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A GENERALIZED INTEGRAL INEQUALITY FOR TWICE DIFFERENTIABLE MAPPINGS

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Abstract. In this paper, a general form of integral inequality of Ostrowski type for twice differentiable mappings whose second derivatives are bounded and first deriviatives are absolutely continuous is established. The generalized integral inequality points some better estimates than some already presented bounds. The inequality is then applied to numerical integration and special means.

1. INTRODUCTION

In recent years, a number of authors have written about generalizations of Ostrowski's inequality. For example, this topic is considered in [4, 6, 8, 9].

In [2], P. Cerone, S. S. Dragomir and J. Roumeliotis, established an integral inequality of Ostrowski type for mappings with bounded second derivatives. A similar inequality has been established by S. S. Dragomir and N. S. Barnett in [3]. In [5], S.

S. Dragomir and A. Sofo, pointed out an integral inequality of Ostrowski type similar in a sense to that of [2] or [3]. However, this inequality contains a minor mistake. The corrected version of the inequality is given in the form of the following theorem [7]:

Theorem CDR. *Let* $g : [a,b] \longrightarrow \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$ *and assume that the second derivative* $g'' \in L_\infty[a,b]$. *Then, we have the inequality*

$$\left| \int_a^b g(t)dt - \frac{1}{2}\left[(b-a)\left(g(x) + \frac{g(a)+g(b)}{2}\right) - (b-a)\left(x - \frac{a+b}{2}\right)g'(x)\right] \right|$$
$$\leq \|g''\|_\infty \left(\frac{1}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48} \right), \tag{1}$$

for all $x \in [a,b]$.

The main aim of this paper is to point out a generalization of (1). It turns out that this generalization can give some better results than the estimations based on (1).

2. MAIN RESULTS

We establish here a general form of integral inequality (1) and apply it to numerical integration and special means. The inequality is given in the form of the following theorem:

Theorem 1. *Let* $g : [a,b] \longrightarrow \mathbb{R}$ *be a mapping whose first derivative is absolutely continuous on* $[a,b]$ *and assume that the second derivative* $g'' \in L_\infty[a,b]$. *Then, we have the inequality*

$$\left| \frac{1}{(b-a)} \int_a^b g(t)dt - \frac{1}{2}\left[(1-h)\,g(x) + (1+h)\left(\frac{g(a)+g(b)}{2}\right)\right.\right.$$
$$\left.\left. - (1-h)\,(x - \frac{a+b}{2})g'(x) - h\frac{b-a}{4}\left(g'(b) - g'(a)\right)\right]\right|$$

$$\leq \quad \|g''\|_\infty \frac{1}{(b-a)} \left[\frac{1}{3} \left| x - \frac{a+b}{2} \right|^3 + \frac{(b-a)^3}{48} \Psi(h) \right], \tag{2}$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2} \right]$,

$$where \quad \Psi(h) = (1-h)\left[2(1-h)^2 - 1\right] + 2h, \; h \in [0,1].$$

Proof. Let us start with the following integral identity,

$$f(x) = \frac{1}{(1-h)} \left[\frac{1}{(b-a)} \int_a^b f(t)dt - \frac{h}{2}(f(a) + f(b)) \right] + \frac{1}{(b-a)(1-h)} \int_a^b p(x,t) f'(t)dt.$$

This implies

$$(1-h) f(x) = \frac{1}{(b-a)} \int_a^b f(t)dt - \frac{h}{2}(f(a) + f(b)) + \frac{1}{b-a} \int_a^b p(x,t) f'(t)dt, \tag{3}$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2} \right]$, $h \in [0,1]$ provided f is absolutely continuous on $[a,b]$ and the kernel $p : [a,b]^2 \longrightarrow \mathbb{R}$ is given by:

$$p(x,t) = \begin{cases} t - \left(a + h\frac{b-a}{2}\right), & \text{if } t \in [a,x] \\ t - \left(b - h\frac{b-a}{2}\right), & \text{if } t \in (x,b]. \end{cases}$$

A simple proof using the integration by parts can be found in [4]. We choose in (3),

$$f(x) = \left(x - \frac{a+b}{2}\right) g'(x),$$

to get

$$(1-h)\left(x - \frac{a+b}{2}\right) g'(x)$$

$$= \frac{1}{(b-a)}\int_a^b \left(t-\frac{a+b}{2}\right)g'(t)dt - \frac{h}{4}(b-a)\left(g'(b)-g'(a)\right)$$
$$+\frac{1}{b-a}\int_a^b p(x,t)\left[g'(t)+\left(t-\frac{a+b}{2}\right)g''(t)\right]dt. \quad (4)$$

