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WEIGHTED ČEBYŠEV TYPE INEQUALITIES INVOLVING FUNCTIONS WHOSE FIRST DERIVATIVES BELONG TO $\mathbf{L}_p(a,b)$, $\forall\,(1 \leq p < \infty)$

Farooq Ahmad[1], Nazir Ahmad Mir[2] and Arif Rafiq[2]

[1] *Centre for Advanced Studies in Pure and Applied Mathematics, B. Z. University, Multan 60800, Pakistan*
(e-mails: farooqgujar@gmail.com)

[2] *Mathematics Department, COMSATS Institute of Information Technology, Plot # 30, Sector H-8/1, Islamabad 44000, Pakistan*
(e-mails: namir@comsats.edu.pk, aarafiq@comsats.edu.pk)

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Abstract. In this paper we establish some new weighted Čebyšev type integral inequalities, via certain integral inequalities, for the functions whose first derivatives belong to $\mathbf{L}_p(a,b)$ spaces $\forall\ \ 1 \leq p < \infty$.

1. INTRODUCTION

In 1882, one of the classical inequalities discovered by P. L. Čebyšev [1] is given in the form the integral inequality (see also ([13], p. 207),

$$T\,(f,g) \leq \frac{1}{12}\,(b-a)^2\,\|f'\|_\infty\,\|g'\|_\infty\,, \tag{1}$$

[1]Corresponding Author

where f, $g : [a,b] \to \mathbb{R}$, are two absolutely continuous functions whose first derivatives $f', g' \in \mathbf{L}_\infty(a,b)$, and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \quad (2)$$

which is called Čebyšev functional, provided, the involved integrals exist.

In the last few years, the study of such inequalities has been the focus of many mathematicians and a number of research papers have appeared which deal with various generalizations, extensions and variants, (see [4], [6]-[15]) and references given therein.

In 2007, Rafiq et. al [14] established some results related to inequalities (1) and (2). Inspired and motivated by the research work going on, we establish here weighted Ostrowski type inequalities for the product of two continuous functions whose first derivatives are in $\mathbf{L}_P(a,b)$ where $1 \leq p < \infty$. Our proofs provide new estimates on these types of inequalities in $\mathbf{L}_P(a,b)$ spaces where $1 \leq p < \infty$.

2. MAIN RESULT

Let the weight $w : [a,b] \to [0,\infty)$, be non-negative, integrable and

$$\int_a^b w(x)dx < \infty$$

The domain of w may be finite or infinite. We denote the zero moment as

$$m(a,b) = \int_a^b w(x)dx$$

Let $[a,b] \subset \mathbb{R}$, $a < b$; and as usual for any function $h \in \mathbf{L}_p[a,b]$, $p \geq 1$, we define

$$\|h\|_p = \left(\int_a^b |h(t)|^p\, dt\right)^{1/p}$$

Let $[a,b]\subset\mathbb{R},\ \ a<b$; we use the following notations to simplify the details of presentation. For suitable functions $f,g:[a,b]\to\mathbb{R}$ we set

$$\begin{aligned} F &= m\left(a,\frac{5a+b}{6}\right)f(a)+m\left(\frac{5a+b}{6},\frac{a+5b}{6}\right)f\left(\frac{a+b}{2}\right)\\ &\quad +m\left(\frac{a+5b}{6},b\right)f(b), \qquad (3)\\ G &= m\left(a,\frac{5a+b}{6}\right)g(a)+m\left(\frac{5a+b}{6},\frac{a+5b}{6}\right)g\left(\frac{a+b}{2}\right)\\ &\quad +m\left(\frac{a+5b}{6},b\right)g(b), \qquad (4)\end{aligned}$$

$$\begin{aligned} S_w(f,g) &= FG-G\frac{1}{m(a,b)}\int_a^b w(t)\,f(t)\,dt-F\frac{1}{m(a,b)}\int_a^b w(t)\,g(t)\,dt\\ &+\left(\frac{1}{m(a,b)}\int_a^b w(t)\,f(t)\,dt\right)\left(\frac{1}{m(a,b)}\int_a^b w(t)\,g(t)\,dt\right), \qquad (5)\end{aligned}$$

$$\begin{aligned} T_w(f,g) &= \frac{1}{m(a,b)}\int_a^b w(x)\,f(x)g(x)dx\\ &-\left(\frac{1}{m(a,b)}\int_a^b w(t)\,f(t)dt\right)\left(\frac{1}{m(a,b)}\int_a^b w(t)\,g(t)\,dt\right), \qquad (6)\\ H_w(f,g) &= \frac{1}{m(a,b)}\int_a^b \left[Fw(x)g(x)+Gw(x)f(x)\right]dx\\ &-2\left(\frac{1}{m(a,b)}\int_a^b w(x)g(x)\,dx\right)\left(\frac{1}{m(a,b)}\int_a^b w(x)\,f(x)\,dt\right). \qquad (7)\end{aligned}$$

Our main results are given in the following theorems:

Case 1: When $f',g'\in\mathbf{L}_1(a,b)$ spaces (for $p=1$).

