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A SIMPLE PROOF OF MONOTONICITY FOR STOLARSKY AND GINI MEANS

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Abstract. We give a new and simple proof of monotonicity for the well known classes of Stolarsky and Gini mean values.

1. INTRODUCTION

There is a huge amount of papers investigating properties of the so-called Stolarsky (or extended) two-parametric mean value, defined as

$$E_{r,s}(x,y)=\begin{cases}\left(\dfrac{r(x^s-y^s)}{s(x^r-y^r)}\right)^{1/(s-r)}, & rs(r-s)\neq 0,\\ \exp\left(-\dfrac{1}{s}+\dfrac{x^s\log x-y^s\log y}{x^s-y^s}\right), & r=s\neq 0,\\ \left(\dfrac{x^s-y^s}{s(\log x-\log y)}\right)^{1/s}, & s\neq 0,\ r=0,\\ \sqrt{xy}, & r=s=0,\\ x, & x=y>0.\end{cases}$$

Therefore E is continuous on the domain

$$\{(r, s; x, y) : r, s \in R;\ x, y \in R^+\},$$

and in this form is introduced by K. Stolarsky in [1].

Most of the classical two variable means are special cases of E. For example, $E_{1,2}$ is the arithmetic mean, $E_{0,0}$ is the geometric mean, $E_{-2,-1}$ is the harmonic mean, $E_{0,1}$ is the logarithmic mean, etc. More generally, the rth power mean is equal to $E_{r,2r}$.

The second family of widely known bivariate means was introduced by C. Gini [7]; they will be denoted by $D_{r,s}(\cdot,\cdot)$ and are defined as follows

$$D_{r,s}(x,y) = \begin{cases} \left(\dfrac{x^s + y^s}{x^r + y^r}\right)^{1/(s-r)}, & r \neq s, \\ \exp\left(\dfrac{x^s \log x + y^s \log y}{x^s + y^s}\right), & r = s \neq 0, \\ \sqrt{xy}, & r = s = 0, \\ x, & x = y > 0. \end{cases}$$

Therefore, $D_{0,-1}$ is the harmonic mean, $D_{0,0}$ is the geometric mean, $D_{1,0}$ is the arithmetic mean, etc. Alzer and Ruscheweyh have proven in [8] that the joint members in the families of the Stolarsky and Gini means are exactly the power means $\left(\dfrac{x^s + y^s}{2}\right)^{1/s} = E_{2s,s}(x,y) = D_{s,0}(x,y)$.

2. RESULTS

We shall give here an elementary proof of one of the crucial properties of the means E and D.

Proposition 1. *The family of Stolarsky means $E_{r,s}(x,y)$ increases with increase in either r or s.*

Proposition 2. *The family of Gini means $D_{r,s}(x,y)$ increases with increase in either r or s.*

3. PROOFS

Proof of Proposition 1. Since $E_{r,s}(x,y) = E_{s,r}(x,y)$ and $E_{r,s}(x,y) = E_{r,s}(y,x)$, we can assume without any loss that $x \geq y > 0; s \geq r \in R$.

Now, consider the function $e_s(x,y)$ defined as

$$e_s(x,y) := \begin{cases} \dfrac{x^s - y^s}{s}, & s \neq 0, \\ \log x - \log y, & s = 0. \end{cases}$$

Therefore, $e_s(x,y)$ is a positive and continuous function in s for $x > y > 0, s \in R$.

Also, there is an evident representation in the form

$$e_s(x,y) = \int_y^x u^{s-1} du, \quad s \in R.$$

Since for arbitrary $a, b, s, t \in R$,

$$a^2 e_s(x,y) + 2ab\ e_{(s+t)/2}(x,y) + b^2 e_t(x,y) = \int_y^x (a^2 u^{s-1} + 2abu^{(s+t)/2-1} + b^2 u^{t-1}) du$$

$$= \int_y^x (au^{(s-1)/2} + bu^{(t-1)/2})^2 du \geq 0,$$

by the discriminant test for the nonnegativity of second-order polynomials, we conclude that

$$e_s(x,y) e_t(x,y) - (e_{(s+t)/2}(x,y))^2 \geq 0,$$

i.e., that $\log e_s(x,y)$ is a convex function in s, $s \in R$. Since it is also continuous, by the well known result (cf [1], p. 74), we obtain that for arbitrary $r < s < t$,

$$(t-s) \log e_r(x,y) + (r-t) \log e_s(x,y) + (s-r) \log e_t(x,y) \geq 0. \qquad (*)$$

This inequality is equivalent to

$$(t-r)(\log e_s(x,y) - \log e_r(x,y)) \leq (s-r)(\log e_t(x,y) - \log e_r(x,y)),$$

i.e., getting rid of the logarithms, we finally obtain

$$E_{r,s}(x,y) = \left(\frac{e_s(x,y)}{e_r(x,y)}\right)^{1/(s-r)} \leq \left(\frac{e_t(x,y)}{e_r(x,y)}\right)^{1/(t-r)} = E_{r,t}(x,y), \quad t > s.$$

Also, letting $s \downarrow r$, we get

$$E_{r,t}(x,y) \geq \lim_{s\to r} E_{r,s}(x,y) = E_{r,r}(x,y),\ t > r.$$

Therefore we prove that $E_{r,s}(x,y)$ is increasing in s.

Since $E_{r,s}(x,y) = E_{s,r}(x,y)$, the same is valid for r and the proof is done. □

Proof of Proposition 2. The proof goes essentially along the same lines as the previous one. Namely, defining

$$d_s(x,y) := x^s + y^s,\ x,y > 0, s \in R,$$

and noting that

$$d_s(x,y)d_t(x,y) - (d_{(s+t)/2}(x,y))^2 = (x^{s/2}y^{t/2} - x^{t/2}y^{s/2})^2 \geq 0,$$

we conclude that $\log d_s(x,y)$ is a convex function in s for $x,y > 0$.

Now, applying the inequality $(*)$ with $\log d_s(x,y)$ instead of $\log e_s(x,y)$, after some calculation we get

$$D_{r,s}(x,y) = \left(\frac{d_s(x,y)}{d_r(x,y)}\right)^{1/(s-r)} \leq \left(\frac{d_t(x,y)}{d_r(x,y)}\right)^{1/(t-r)} = D_{r,t}(x,y), \quad t > s.$$

Therefore $D_{r,s}(x,y)$ is increasing in s. Since $D_{r,s}(x,y) = D_{s,r}(x,y)$, the same is valid for r and the proof is completed. □

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