Integrating by parts, we have

$$\int_a^b \left(t-\frac{a+b}{2}\right)g'(t)dt = (b-a)\left(\frac{g(a)+g(b)}{2}\right) - \int_a^b g(t)dt. \quad (5)$$

Also,

$$\int_a^b p(x,t)g'(t)dt = (1-h)(b-a)g(x) + h\frac{b-a}{2}\left(g(a)+g(b)\right) - \int_a^b g(t)dt. \quad (6)$$

Using (5) and (6) in (4), we get:

$$\begin{aligned}
&(1-h)\left(x-\frac{a+b}{2}\right)g'(x) \\
= \ &\frac{(1+h)}{2}\left(g(a)+g(b)\right) - h\frac{b-a}{4}\left(g'(b)-g'(a)\right) \\
&+(1-h)g(x) - \frac{2}{b-a}\int_a^b g(t)dt \\
&+\frac{1}{b-a}\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt
\end{aligned}$$

or

$$\begin{aligned}
\frac{1}{b-a}\int_a^b g(t)dt = \ &\frac{(1+h)}{4}\left(g(a)+g(b)\right) - h\frac{b-a}{8}\left(g'(b)-g'(a)\right) \\
&+\frac{(1-h)}{2}g(x) - \frac{(1-h)}{2}\left(x-\frac{a+b}{2}\right)g'(x) \\
&+\frac{1}{2(b-a)}\int_a^b p(x,t)\left(t-\frac{a+b}{2}\right)g''(t)dt,
\end{aligned}$$

for all $x \in \left[a + h\,\frac{b-a}{2}, b - h\,\frac{b-a}{2}\right]$. This implies

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b g(t)dt\right. \\
&-\frac{1}{2}\left[(1-h)\,g(x) + (1+h)\left(\frac{g(a)+g(b)}{2}\right)\right. \\
&\left.\left.-(1-h)\left(x - \frac{a+b}{2}\right)g'(x) - h\frac{b-a}{4}\left(g'(b) - g'(a)\right)\right]\right| \\
= &\left|\frac{1}{2(b-a)}\int_a^b p(x,t)\left(t - \frac{a+b}{2}\right)g''(t)dt\right| \\
\leq &\frac{1}{2(b-a)}\int_a^b |p(x,t)|\left|t - \frac{a+b}{2}\right| |g''(t)|\,dt. \qquad (7)
\end{aligned}$$

Obviously, we have

$$\begin{aligned}
&\int_a^b |p(x,t)|\left|t - \frac{a+b}{2}\right| |g''(t)|\,dt \\
\leq\ &\|g''\|_\infty \int_a^b |p(x,t)|\left|t - \frac{a+b}{2}\right| dt, \qquad (8)
\end{aligned}$$

where

$$\|g''\|_\infty = \sup_{t\in(a,b)} |g''(t)| < \infty.$$

Also,

$$I = \int_a^b |p(x,t)|\left|t - \frac{a+b}{2}\right| dt$$

or

$$I = \int_a^b \left|t - \left(a + h\frac{b-a}{2}\right)\right|\left|t - \frac{a+b}{2}\right| dt + \int_x^b \left|t - \left(b - h\frac{b-a}{2}\right)\right|\left|t - \frac{a+b}{2}\right| dt. \quad (9)$$

We have two cases: **1**$^\circ$ Case when $x \in \left[a + h\,\frac{b-a}{2}, \frac{a+b}{2}\right]$, then we obtain:

$$
\begin{aligned}
I \;=\; & \int_a^{a+h\frac{b-a}{2}} \left(a+h\frac{b-a}{2}-t\right)\left(\frac{a+b}{2}-t\right)dt \\
& + \int_{a+h\frac{b-a}{2}}^{x} \left[t-\left(a+h\frac{b-a}{2}\right)\right]\left(\frac{a+b}{2}-t\right)dt \\
& + \int_x^{\frac{a+b}{2}} \left(b-h\frac{b-a}{2}-t\right)\left(\frac{a+b}{2}-t\right)dt \\
& + \int_{\frac{a+b}{2}}^{b-h\frac{b-a}{2}} \left(b-h\frac{b-a}{2}-t\right)\left(t-\frac{a+b}{2}\right)dt \\
& + \int_{b-h\frac{b-a}{2}}^{b} \left[t-\left(b-h\frac{b-a}{2}\right)\right]\left(t-\frac{a+b}{2}\right)dt.
\end{aligned}
$$