Theorem 1. *Let* $f,\ g:[a,b]\to\mathbb{R}$ *be absolutely continuous functions whose first* *derivatives* $f', g' : (a,b) \to \mathbb{R}$ *belong to* $\mathbf{L}_1(a,b)$ *spaces* $(p=1)$. *Then,*

$$|T_w(f,g)| \leq \frac{\|f'\|_1 \|g'\|_1}{4m^3(a,b)} \int_a^b w(x) \left(m(a,b) - \int_a^b sgn(t-x)\, w(t)\, dt \right)^2 dx, \qquad (8)$$

$$\begin{aligned} |T_w(f,g)| &= \frac{1}{4m^2(a,b)} \left[\int_a^b |w(x)| \left(|g(x)| \|f'\|_1 + |f(x)| \|g'\|_1 \right) \right. \\ &\qquad \left. \times \left(m(a,b) - \int_a^b sgn(t-x)\, w(t)\, dt \right) dx \right], \end{aligned} \qquad (9)$$

for all $x \in [a,b]$.

Proof. From the hypothesis we have the following identities proved in ([3], p. 319, see also [14], p. 903),

$$f(x) - \frac{1}{m(a,b)} \int_a^b w(t) f(t)\, dt = \frac{1}{m(a,b)} \int_a^b p(x,t) f'(t)\, dt, \qquad (10)$$

and

$$g(x) - \frac{1}{m(a,b)} \int_a^b w(t) g(t)\, dt = \frac{1}{m(a,b)} \int_a^b p(x,t) g'(t)\, dt, \qquad (11)$$

for all $x \in [a,b]$ and the kernel

$$p(x,t) = \begin{cases} \int_a^t w(u)\, du, & \text{if } t \in [a,x] \\ \int_b^t w(u)\, du, & \text{if } t \in (x,b]. \end{cases} \qquad (12)$$

We have

$$\begin{aligned} \sup_t |p(x,t)| &= \max \left\{ \int_a^t w(u)\, du, \int_t^b w(u)\, du \right\} \\ &= \frac{1}{2} m(a,b) - \frac{1}{2} \int_a^b sgn(t-x)\, w(t)\, dt. \end{aligned} \qquad (13)$$

Multiplying the left and right hand sides of (10) and (11), we get

$$\begin{aligned} & f(x)g(x)-\frac{f(x)}{m(a,b)}\int_a^b w(t)\,g(t)\,dt-\frac{g(x)}{m(a,b)}\int_a^b w(t)\,f(t)\,dt \\ & \quad +\frac{1}{m^2(a,b)}\left(\int_a^b w(t)\,f(t)\,dt\right)\left(\int_a^b w(t)\,g(t)\,dt\right) \\ = & \frac{1}{m^2(a,b)}\left(\int_a^b p(x,t)\,f'(t)\,dt\right)\left(\int_a^b p(x,t)\,g'(t)\,dt\right). \end{aligned}$$

Multiplying with $\frac{w(x)}{m(a,b)}$ and integrating both sides with respect to x over $[a,b]$ and using notation, we have (see [14]):

$$T_w(f,g)=\frac{1}{m^3(a,b)}\int_a^b w(x)\left(\int_a^b p(x,t)\,f'(t)\,dt\right)\left(\int_a^b p(x,t)\,g'(t)\,dt\right)dx. \tag{14}$$

Using the properties of modulus, we have:

$$\begin{aligned} & |T_w(f,g)| \\ \leq & \frac{1}{m^3(a,b)}\int_a^b |w(x)|\left(\int_a^b |p(x,t)|\,|f'(t)|\,dt\right)\left(\int_a^b |p(x,t)|\,|g'(t)|\,dt\right)dx \\ \leq & \frac{1}{m^3(a,b)}\int_a^b \Bigg(|w(x)|\sup_t|p(x,t)|\int_a^b |f'(t)|\,dt \\ & \qquad \sup_t|p(x,t)|\int_a^b |g'(t)|\,dt\Bigg)dx \\ = & \frac{\|f'\|_1\,\|g'\|_1}{4m^3(a,b)}\int_a^b w(x)(\sup_t|p(x,t)|)^2dx. \end{aligned}$$

Using (13) and (15), we get the desired (8).

