon_1 R_{1221} E_1 + \varepsilon_3 R_{1223} E_3 + \varepsilon_4 R_{1224} E_4 = \varepsilon_2 \lambda E_1$, and duality principle holds. □

Moreover, under the same initial conditions strong duality principle holds. In the Riemannian settings (signature (0,4) or (4,0)) we know that four-dimensional zwei-stein is necessarily Osserman [4, 9]. For Riemannian Osserman manifold duality principle holds [10], and (because Riemannian R is diagonalizable) by Theorem 2 strong duality principle holds. In the Lorentzian settings (signature (1,3) or (3,1)) R is necessarily of constant sectional curvature [5], and it is easy to check that strong duality principle holds. This is why we should check only the signature (2,2). Without loss of generality we can set $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4$.

If we apply (5) for $i = 2, j = 3$ and for $i = 3, j = 2$, we get

$$\begin{aligned} R_{1221}R_{1231} - R_{1224}R_{1234} - R_{3221}R_{3231} + R_{3224}R_{3234} \\ - R_{4221}R_{4231} + R_{4224}R_{4234} = 0, \\ R_{1331}R_{1321} - R_{1334}R_{1324} + R_{2331}R_{2321} - R_{2334}R_{2324} \\ - R_{4331}R_{4321} + R_{4334}R_{4324} = 0, \end{aligned}$$

and after some arrangement and use of (8) and (9)

$$\begin{aligned} R_{2112}R_{2113} - R_{1224}R_{1234} + R_{1223}R_{1332} + R_{3114}R_{2114} \\ - R_{1224}R_{1324} + R_{3113}R_{2113} = 0, \\ R_{3113}R_{2113} - R_{1224}R_{1324} - R_{1332}R_{1223} - R_{2114}R_{3114} \\ - R_{1224}R_{1234} + R_{2112}R_{2113} = 0. \end{aligned}$$

The sum and the difference of the previous equations are

$$R_{2112}R_{2113} + R_{3113}R_{2113} - R_{1224}R_{1234} - R_{1224}R_{1324} = 0, \qquad (11)$$

$$R_{1223}R_{1332} + R_{2114}R_{3114} = 0. \qquad (12)$$

From (6) for $i = 2, j = 3$ we get

$$\begin{aligned} &2(R_{1221}R_{1331} - 2R_{1224}R_{1334} + R_{4224}R_{4334}) \\ &+ 2(R_{2112}^2 + R_{3113}^2 + R_{4114}^2 - 2R_{2113}^2 - 2R_{2114}^2 + 2R_{3114}^2) + (R_{1231} + R_{1321})^2 \\ &+ R_{1232}^2 - R_{1323}^2 - (R_{1234} + R_{1324})^2 + R_{2321}^2 - R_{2323}^2 - R_{2324}^2 - R_{3231}^2 - R_{3232}^2 \\ &+ R_{3234}^2 - (R_{4231} + R_{4321})^2 - R_{4232}^2 + R_{4323}^2 + (R_{4234} + R_{4324})^2 = 0, \end{aligned}$$

which after use of (8), (9), and (10) becomes

$$\begin{aligned}
&2R_{2112}R_{3113} - 4R_{2113}^2 + 2R_{3113}R_{2112}\\
&+ 2R_{2112}^2 + 2R_{3113}^2 + 2R_{4114}^2 - 4R_{2113}^2 - 4R_{2114}^2 + 4R_{3114}^2 + 4R_{2113}^2\\
&+ R_{2114}^2 - R_{3114}^2 - (R_{1234} + R_{1324})^2 + R_{2114}^2 - R_{4114}^2 - R_{3114}^2 - R_{3114}^2 - R_{4114}^2\\
&+ R_{2114}^2 - (R_{1324} + R_{1234})^2 - R_{3114}^2 + R_{2114}^2 + 4R_{2113}^2 = 0.
\end{aligned}$$

The easy calculation gives

$$4R_{2112}R_{3113} + 2R_{2112}^2 + 2R_{3113}^2 - 2(R_{1234} + R_{1324})^2 = 0,$$

and finally

$$(R_{2112} + R_{3113})^2 = (R_{1234} + R_{1324})^2. \tag{13}$$

Theorem 4. *For four-dimensional zwei-stein curvature tensor strong duality principle holds.*