After some simple calculations, we obtain

$$
I = \frac{2}{3}\left(\frac{a+b}{2}-x\right)^3 + \frac{(b-a)^3}{24}\left[3h+2\,(1-h)^3-1\right], \tag{10}
$$

for all $x \in \left[a+h\,\frac{b-a}{2}, \frac{a+b}{2}\right]$.

2° Case when $x \in \left[\frac{a+b}{2}, b-h\frac{b-a}{2}\right]$, we take

$$
\begin{aligned}
I \;=\; & \int_a^{a+h\frac{b-a}{2}} \left(a+h\frac{b-a}{2}-t\right)\left(\frac{a+b}{2}-t\right)dt \\
& + \int_{a+h\frac{b-a}{2}}^{\frac{a+b}{2}} \left[t-\left(a+h\frac{b-a}{2}\right)\right]\left(\frac{a+b}{2}-t\right)dt \\
& + \int_{\frac{a+b}{2}}^{x} \left[t-\left(a+h\frac{b-a}{2}\right)\right]\left(t-\frac{a+b}{2}\right)dt
\end{aligned}
$$

$$+ \int_{x}^{b-h\frac{b-a}{2}} \left(b - h\frac{b-a}{2} - t\right)\left(t - \frac{a+b}{2}\right) dt$$

$$+ \int_{b-h\frac{b-a}{2}}^{b} \left[t - \left(b - h\frac{b-a}{2}\right)\right]\left(t - \frac{a+b}{2}\right) dt.$$

$$I = \frac{2}{3}\left(x - \frac{a+b}{2}\right)^3 + \frac{(b-a)^3}{24}\left[3h + 2(1-h)^3 - 1\right], \tag{11}$$

for all $x \in \left[\frac{a+b}{2}, b - h\frac{b-a}{2}\right]$.

Using (8), (9), (10) and (11) in (7), we obtain

$$\begin{aligned} &\Bigg|\frac{1}{b-a}\int_a^b g(t)dt - \frac{1}{2}\left[(1-h)\,g(x) + (1+h)\left(\frac{g(a)+g(b)}{2}\right)\right. \\ &\left.\left. - (1-h)\left(x - \frac{a+b}{2}\right)g'(x) - h\frac{b-a}{4}\left(g'(b) - g'(a)\right)\right]\right| \\ \leq\ & \frac{\|g''\|_\infty}{2(b-a)}\left[\frac{2}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{24}\left[3h + 2(1-h)^3 - 1\right]\right] \\ =\ & \frac{\|g''\|_\infty}{(b-a)}\left[\frac{1}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48}\Psi(h)\right] \end{aligned}$$

or equivalently,

$$\begin{aligned} &\Bigg|\int_a^b g(t)dt - \frac{(b-a)}{2}\left[(1-h)\,g(x) + (1+h)\left(\frac{g(a)+g(b)}{2}\right)\right. \\ &\left.\left. - (1-h)\left(x - \frac{a+b}{2}\right)g'(x) - h\frac{b-a}{4}\left(g'(b) - g'(a)\right)\right]\right| \\ \leq\ & \|g''\|_\infty\left[\frac{1}{3}\left|x - \frac{a+b}{2}\right|^3 + \frac{(b-a)^3}{48}\Psi(h)\right], \end{aligned} \tag{12}$$

for all $x \in \left[a + h\frac{b-a}{2}, b - h\frac{b-a}{2}\right]$,

$$\begin{aligned} \text{where}\quad \Psi(h) &= 3h + 2(1-h)^3 - 1 \\ &= (1-h)\left[2(1-h)^2 - 1\right] + 2h,\ \ h \in [0,1]. \end{aligned}$$

Special Case 1. Choosing $h = 0$ in (12) gives us the inequality (1).