Multiplying (10) and (11) with $g(x)$ and $f(x)$ respectively then adding the two results and multiplying the final result with $\frac{w(x)}{2m(a,b)}$, we have:

$$\frac{w(x)\,f(x)g(x)}{m(a,b)}-\frac{w(x)\,f(x)}{2m^2(a,b)}\int_a^b w(t)\,g(t)\,dt-\frac{w(x)\,g(x)}{2m^2(a,b)}\int_a^b w(t)\,f(t)\,dt$$

$$= \frac{w(x)}{2m^2(a,b)}\left(g(x)\int_a^b p(x,t)\,f'(t)\,dt + f(x)\int_a^b p(x,t)\,g'(t)\,dt\right).$$

Integrating both sides with respect to x over $[a,b]$, we have:

$$\begin{aligned}\frac{1}{m(a,b)}\int_a^b w(x)\,f(x)g(x)dx &- \left(\frac{1}{m(a,b)}\int_a^b w(t)\,f(t)dt\right) \\ &\times\left(\frac{1}{m(a,b)}\int_a^b w(t)\,g(t)dt\right) \\ = \frac{1}{2m^2(a,b)}\int_a^b &\left(w(x)\,g(x)\int_a^b p(x,t)\,f'(t)\,dt\right. \\ &\left.+\,w(x)\,f(x)\int_a^b p(x,t)\,g'(t)\,dt\right)dx,\end{aligned}$$

using notations it becomes

$$\begin{aligned}T_w(f,g) &= \frac{1}{2m^2(a,b)}\int_a^b\left(w(x)\,g(x)\int_a^b p(x,t)\,f'(t)\,dt\right. \\ &\quad\left.+\,w(x)\,f(x)\int_a^b p(x,t)\,g'(t)\,dt\right)dx. \qquad (15)\end{aligned}$$

Using the properties of modulus, we have:

$$\begin{aligned}|T_w(f,g)| &\le \frac{1}{2m^2(a,b)}\int_a^b\left[\left(w(x)\,|g(x)|\int_a^b |p(x,t)|\,|f'(t)|\,dt\right)\right. \\ &\quad\left.+\left(w(x)\,|f(x)|\int_a^b |p(x,t)|\,|g'(t)|\,dt\right)\right]dx \\ &\le \frac{1}{2m^2(a,b)}\left[\int_a^b\left(w(x)\,|g(x)|\,\|f'\|_1\sup_t|p(x,t)|\right)dx\right. \\ &\quad\left.+\int_a^b\left(|w(x)|\,|f(x)|\,\|g'\|_1\sup_t|p(x,t)|\right)\right]dx,\end{aligned}$$

implies

$$|T_w(f,g)| = \frac{1}{4m^2(a,b)} \int_a^b |w(x)| \left(|g(x)| \|f'\|_1 + |f(x)| \|g'\|_1\right) \times \left(m(a,b) - \int_a^b sgn(t-x)\, w(t)\, dt \right) dx.$$

This completes the proof.

Theorem 2. *Let $f,\ g : [a,b] \to \mathbb{R}$ be absolutely continuous functions whose first derivatives $f', g' : (a,b) \to \mathbb{R}$ belong to $\mathbf{L}_1(a,b)$ spaces $(p=1)$. Then,*

$$|S_w(f,g)| \leq \frac{\|f'\|_1 \|g'\|_1}{4m^2(a,b)} \times \left(m\left(\frac{5a+b}{6}, \frac{a+5b}{6}\right) - \int_{\frac{5a+b}{6}}^{\frac{a+5b}{6}} sgn(t-x)\, w(t)\, dt \right)^2, \quad (16)$$

and

$$|H_w(f,g)| \leq \frac{1}{2m^2(a,b)} \int_a^b w(x) \left(|g(x)| \|f'\|_1 + |f(x)| \|g'\|_1\right) \times \left(m\left(\frac{5a+b}{6}, \frac{a+5b}{6}\right) - \int_{\frac{5a+b}{6}}^{\frac{a+5b}{6}} sgn(t-x)\, w(t)\, dt \right) dx, \quad (17)$$

for all $x \in [a,b]$.

