Kragujevac J. Math. 32 (2009) 37–46.

ON THE CHEN CHARACTER OF $\delta(2)$–IDEAL SUBMANIFOLDS

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(Received May 29, 2008)

Abstract. In this paper we characterized $\delta(2)$*–ideal submanifolds* M^n in ambient Euclidean spaces E^{n+m}, $(n \geq 2, m \geq 1)$, which are at the same time *Chen submanifolds*

1. $\delta(2)$–IDEAL SUBMANIFOLDS

Let M^n be an $n-$dimensional Riemannian submanifold of an $(n+m)$–dimensional Euclidean space E^{n+m}, $(n \geq 2, m \geq 1)$. Let g and ∇, and, respectively, $\tilde{g}$ and

$\tilde{\nabla}$, denote the *Riemannian metrics* and the corresponding *Levi–Civita connections* of M^n and of E^{n+m}. The *formulae of Gauss and Weingarten* are then given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{1}$$

and

$$\tilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{2}$$

whereby h, A_ξ and $\nabla^\perp$ denote the *second fundamental form*, the *shape operator* or *Weingarten map* with respect to ξ and the *normal connection* of M^n in E^{n+m}, respectively, (X, Y, etc. stand for *tangent vector fields* and ξ etc. for *normal vector fields* on M^n in E^{n+m}).

From (1) and (2) it follows that

$$\tilde{g}(h(X, Y), \xi) = g(A_\xi(X), Y), \tag{3}$$

such that, for any *orthonormal local normal frame* $\{\xi_\alpha\}$ on M^n, $(\alpha = 1, \ldots, m)$,

$$h(X, Y) = \sum_\alpha g(A_\alpha(X), Y)\xi_\alpha, \tag{4}$$

whereby $A_\alpha = A_{\xi_\alpha}$.

The *mean curvature vector field* $\overrightarrow{H}$ of M^n in E^{n+m} is defined as $\overrightarrow{H} = \frac{1}{n}\, tr\ h = \frac{1}{n}\sum_{i=1}^n h(E_i, E_i)$, for any *orthonormal local tangent frame* $\{E_i\}$ on M^n, and its length $H = \|\overrightarrow{H}\|$ is the *mean curvature* of M^n in E^{n+m}.

Let R denote the $(0, 4)$ *Riemann–Christoffel curvature tensor* of (M^n, g). Then, according to the *equation of Gauss*,

$$R(X, Y, Z, W) = \tilde{g}(h(X, W), h(Y, Z)) - \tilde{g}(h(X, Z), h(Y, W)). \tag{5}$$

Denoting by τ the *scalar curvature function* of (M^n, g), we put

$$\tau(p) := \sum_{i<j} K(p, E_i(p) \wedge E_j(p)), \tag{6}$$

where K denotes *sectional curvature* $K(p, \pi) = K(p, E_i \wedge E_j) = R(E_i, E_j, E_j, E_i)$, $(i \neq j)$ and $E_i \wedge E_j = \pi$ is a *plane section* in $T_p(M^n)$. For each point p in M^n, consider the real function $(\inf K)(p) := \inf\{K(p, \pi)\, |\pi \text{ is a plane section in } T_p(M^n)\}$.

Then B. Y. Chen introduced in [1] the $\delta(2)$–*curvature* by

$$\delta := \tau - \inf K. \tag{7}$$

This $\delta(2)$–*curvature*, or, for short, δ–*curvature of Chen* is a well defined real function on M^n, $\delta : M^n \to R$, which is a *scalar valued intrinsic invariant of the Riemannian manifold.*

Later B. Y. Chen introduced many more new scalar valued *Riemannian curvature invariants* (δ–curvatures), based on the scalar curvature of the manifold and sectional or scalar curvatures of certain subspaces. Recent works of Chen ([2], [3],..) dealing with the problem by obtaining optimal general inequalities between *intrinsic* and *extrinsic* curvature invariants, give ample solutions to the question of S. S. Chern [4] to look for *necessary conditions of intrinsic type on a Riemannian manifold* in order to be able to admit an *isometric immersion into some Euclidean space as a minimal submanifold*, (cfr. [3] for an up to date survey).