Remark 1. In (2), if we investigate the estimates for the end points $x = a$, $x = b$ and the midpoint $x = \frac{a+b}{2}$, we find that the midpoint gives us the best estimate, so that from inequality (12), we have

$$\begin{aligned} &\left| \int_a^b g(t)dt - \frac{(b-a)}{2} \left[(1-h)\, g\left(\frac{a+b}{2}\right) \right.\right. \\ &\left.\left. + (1+h)\left(\frac{g(a)+g(b)}{2}\right) - h\frac{b-a}{4}\left(g'(b)-g'(a)\right) \right] \right| \\ \le\ & \|g''\|_\infty \frac{(b-a)^3}{48} \Psi(h). \qquad (13) \end{aligned}$$

Remark 2. If we investigate $\Psi(h)$ for different values of $h \in [0,1]$, we find that

$$\Psi(h) < 1, \quad \text{for} \quad 0 < h < \frac{6}{10}, \qquad (14)$$

and it is minimum for $h = \frac{3}{10}$.

Thus, for the specified range of h as mentioned in (14), our result gives us better estimate than as given in [3] i.e.

$$\frac{\Psi(h)}{48} < \frac{1}{48}, \quad \text{for} \quad 0 < h < \frac{6}{10}.$$

Special Case 2. Further, putting $h = \frac{3}{10}$ in the inequality (13) gives us the best estimate:

$$\begin{aligned} &\left| \int_a^b g(t)dt - \frac{(b-a)}{20}\left[7g(\frac{a+b}{2}) + \frac{13}{2}\left(g(a)+g(b)\right) - \frac{3}{4}(b-a)\left(g'(b)-g'(a)\right)\right]\right| \\ \le\ & \frac{293}{24000}(b-a)^3 \|g''\|_\infty, \qquad (15) \end{aligned}$$

which has a better estimate than the three-point quadrature inequalities presented in [1] and [8] for $\|.\|_\infty$-norm.

Special Case 3. If we choose $h = 1$ in the inequality (13), we get a perturbed trapezoid inequality as follows:

$$\left| \int_a^b g(t)dt - (b-a)\left[\left(\frac{g(a)+g(b)}{2} \right) - \frac{b-a}{8}\left(g'(b)-g'(a)\right) \right] \right|$$
$$\leq \|g''\|_\infty \frac{(b-a)^3}{24}, \tag{16}$$

which has a better estimate than the perturbed trapezoid inequalities presented in [1] and [8] for $\|.\|_\infty$-norm.

3. APPLICATIONS TO COMPOSITE QUADRATURE RULES

We may use the inequality (2) to get the estimates of composite quadrature rules with smaller error than that which may be obtained by the classical results.

Theorem 2. *Let* $I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ *be a partition of the interval* $[a,b]$, $h_i = x_{i+1} - x_i$, $\delta \in [0,1]$, $x_i + \delta\frac{h_i}{2} \leq \xi_i \leq x_{i+1} - \delta\frac{h_i}{2}$, $\quad i = 0, \cdots, n-1$, *then*

$$\int_a^b g(t)\ dt = S(g, g', I_n, \xi, \delta) + R(g, g', I_n, \xi, \delta),$$

where

$$S(g, g', I_n, \xi, \delta) = \frac{1}{2}\sum_{i=0}^{n-1}\left[(1-\delta)\, g(\xi_i) + (1+\delta)\left(\frac{g(x_i)+g(x_{i+1})}{2}\right) \right.$$
$$\left. - (1-\delta)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) g'(\xi_i) - \frac{\delta}{4} h_i \left(g'(x_{i+1}) - g'(x_i)\right) \right] h_i \tag{17}$$

and

$$|R(g, g', I_n, \xi, \delta)|$$
$$\leq \|g''\|_\infty \left[\sum_{i=0}^{n-1} \left(\frac{1}{3}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|^3 + \frac{h_i^3}{48}\Psi(\delta) \right) \right]$$
$$= \|g''\|_\infty \left[\frac{1}{3}\sum_{i=0}^{n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|^3 + \frac{\Psi(\delta)}{48}\sum_{i=0}^{n-1} h_i^3 \right], \tag{18}$$

$$where \;\; \Psi(\delta) = (1-\delta)\left[2(1-\delta)^2 - 1\right] + 2\delta, \;\; \delta \in [0,1].$$

Proof. Applying inequality (12) on $\xi_i \in \left[x_i + \delta\frac{h_i}{2}, x_{i+1} - \delta\frac{h_i}{2}\right]$ and summing over i from 0 to $n-1$ and using triangular inequality, we get (18).