Proof. From the hypothesis we have the following weighted identities

$$F - \frac{1}{m(a,b)} \int_a^b w(x)\, f(x)\, dx = \frac{1}{m(a,b)} \int_a^b m(x) f'(x)\, dx, \quad (18)$$

$$G - \frac{1}{m(a,b)} \int_a^b w(x)\, g(x)\, dx = \frac{1}{m(a,b)} \int_a^b m(x) g'(x)\, dx, \quad (19)$$

where

$$m(x)=\begin{cases}\int\limits_{\frac{5a+b}{6}}^{x} w(u)\,du, & \text{if } x\in[a,\frac{a+b}{2}]\\ \int\limits_{\frac{a+5b}{6}}^{x} w(u)\,du, & \text{if } x\in(\frac{a+b}{2},b].\end{cases} \tag{20}$$

We have:

$$\begin{aligned}\sup_x|p(x,t)| &= \max\left\{\int\limits_{\frac{5a+b}{6}}^{x} w(t)\,dt,\ \int\limits_{x}^{\frac{a+5b}{6}} w(t)\,dt\right\}\\ &= \frac{1}{2}m\left(\frac{5a+b}{6},\frac{a+5b}{6}\right)-\frac{1}{2}\int\limits_{\frac{5a+b}{6}}^{\frac{a+5b}{6}} sgn\,(t-x)\,w(t)\,dt. \end{aligned}\tag{21}$$

Multiplying the left and right hand sides of (18) and (19) with each others, we have:

$$\begin{aligned}&FG-G\frac{1}{m(a,b)}\int\limits_a^b w(x)\,f(x)\,dx-F\frac{1}{m(a,b)}\int\limits_a^b w(x)\,g(x)\,dx\\ +&\left(\frac{1}{m(a,b)}\int\limits_a^b w(x)\,f(x)\,dx\right)\left(\frac{1}{m(a,b)}\int\limits_a^b w(x)\,g(x)\,dx\right)\\ =&\left(\frac{1}{m(a,b)}\int\limits_a^b m(x)f'(x)\,dx\right)\left(\frac{1}{m(a,b)}\int\limits_a^b m(x)g'(x)\,dx\right),\end{aligned}$$

implies

$$S_w(f,g)=\frac{1}{m^2(a,b)}\left(\int\limits_a^b m(x)f'(x)\,dx\right)\left(\int\limits_a^b m(x)g'(x)\,dx\right). \tag{22}$$

Using the properties of modulus, we have:

$$|S_w(f,g)|\le\frac{\|f'\|_1\,\|g'\|_1}{m^2(a,b)}\left(\sup_x|m(x)|\right)^2. \tag{23}$$

From (21) and (23), we get the required inequality (16) .

Multiplying (18) and (19) with $g(x)$ and $f(x)$ respectively then adding the two results, we have:

$$
\begin{aligned}
& Fg(x) + Gf(x) - \frac{g(x)}{m(a,b)} \int_a^b w(x) f(x)\, dx - \frac{f(x)}{m(a,b)} \int_a^b w(x) g(x)\, dx \\
= & \frac{g(x)}{m(a,b)} \int_a^b m(x) f'(x)\, dx + \frac{f(x)}{m(a,b)} \int_a^b m(x) g'(x)\, dx.
\end{aligned}
$$

Multiplying with $\frac{w(x)}{m(a,b)}$ and integrating with respect to x over $[a,b]$, we have:

$$
\begin{aligned}
& H_w(f,g) \\
= & \frac{1}{m^2(a,b)} \int_a^b w(x) \left(g(x) \int_a^b m(x) f'(x)\, dx + f(x) \int_a^b m(x) g'(x)\, dx \right) dx.
\end{aligned} \tag{24}
$$

Using the properties of modulus, we have:

$$
\begin{aligned}
|H_w(f,g)| \le & \frac{1}{m^2(a,b)} \int_a^b w(x) \left(|g(x)| \sup_x |m(x)| \int_a^b |f'(x)|\, dx \right. \\
& \left. + |f(x)| \sup_x |m(x)| \int_a^b |g'(x)|\, dx \right) dx \\
\le & \frac{1}{2m^2(a,b)} \int_a^b w(x) \left(|g(x)| \, \|f'\|_1 + |f(x)| \, \|g'\|_1 \right) \\
& \times \left(m\left(\frac{5a+b}{6}, \frac{a+5b}{6} \right) - \int_{\frac{5a+b}{6}}^{\frac{a+5b}{6}} sgn(t-x)\, w(t)\, dt \right) dx.
\end{aligned}
$$

This completes the proof.

Case 2: When $f', g' \in \mathbf{L}_p(a,b)$ spaces where $1 < p < \infty$.

