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A TAXİCAB VERSİON OF THE ERDŐS-MORDELL THEOREM

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Abstract. In this work, we give taxicab version of Erdős-Mordell theorem.

1. INTRODUCTION

The taxicab plane geometry has been introduced by Menger and developed by Krause (see [8, 9]). Taxicab plane R_T^2 is almost the same as the Euclidean analytical plane R^2. The points are the same, the lines are the same and the angles are measured in the same way. However, the distance function is different. Taxicab distance between the points P and Q is the length of a shortest path from P to Q composed of the line segments parallel to the coordinate axes. That is, if $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ than the taxicab distance from P to Q is $d_T(P, Q) = |x_1 - x_2| + |y_1 - y_2|$.

The taxicab plane geometry is a non-Euclidean, since it fails to satisfy the side-angle-side axiom but satisfies all the remaining twelve axioms of the Euclidean plane geometry [8]. Since the taxicab plane geometry has a different function it seems interesting to study the taxicab analogues of the topics that include the concept of distance in the Euclidean geometry. A few of such topics have been studied by some authors (see [1–6; 8–15]). Here in this study, we give taxicab version of Erdős-Mordell theorem.

2. A TAXİCAB VERSİON OF THE ERDŐS-MORDELL THEOREM

Theorem 1. **(Pisagor Theorem)** *Let a denote the length of the hypotenuse, b and c denote the lengths of the legs of a triangle ABC with right angle A in the taxicab plane* (see Figure 1a,1b). *Then,*

$$a = \begin{cases} b + c - 2\gamma \ ; & \text{if there exists only one base line through the vertex } A, \\ b + c \ ; & \text{if there exists two base lines through the vertex } A, \end{cases}$$

where $\gamma = d_T(A, H)$ and $H =$ The point of orthogonal projection of B or C to the base segment through A (see [6]).

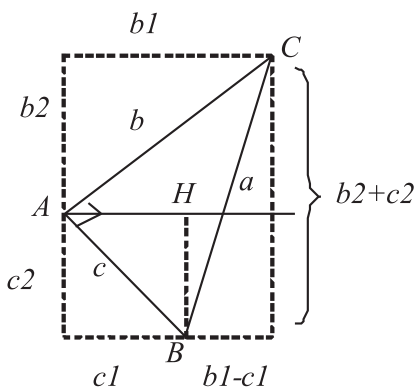

Figure 1a.

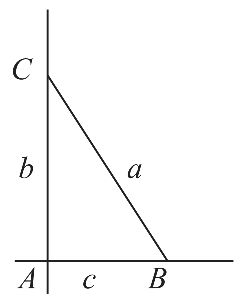

Figure 1b.

It is known that for any triangle ABC in the Euclidean plane, if P is a point inside a triangle ABC, R_a, R_b, R_c are its distances from the vertices A, B, C and r_a, r_b, r_c are its distances from the lines of the sides $a = d(B, C)$, $b = d(C, A)$, $c = d(A, B)$, then

$$R_a + R_b + R_c \geq 2(r_a + r_b + r_c)$$

which is known as **Erdős-Mordell Theorem** (see [7]), (see Figure 2).

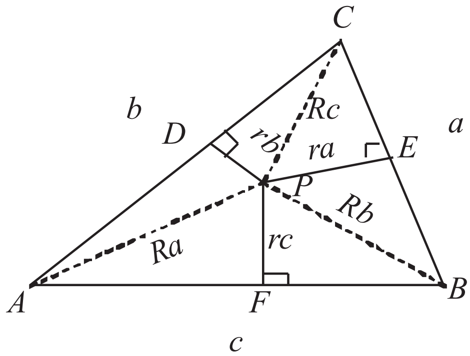

Figure 2.

We give to a taxicab version of this theorem in the following result.

Theorem 2. *Given a point P inside a triangle ABC, let us denote by R_a, R_b, R_c its distance from the vertices A, B, C and by r_a, r_b, r_c its distance from the lines of the sides $a = d_T(B,C)$, $b = d_T(C,A)$, $c = d_T(A,B)$. Then,*

$$R_a + R_b + R_c > 2(r_a + r_b + r_c) \ .$$

Proof: Let's apply to the Pisagor theorem in the triangles CEP, CDP, ADP, AFP, BFP and BEP, respectively (see Theorem 1). If we apply Pisagor theorem in the triangle CEP, then

$$R_c = x + r_a - 2x_1 \ .$$

Hence $R_c > r_a$. If $x_1 = x_2$, then $R_c = r_a$ (see Figure 3).

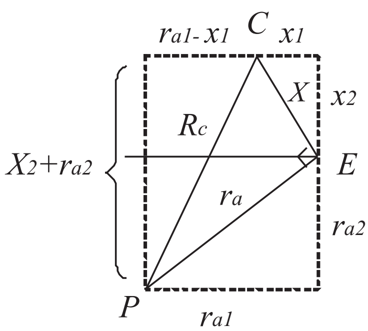

Figure 3.

If we apply Pisagor theorem in the triangle CDP, then

$$R_c = y + r_b - 2r_{b1} \ .$$

Hence $R_c > r_b$. If $y = 2r_{b1}$, then $R_c = r_b$. This is impossible. So, $R_c \neq r_b$ (see Figure 4).