Special Case 4. Putting $\delta = 0$ in (17) and (18) gives us as a special case [5], the corrected version of estimates of composite quadrature rules.

Special Case 5. For $\xi_i = \dfrac{x_i + x_{i+1}}{2}$ in (17) and (18), $(i = 0, \cdots n-1)$, we have the following quadrature rule:

$$\begin{aligned} S(g, g', I_n, \delta) &= \frac{1}{2}\sum_{i=0}^{n-1}\left[(1-\delta)\, g\left(\frac{x_i + x_{i+1}}{2}\right) + (1+\delta)\left(\frac{g(x_i) + g(x_{i+1})}{2}\right)\right. \\ &\quad \left. - \frac{\delta}{4} h_i \left(g'(x_{i+1}) - g'(x_i)\right)\right] h_i \end{aligned} \tag{19}$$

and

$$|R(g, g', I_n, \delta)| \leq \frac{\Psi(\delta)}{48} \|g''\|_\infty \sum_{i=0}^{n-1} h_i^3, \;\; \delta \in [0,1]. \tag{20}$$

Special Case 6. If we choose $\delta = 0$ in (19) and (20), $(i = 0, \cdots n-1)$, then

$$\overline{S}(g, I_n) = \frac{1}{2}\sum_{i=0}^{n-1}\left[g\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{g(x_i) + g(x_{i+1})}{2}\right] h_i \tag{21}$$

and

$$\overline{R}(g, I_n) \leq \frac{\|g''\|_\infty}{48} \sum_{i=0}^{n-1} h_i^3. \tag{22}$$

It may be noted that $\overline{S}(g, I_n)$ is an arithmetic mean of the midpoint and trapezoidal quadrature rules.

Special Case 7. If we choose $\delta = \dfrac{3}{10}$ in (19) and (20), $(i = 0, \cdots n-1)$, then

$$\begin{aligned} S(g, g', I_n) &= \frac{1}{20}\sum_{i=0}^{n-1}\left[7g\left(\frac{x_i + x_{i+1}}{2}\right) + \frac{13}{2} g(x_i) + g(x_{i+1})\right] h_i \\ &\quad - \frac{3}{80}\sum_{i=0}^{n-1}\left[g'(x_{i+1}) - g'(x_i)\right] h_i \end{aligned} \tag{23}$$

and

$$R(g,g',I_n) \leq \frac{293}{24000} \|g''\|_\infty \sum_{i=0}^{n-1} h_i^3, \tag{24}$$

which is a perturbed composite three point quadrature inequality of Simpson type.

Special Case 8. If we choose $\delta = 1$ in (19) and (20), $(i = 0, \cdots n-1)$, then

$$\begin{aligned} S(g,g',I_n) &= \frac{1}{2}\sum_{i=0}^{n-1} (g(x_i) + g(x_{i+1}))\, h_i \\ &\quad -\frac{1}{8}\sum_{i=0}^{n-1} [g'(x_{i+1}) - g'(x_i)]\, h_i \end{aligned} \tag{25}$$

and

$$R(g,g',I_n) \leq \frac{1}{24} \|g''\|_\infty \sum_{i=0}^{n-1} h_i^3, \tag{26}$$

which is a perturbed composite trapezoid inequality.

4. APPLICATIONS TO SPECIAL MEANS

The inequality (2) may be written as

$$\begin{aligned} &\Bigg| \frac{(1-h)}{2} g(x) + \frac{(1+h)}{2} \left(\frac{g(a)+g(b)}{2} \right) \\ &-\frac{(1-h)}{2} (x - A(a,b))\, g'(x) - h\frac{b-a}{8} (g'(b) - g'(a)) - \frac{1}{b-a} \int_a^b g(t)dt \Bigg| \\ \leq\ & \frac{\|g''\|_\infty}{(b-a)} \left[\frac{1}{3} |x - A(a,b)|^3 + \frac{(b-a)^3}{48} \Psi(h) \right], \end{aligned} \tag{27}$$

where $\Psi(h) = (1-h)\left[2(1-h)^2 - 1\right] + 2h,\ h \in [0,1]$.