Here, we mention, that for surfaces M^2 in E^3, the *Euler inequality* (1760), $K \leq H^2$ holds, whereby K is the intrinsic *Gauss curvature* of M^2 and H^2 is the extrinsic *squared mean curvature* of M^2 in E^3. And, obviously, $K = H^2$ everywhere on M^2 if and only if the surface M^2 is *totally umbilical* in E^3, or still, by a *theorem of Meusnier*, if and only if M^2 is a part of a *plane* E^2 or a *round sphere* S^2 in E^3.

The *inequalities of Chen*, alluded to above, do generalize this Euler inequality for Riemannian manifolds M^n in ambient Riemannian manifolds $\tilde{M}^{n+m}$, whereby instead of K comes some δ–curvature and whereby in the right hand side, besides H^2, some additional terms relating to the curvature of $\tilde{M}^{n+m}$ may appear.

From [1] we recall the following first answer to the above question of Chern.

Theorem A. *For any submanifold M^n in E^{n+m},*

$$\delta \leq \frac{n^2(n-2)}{2(n-1)} H^2 \qquad (*)$$

and equality holds at a point p of M^n if and only if, with respect to a suitable adapted *orthonormal frame* $\{E_i, \xi_\alpha\}$*, the shape operators* A_α *at* p *are given by*

$$A_1 = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix} \qquad A_\beta = \begin{pmatrix} c_\beta & d_\beta & 0 & \dots & 0 \\ d_\beta & -c_\beta & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, (\beta > 1),$$

whereby $\mu = a + b$ *and* $\inf K(p) = ab - \sum_{\beta>1} (c_\beta^2 + d_\beta^2)$*.* □

Submanifolds M^n of E^{n+m} for which the basic inequality $(*)$ at all points of M^n is actually an equality are called $\delta(2)$*–ideal submanifolds* or *Chen ideal submanifolds* ([12]). We also quote the following result from [1].

Theorem B. *Let* M^n *be a minimal submanifold of* E^{n+m}*. Then* $\delta \leq 0$*. If* $\delta = 0$*, then for each point* p *in* M*, either* $\dim \mathcal{D}(p) = n$ *or* $\dim \mathcal{D}(p) = n - 2$*, whereby* $\mathcal{D}(p) := \{\vec{z} \in T_pM \,|\, \forall \vec{v} \in T_pM : h(\vec{z}, \vec{v}) = \vec{o}\}$*. Moreover, if dim* $\mathcal{D} \equiv n$ *then* M^n *is totally geodesic, and, if dim* $\mathcal{D} \equiv n-2$*, then* M^n *is a ruled submanifold with* $(n-2)$*–dimensional rulers. In particular, if* M^n *is a normal* $(n-2)D$ *ruled submanifold, then* M^n *is a piece of one of the following minimal submanifolds: (i) the product submanifold* $N^2 \times E^{n-2}$ *of a minimal surface* N^2 *in* E^{2+m} *with an* $(n-2)$*–dimensional linear subspace* E^{n-2} *in* E^{n+m}*, or, (ii) the product submanifold* $CN^2 \times E^{n-3}$ *of a* $3D$ *minimal cone* CN^2 *in* E^{3+m}*, (*N^2 *now is a minimal surface of the unit hypersphere* S^{2+m} *in* E^{3+m}*, centered at the origin, and* CN^2 *is the cone over* N^2 *in* E^{3+m} *with vertex at the origin) with an* $(n-3)$*– dimensional linear subspace* E^{n-3} *of* E^{n+m}*. And, conversely,* $\delta = 0$ *for all minimal submanifolds* M^n *in* E^{n+m} *described in (i) and (ii).* □

2. SYMMETRIES OF $\delta(2)$–IDEAL SUBMANIFOLDS

For a Riemannian manifold (M^n, g), let R also denote the $(1,1)$ *curvature operator* $R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, besides the $(0,4)$ curvature tensor, such that, by definition

$$R(X,Y,Z,W) = g(R(X,Y)Z, W), \tag{8}$$

$[.\,,.]$ denoting the *Lie– bracket* on the differential manifold M^n.