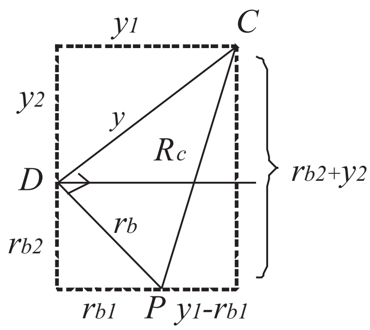

Figure 4.

If we apply Pisagor theorem in the triangle ADP, then

$$R_a = z + r_b - 2r_{b2} \ .$$

Hence $R_a > r_b$. If $z = 2r_{b2}$, then R_a=r_b. This is impossible. So, $R_a \neq r_b$ (see Figure 5).

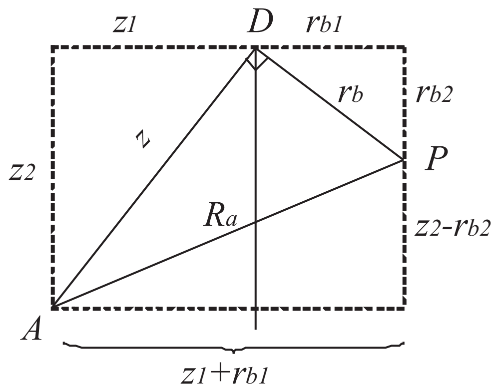

Figure 5.

If we apply Pisagor theorem in the triangle AFP, then

$$R_a = m + r_c \ .$$

Hence $R_a > r_c$. If $m = 0$, then $R_a = r_c$. This is impossible. Hence, $R_a \neq r_c$ (see Figure 6).

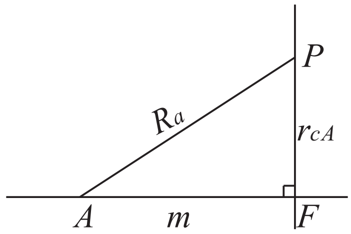

Figure 6.

If there exists one base line of the triangle AFP, then

$$R_a = m + r_c - 2r_{c1} \ .$$

Hence $R_a > r_c$. If $m = 2r_{c1}$, then $R_a = r_c$. This is impossible. $R_a \neq r_c$ (see Figure 7).

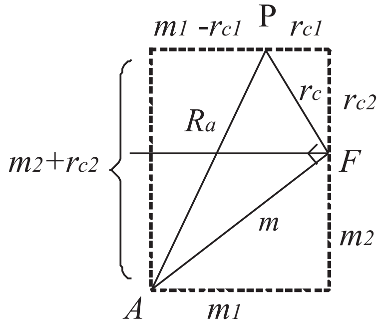

Figure 7.

If we apply Pisagor theorem in the triangle BFP, then

$$R_b = n + r_c \ .$$

Hence $R_b > r_c$. If $n = 0$, then $R_b = r_c$. This is impossible. So, $R_b \neq r_c$ (see Figure 8).

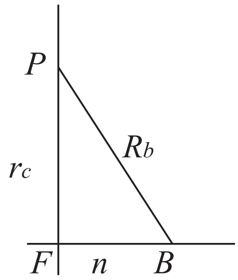

Figure 8.

If there exists one base line of the triangle BFP, then

$$R_b = n + r_c - 2n_2\ .$$

Hence $R_b > r_c$. If $n = 2n_2$, then $R_b = r_c$ (see Figure 9).

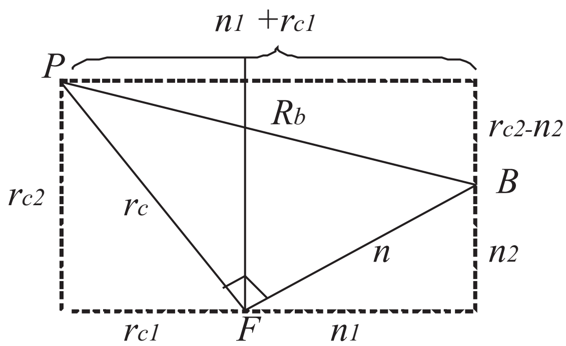

Figure 9.

If we apply Pisagor theorem in the triangle BEP, then

$$R_b = k + r_a - 2r_{a2}\ .$$

Hence $R_b > r_a$. If $k = 2r_{a2}$, then R_b=r_a. This is impossible. Hence $R_b \neq r_a$ (see Figure 10).

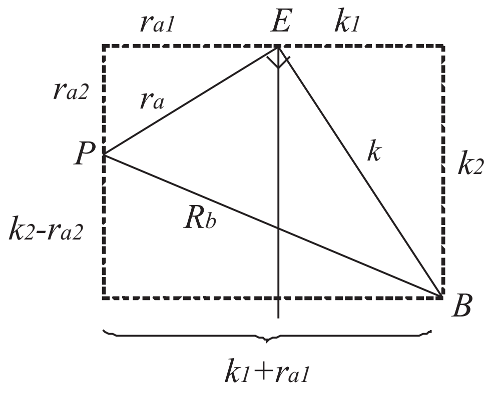

Figure 10.

If we investigate all cases, we find the following result

$$R_a + R_b + R_c > 2\left(r_a + r_b + r_c\right)\ .$$

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