Special Case 9. Choosing $h = 0$ gives us a special case, the corrected version of the inequality given in [5] as follows:

$$\begin{aligned} &\left| \frac{1}{2} \left[g(x) + \left(\frac{g(a)+g(b)}{2} \right) - (x - A(a,b))\, g'(x) \right] - \frac{1}{b-a} \int_a^b g(t)dt \right| \\ \leq\ & \|g''\|_\infty \left[\frac{1}{3(b-a)} |x - A(a,b)|^3 + \frac{(b-a)^2}{48} \right]. \end{aligned} \tag{28}$$

We may now apply (27), to deduce some inequalities for special means using some particular mappings. The results of the special means are as follows:

1. Consider $g(t) = \ln t$, $g : (0, \infty) \to \mathbb{R}$, then

$$\begin{aligned} \frac{1}{b-a}\int_a^b g(t)dt &= \ln I(a,b), \\ \frac{g(a)+g(b)}{2} &= \ln G(a,b), \\ g'(b) - g'(a) &= -\frac{b-a}{G^2(a,b)} \end{aligned}$$

and

$$\|g''\|_\infty = \sup_{t\in(a,b)} \|g''(t)\| = \frac{1}{a^2}.$$

From (27), we have:

$$\begin{aligned} &\left|(1-h)\ln x + (1+h)\ln G(a,b) - (1-h)\left(1 - \frac{A(a,b)}{x}\right)\right. \\ &\left. +h\frac{(b-a)^2}{4}\frac{1}{G^2(a,b)} - 2\ln I(a,b)\right| \\ \leq\ & \frac{2}{a^2}\left(\frac{1}{3(b-a)}\left|x - A(a,b)\right|^3 + \frac{(b-a)^2}{48}\Psi(h)\right), \end{aligned}$$

from which we obtain the estimate at the centre $x = \dfrac{a+b}{2} = A(a,b)$, so that

$$\begin{aligned} &\left|(1-h)\ln A(a,b) + (1+h)\ln G(a,b) + h\frac{(b-a)^2}{4}\frac{1}{G^2} - 2\ln I(a,b)\right| \\ \leq\ & \frac{(b-a)^2}{24a^2}\Psi(h) \end{aligned}$$

or

$$\left|\ln\left(\frac{A^{(1-h)}G^{(1+h)}}{I^2}\right) + h\frac{(b-a)^2}{4}\frac{1}{G^2(a,b)}\right| \leq \frac{(b-a)^2}{24a^2}\Psi(h),$$

from which we obtain the best estimate if we choose $h = \dfrac{3}{10}$, that is

$$\left|\ln\left(\frac{A^{\frac{7}{10}}G^{\frac{13}{10}}}{I^2}\right) + \frac{3(b-a)^2}{40}\frac{1}{G^2(a,b)}\right| \leq \frac{293}{12000a^2}(b-a)^2.$$

And for $h=0$, we have

$$\begin{aligned} & \left|\ln A(a,b)+\ln G(a,b)-2\ln I(a,b)\right| \\ \leq & \frac{(b-a)^2}{24a^2}. \end{aligned}$$

2. Consider $g(x)=\frac{1}{t},\ g:(0,\infty)\rightarrow(0,\infty)$, then

$$\begin{aligned} \frac{1}{b-a}\int_a^b g(t)dt &= L^{-1}(a,b), \\ \frac{g(a)+g(b)}{2} &= \frac{A(a,b)}{G^2(a,b)}, \\ g'(b)-g'(a) &= \frac{2(b-a)}{H(a,b)\,G^2(a,b)} \end{aligned}$$

and

$$\|g''\|_\infty = \sup_{t\in(a,b)}\|g''(t)\| = \frac{2}{a^3}.$$

From (27), we have

$$\begin{aligned} & \left|\frac{(1+h)\ A(a,b)}{2G^2(a,b)}+\frac{(1-h)}{2x}\left(2-\frac{A(a,b)}{x}\right)\right. \\ & \left.-h\frac{(b-a)^2}{4}\frac{1}{H(a,b)\,G^2(a,b)}-L^{-1}(a,b)\right| \\ \leq & \frac{2}{a^3}\left(\frac{1}{3(b-a)}\left|x-A(a,b)\right|^3+\frac{(b-a)^2}{48}\Psi(h)\right) \end{aligned}$$

and the estimate at the centre point $x=\dfrac{a+b}{2}=A(a,b)$, so that

$$\begin{aligned} & \left|\frac{(1+h)\ A(a,b)}{2G^2(a,b)}+\frac{(1-h)}{2A(a,b)}-h\frac{(b-a)^2}{4}\frac{1}{H(a,b)\,G^2(a,b)}-L^{-1}(a,b)\right| \\ \leq & \frac{(b-a)^2}{24a^3}\Psi(h), \end{aligned}$$

which becomes best by choosing $h=\dfrac{3}{10}$ in the above inequality,

$$\begin{aligned} & \left|\frac{13\ A(a,b)}{20G^2(a,b)}+\frac{7}{20A(a,b)}-\frac{3(b-a)^2}{40}\frac{1}{H(a,b)\,G^2(a,b)}-L^{-1}(a,b)\right| \\ \leq & \frac{293}{12000a^3}(b-a)^2. \end{aligned}$$