By the action of the curvature operator R *working as a derivation* on the curvature tensor R, the following $(0,6)$ tensor $R\cdot R$ is obtained:

$$\begin{aligned}(R\cdot R\,)(\,X_1,X_2,X_3,X_4;X,Y) &:= (R(X,Y)\cdot R)(X_1,X_2,X_3,X_4)\\ &= -R(R(X,Y)X_1,X_2,X_3,X_4)-R(X_1,R(X,Y)X_2,X_3,X_4)\\ &\quad -R(X_1,X_2,R(X,Y)X_3,X_4)-R(X_1,X_2,X_3,R(X,Y)X_4).\end{aligned}$$

It was recently shown by S. Haesen and one of the authors [5], that this tensor $R\cdot R$ can be geometrically interpreted as giving the second order measure of *the change of the sectional curvatures* $K(p,\pi)$ *for tangent* $2D$*–planes* π *at points* p *after the parallel transport of* π *all around infinitesimal co–ordinate parallelograms in* M *cornered at* p.

Semi–symmetric or *Szabó symmetric manifolds* ([6] [7]), are characterized by the property that $R\cdot R=0$. According to [5], Szabó symmetric spaces are the Riemannian manifolds for which all sectional curvatures remain preserved after parallel transport of their planes around all infinitesimal co–ordinate parallelograms in M.

Under projective transformations, the Szabó symmetric spaces give rise to the Deszcz symmetric spaces.

Deszcz symmetric spaces or *pseudo–symmetric spaces* ([5] [8] [9] [10]) are defined by

$$R\cdot R = L\ Q(g,R) \tag{9}$$

for some function $L: M^n \to R$ (whenever $Q(g,R)\neq 0$), whereby $Q(g,R) := -\wedge_g\cdot R$, is $(0,6)$ *Tachibana tensor*, and $\wedge_g$ (denoting the *metrical endomorphism*: $(X\wedge_g Y)Z = g(Y,Z)X - g(X,Z)Y$) acts on the $(0,4)$ tensor R as a derivation.

A geometrical interpretation of L is given as *the isotropic double sectional curvature* function on (M^n,g) [5].

Clearly, every semi–symmetric manifold is also Deszcz symmetric, but the converse is not true. There do exist proper Deszcz symmetric manifolds (see e.g. [8]).

The Deszcz symmetry (10) corresponds to *proper Deszcz symmetry* in case $L\neq 0$ and to *semi–symmetry* in case $L=0$. And, as already observed by Cartan, every

$2D$ Riemannian manifold is automatically semi–symmetric, we further only consider $\delta(2)$–ideal submanifolds M^n of dimension $n \geq 3$.

In [11], F. Dillen and two of the authors classified the *semi–symmetric Chen ideal submanifolds*, i.e. the *semi–symmetric* $\delta(2)$*–ideal submanifolds* as follows.

Theorem C. *A Chen ideal submanifold* M^n*,* $n \geq 3$*, in* E^{n+m} *is semi–symmetric if and only if it is minimal (cfr. Theorem B) or it is a round hypercone in some totally geodesic* E^{n+1} *in* E^{n+m}*, (including as "degenerate cases" the totally geodesic and the totally umbilical submanifolds).* □

Recently, R. Deszcz, G. Zafindratafa and two of the authors [12] studied the intrinsic symmetry property to be Deszcz symmetric for Chen ideal submanifolds M^n in E^{n+m}.