Proof. As we know that duality principle holds by Theorem 3, it is enough to prove (1) for $X \perp Y$ with $\varepsilon_X^2 = 1$ and $\varepsilon_Y = 0$ in the view of Lemma 2. Let us set $E_1 = X$, and decompose $Y = E_2 + E_3$ by Lemma 1, with mutually orthogonal units E_2 and E_3, which are orthogonal to X, such that $\varepsilon_{E_1} = \varepsilon_{E_2} = -\varepsilon_{E_3}$. Let E_4 be a vector which extend them to an orthonormal basis (E_1, E_2, E_3, E_4). Because of the signature $(2, 2)$, we have $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = -\varepsilon_4$, which enable use of the previous derived equations. Our aim is to prove the formula

$$\mathcal{J}_{E_1}(E_2 + E_3) = \varepsilon_1\lambda(E_2 + E_3) \Rightarrow \mathcal{J}_{E_2+E_3}E_1 = 0.$$

The assumption is

$$\begin{aligned}
&\varepsilon_1\lambda(E_2 + E_3) = \mathcal{J}_{E_1}(E_2 + E_3)\\
&= \varepsilon_2(R_{2112} + R_{3112})E_2 + \varepsilon_3(R_{2113} + R_{3113})E_3 + \varepsilon_4(R_{2114} + R_{3114})E_4,
\end{aligned}$$

hence

$$\varepsilon_2(R_{2112} + R_{3112}) = \varepsilon_1\lambda = \varepsilon_3(R_{2113} + R_{3113}),\ \varepsilon_4(R_{2114} + R_{3114}) = 0,$$

and finally

$$R_{2112} + R_{3113} + 2R_{2113} = 0, \tag{14}$$

$$R_{2114} + R_{3114} = 0. \tag{15}$$

The substitution (15) into (12) gives

$$R_{1223}R_{1332} = -R_{2114}R_{3114} = R_{2114}^2 = R_{1223}^2,$$

and therefore $R_{1223} = 0$ or $R_{1332} = R_{1223}$. However from (10) and (15)

$$R_{1223} = 0 \;\Rightarrow\; R_{2114} = 0 \;\Rightarrow\; R_{3114} = 0 \;\Rightarrow\; R_{1332} = 0$$

and

$$R_{1332} = R_{1223}, \tag{16}$$

holds anyway. By substitution (14) into (11) we get

$$(R_{1234} + R_{1324})R_{1224} = (R_{2112} + R_{3113})R_{2113} = -2R_{2113}^2 = -2R_{1224}^2.$$

Thus arise $R_{1224} = 0$ or $R_{1234} + R_{1324} = -2R_{1224}$. Similarly we use (10), (14), and (13) for

$$R_{1224} = 0 \;\Rightarrow\; R_{2113} = 0 \;\Rightarrow\; R_{2112} + R_{3113} = 0 \;\Rightarrow\; R_{1234} + R_{1324} = 0,$$

and undoubtedly

$$R_{1234} + R_{1324} = -2R_{1224} = -(R_{1224} + R_{1334}). \tag{17}$$

Everything is ready for the final computation of $\mathcal{J}_{E_2+E_3}E_1$.

$$\begin{aligned}
\mathcal{J}_{E_2+E_3}E_1 &= \sum_p \varepsilon_p(R_{122p} + R_{123p} + R_{132p} + R_{133p})E_p \\
&= \varepsilon_1(R_{2112} + R_{3113} + 2R_{2113})E_1 + \varepsilon_2(R_{1332} - R_{1223})E_2 \\
&+ \varepsilon_3(R_{1223} - R_{1332})E_3 + \varepsilon_4(R_{1224} + R_{1334} + R_{1234} + R_{1324})E_4
\end{aligned}$$

The equations (14), (16), and (17) finally gives $\mathcal{J}_{E_2+E_3}E_1 = 0$, which completes the proof. □

Of course, Osserman curvature tensor is zwei-stein and strong duality principle holds in four-dimensional case.

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