Also for $h=0$, we have

$$\left|\frac{A(a,b)}{2G^2(a,b)}+\frac{1}{2A(a,b)}-L^{-1}(a,b)\right|\leq\frac{(b-a)^2}{24a^3}.$$

3. Consider $g(t)=t^p, g:(0,\infty)\longrightarrow(0,\infty)$, where $p\in\mathbb{R}\backslash\{-1,0\}$ then for $a<b$

$$\begin{aligned}\frac{1}{b-a}\int_a^b g(t)dt &= L_p^p(a,b),\\ \frac{g(a)+g(b)}{2} &= A(a^p,b^p),\\ g'(b)-g'(a) &= p(p-1)(b-a)L_{p-2}^{p-2}(a,b)\end{aligned}$$

and

$$\|g''\|_\infty=|p(p-1)|\begin{cases} b^{p-2} & \text{if } p\in[2,\infty)\\ a^{p-2} & \text{if } p\in(-\infty,2]\backslash\{-1,0\}.\end{cases}$$

From (27), we obtain

$$\begin{aligned}&\left|\frac{(1-h)}{2}x^{p-1}\left[(1-p)x+pA(a,b)\right]+\frac{(1+h)}{2}A(a^p,b^p)\right.\\ &\left.-h\frac{(b-a)^2}{8}p(p-1)L_{p-2}^{p-2}(a,b)-L_p^p(a,b)\right|\\ \leq\ & |p(p-1)|\,\delta_p(a,b)\left(\frac{1}{3(b-a)}\left|x-A(a,b)\right|^3+\frac{(b-a)^2}{48}\Psi(h)\right),\end{aligned}$$

where

$$\delta_p(a,b)=\begin{cases} b^{p-2} & \text{if } p\in[2,\infty)\\ a^{p-2} & \text{if } p\in(-\infty,2]\backslash\{-1,0\}.\end{cases}$$

At $x=\dfrac{a+b}{2}=A(a,b)$, we get

$$\begin{aligned}&\left|\frac{(1-h)\ A^p(a,b)}{2}+\frac{(1+h)}{2}A(a^p,b^p)\right.\\ &\left.-h\frac{(b-a)^2}{8}p(p-1)L_{p-2}^{p-2}(a,b)-L_p^p(a,b)\right|\\ \leq\ & |p(p-1)|\ \delta_p(a,b)\frac{(b-a)^2}{48}\Psi(h)\end{aligned}$$

or

$$\begin{aligned} & \left| \ (1-h)\, A^p(a,b) + (1+h)\, A(a^p,b^p) \right. \\ & \left. -h\frac{(b-a)^2}{4} p\,(p-1)\, L_{p-2}^{p-2}(a,b) - 2L_p^p(a,b) \ \right| \\ \leq \ & |p(p-1)|\, \delta_p(a,b) \frac{(b-a)^2}{24} \Psi\,(h)\,, \end{aligned}$$

which gives us the best estimate at $h = \dfrac{3}{10}$,

$$\begin{aligned} & \left| \frac{7}{10} A^p(a,b) + \frac{13}{10} A(a^p,b^p) \right. \\ & \left. -3\frac{(b-a)^2}{10} p\,(p-1)\, L_{p-2}^{p-2}(a,b) - 2L_p^p(a,b) \right| \\ \leq \ & |p(p-1)|\, \delta_p(a,b) \frac{293\,(b-a)^2}{12000}. \end{aligned}$$

Moreover, at $h = 0$

$$\begin{aligned} & \left| A^p(a,b) + A(a^p,b^p) - 2L_p^p(a,b) \right| \\ \leq \ & |p(p-1)|\, \delta_p(a,b) \frac{(b-a)^2}{24}. \end{aligned}$$

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