Theorem D. *A Chen ideal submanifold* M^n*,* $n \geq 3$*, in* E^{n+m} *is properly pseudo–symmetric,* $R \cdot R = L\ Q(g, R)$*,* $0 \neq L : M \to R$*, if and only if, at every point* p *of* M^n*, the* $2D$ *normal section* $\sum_{\tilde{\pi}}^2$ *of* M^n *in the planar direction* $\tilde{\pi}$ *for which* $K(p, \tilde{\pi})$ *attains its minimal value* $K_{\inf}(p)$*, is pseudo–umbilical at* p*, or, equivalently, if* p *is a spherical point of the projection* $\widetilde{\sum_{\tilde{\pi}}^2}$ *of this normal section* $\sum_{\tilde{\pi}}^2$ *on the Euclidean space* E^3 *spanned by* $\tilde{\pi}$ *and the mean curvature vector* $\overrightarrow{H}(p)$ *of* M^n *in* E^{n+m} *at* p*, and in this case* $L = \frac{n^2}{2(n-1)^2}\ H^2$*.* □

We recall that the *non–planar umbilical points of surfaces* M^2 *in* E^3 are called *spherical points* and that a submanifold which is not minimal at a point is called *pseudo–umbilical* at this point when its mean curvature normal direction there is an umbilical one [13].

3. CHEN SUBMANIFOLDS

For submanifolds M^n of E^{n+m} the notion of *allied vector field* of a given normal vector field of M^n is defined in [13]. According to that, for $\delta(2)$*–ideal submanifolds* M^n in E^{n+m} we take a local orthonormal frame $\{\xi_1 = \dfrac{\overrightarrow{H}}{\|\overrightarrow{H}\|}, \xi_2, \ldots, \xi_m\}$ where $\overrightarrow{H}$ is the *mean curvature vector* of M^n in E^{n+m}, and then a normal canonical vector field is defined in

$$a(\overrightarrow{H}) = \frac{1}{n}\sum_{\beta=2}^{m} tr(A_1\, A_\beta)\xi_\beta, \tag{10}$$

which is called the *allied vector field of* $\overrightarrow{H}$ or *allied mean curvature vector field* of M^n in E^{n+m}.

A submanifold M^n is called an $\mathcal{A}$*–submanifold* or a *Chen submanifold* if the allied mean curvature vector field of M^n, $a(\overrightarrow{H}) \equiv \overrightarrow{0}$. By a result of B. Rouxel [14], a submanifold M^n of E^{n+m} is a Chen submanifold if and only if the mean curvature vector at a point p, $\overrightarrow{H}(p)$ is an axis of symmetry of the $(m-2)$–nd polar of its *Kommerell hyperquadric curvature image* in the normal space $T_p^\perp M$, for all points p of M.

Minimal submanifolds, pseudo–umbilical submanifolds and hypersurfaces are *Chen submanifolds* in a trivial way.

In order to find the allied mean curvature vector field of such $\delta(2)$*–ideal submanifolds* M^n in E^{n+m}, from the specific forms of the shape operators at a point p of these submanifolds being given in Theorem A, we take

$$A_1\; A_\beta = \begin{pmatrix} ac_\beta & ad_\beta & 0 & \dots & 0 \\ bd_\beta & -bc_\beta & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad (\beta > 1).$$

Then the allied mean curvature vector field $a(\overrightarrow{H}) = \overrightarrow{0}$ if and only if $c_\beta(a-b) = 0$, $(\forall \beta > 1)$.

Thus, a $\delta(2)$*–ideal submanifold* M^n in E^{n+m} is a *Chen submanifold*, if and only if $c_\beta = 0$, $(\forall\beta)$ in which case the shape operators A_α, $(\alpha = 1, \dots, m)$ at p of M^n in Euclidean spaces E^{n+m} are given by

$$A_1 = \begin{pmatrix} a & 0 & 0 & \dots & 0 \\ 0 & b & 0 & \dots & 0 \\ 0 & 0 & \mu & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & \mu \end{pmatrix}, \qquad A_\beta = \begin{pmatrix} 0 & d_\beta & 0 & \dots & 0 \\ d_\beta & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \qquad (\beta > 1),$$

or $a = b$, which is, by Theorem D, equivalent with the property of being properly Deszcz symmetric M^n in E^{n+m}, $(L \neq 0)$, with $L = \dfrac{n^2}{2(n-1)^2} H^2$ ([12]).

In order to characterize $c_\beta = 0$, $(\forall\beta)(\beta > 1)$, we take the $2D$*–normal section* $\sum_\pi^2$ at any point p of M^n in E^{n+m} determined by the tangent 2–plane $\pi = E_1(p) \wedge E_2(p) = R^2$ for which the sectional curvature $K(p, \pi)$ of M^n reaches its minimal value at p. Then $\sum_\pi^2$ is the surface which is the local intersection around p of M^n with the space R^{2+m} through p and spanned by π and the *normal space* $T_p^\perp(M^n) = R^m$ of M^n in E^{n+m} at p, i.e. $\sum_\pi^2$ is a surface in $E^{2+m} = \pi \oplus T_p^\perp(M^n)$. For such surface $\sum_\pi^2$ in E^{2+m}, from the specific forms of the shape operators at a point p of an $\delta(2)$–ideal submanifold M^n in E^{n+m} given in Theorem A, the *shape operators at* p are given by

$$\tilde{A}_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, (a \neq b), \qquad \tilde{A}_\beta = \begin{pmatrix} c_\beta & d_\beta \\ d_\beta & -c_\beta \end{pmatrix}, (\beta > 1).$$

Using the notion of *curvature ellipse* [15], for the $2D$–normal section $\sum_\pi^2$ in E^{2+m}, by virtue of the shape operators $\tilde{A}_1, \tilde{A}_\beta, (\beta > 1)$, we obtain that the curvature ellipse $\mathcal{E}_p$ of $\sum_\pi^2$ in E^{2+m} at $p \in \sum_\pi^2$ is given by

$$\begin{aligned} \tilde{h}(u,u) &= \overrightarrow{H} + \cos 2\theta\Big(\frac{h_{11} - h_{22}}{2}\Big) + \sin 2\theta\, h_{12} \qquad (11) \\ &= \overrightarrow{H} + \cos 2\theta\Big(\frac{a-b}{2}\xi_1 + \sum_\beta c_\beta \xi_\beta\Big) + \sin 2\theta\Big(\sum_\beta d_\beta \xi_\beta\Big), \qquad (a \neq b), \end{aligned}$$

$(u \in T_p(M^n), \|u\| = 1)$, whereby $h_{11} = h(E_1, E_1)$, $h_{22} = h(E_2, E_2)$ and $h_{12} = h(E_1, E_2)$.

Thus the *mean curvature direction* $\overrightarrow{H}$ of M^n in E^{n+m} at p is an *principal axis of the curvature ellipse* $\mathcal{E}$ of $\sum_\pi^2$ in E^{2+m}, if and only if, $\sum_\beta c_\beta \xi_\beta = 0$, i.e. $c_\beta = 0, (\forall\beta)$.

From all above, we thus obtained the following theorem.

Theorem. *Let M^n be a $\delta(2)$–ideal submanifold in Euclidean ambient spaces E^{n+m}, of arbitrary dimensions $n \geq 3$ and codimensions $m \geq 1$, and let $\sum_\pi^2$ be the $2D$–normal section at any point p of M^n for the tangent 2–plane π in which M^n reaches its minimal sectional curvature at p. Then M^n is a Chen submanifold M^n of E^{n+m}, if and only if, M^n is a minimal submanifold of E^{n+m}, or the curvature ellipse $\mathcal{E}$ at p of $\sum_\pi^2$ in E^{2+m} lies in a 2–plane in $T_p^\perp(M^n)$ which is perpendicular* *to the mean curvature vector* $\overrightarrow{H}(p)$ *of* M^n *in* E^{n+m} *at* p *(or equivalently, when* M^n *is properly pseudo–symmetric), or* $\overrightarrow{H}(p)$ *determines a principal axis of the curvature ellipse* $\mathcal{E}$ *of* $\sum_\pi^2$ *at* p*.* □

Acknowledgements: S. Decu was partially supported by the Simon Stevin Institute for Geometry. L. Verstraelen was partially supported by the Research Foundation - Flanders project G.0432.07.

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