Theorem 3. *Let* $f,\ g : [a,b] \to \mathbb{R}$ *be absolutely continuous functions whose first* *derivatives* $f', g' : (a,b) \to \mathbb{R}$ *belong to* $\mathbf{L}_p(a,b)$ *spaces where* $1 < p < \infty$. *Then,*

$$|T_w(f,g)| \leq \frac{\|f'\|_p \|g'\|_p}{m^3(a,b)} \int_a^b w(x) \left(\int_a^x m^q(a,t)dt + \int_x^b m^q(t,b)dt \right)^{2/q} dx, \tag{25}$$

and

$$\begin{aligned} |T_w(f,g)| \leq & \frac{1}{2m^2(a,b)} \left[\int_a^b w(x) \left(|g(x)| \|f'\|_p + |f(x)| \|g'\|_p \right) \right. \\ & \left. \times \left(\int_a^x m^q(a,t)dt + \int_x^b m^q(t,b)dt \right)^{1/q} dx \right], \end{aligned} \tag{26}$$

for all $x \in [a,b]$.

Proof. From (12), we have:

$$\begin{aligned} \left(\int_a^b |p(x,t)|^q dt \right)^{1/q} &= \left(\int_a^x \left| \int_a^t w(u)\,du \right|^q dt + \int_x^b \left| \int_b^t w(u)\,du \right|^q dt \right)^{1/q} \\ &= \left[\int_a^x \left(\int_a^t w(u)\,du \right)^q dt + \int_x^b \left(\int_t^b w(u)\,du \right)^q dt \right]^{1/q} \\ &= \left(\int_a^x m^q(a,t)dt + \int_x^b m^q(t,b)dt \right)^{1/q}. \end{aligned} \tag{27}$$

By using (14) and applying the properties of modulus and Hölder's inequality, we have:

$$\begin{aligned} & |T_w(f,g)| \\ \leq & \frac{1}{m^3(a,b)} \int_a^b |w(x)| \left(\int_a^b |p(x,t)| |f'(t)| dt \right) \left(\int_a^b |p(x,t)| |g'(t)| dt \right) dx \\ \leq & \frac{1}{m^3(a,b)} \int_a^b |w(x)| \|f'\|_p \left(\int_a^b |p(x,t)|^q dt \right)^{1/q} \|g'\|_p \left(\int_a^b |p(x,t)|^q dt \right)^{1/q} dx. \end{aligned} \tag{28}$$

From (27) and (28), we get desired (25).

Using (15) and applying the properties of modulus and Hölder's inequality, we have:

$$\begin{aligned}
|T_w(f,g)| &\leq \frac{1}{2m^2(a,b)}\int_a^b \left[|w(x)||g(x)|\int_a^b |p(x,t)||f'(t)|\,dt\right. \\
&\qquad \left. + |w(x)||f(x)|\int_a^b |p(x,t)||g'(t)|\,dt\right] dx \\
&\leq \frac{1}{2m^2(a,b)}\int_a^b \left[w(x)|g(x)|\|f'\|_p\left(\int_a^b |p(x,t)|^q\,dt\right)^{1/q}\right. \\
&\qquad \left. + w(x)|f(x)|\|g'\|_p\left(\int_a^b |p(x,t)|^q\,dt\right)^{1/q}\right] dx \\
&= \frac{1}{2m^2(a,b)}\left[\int_a^b w(x)\left(|g(x)|\|f'\|_p + |f(x)|\|g'\|_p\right)\right. \\
&\qquad \left. \times\left(\int_a^x m^q(a,t)dt + \int_x^b m^q(t,b)dt\right)^{1/q} dx\right].
\end{aligned}$$

Hence we have (26).

Theorem 4. *Let $f,\ g : [a,b] \to \mathbb{R}$ be absolutely continuous functions whose first derivatives $f', g' : (a,b) \to \mathbb{R}$ belong to $\mathbf{L}_p(a,b)$ spaces where $1 < p < \infty$. Then,*

$$|S_w(f,g)| \leq \frac{\|f'\|_p\|g'\|_p}{m^2(a,b)}N, \tag{29}$$

and

$$|H_w(f,g)| \leq \frac{1}{m^2(a,b)}\int_a^b w(x)\left(|g(x)|\|f'\|_p + |f(x)|\|g'\|_p\right)N dx, \tag{30}$$

where

$$N = \left(\int_a^b |m(x)|^q\,dx\right)^{1/q}$$

$$= \left(\int_a^{\frac{5a+b}{6}} m^q \left(x, \frac{5a+b}{6} \right) dx + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, x \right) dx \right.$$
$$\left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(x, \frac{a+5b}{6} \right) dt + \int_{\frac{a+5b}{6}}^{b} m^q \left(\frac{a+5b}{6}, x \right) dt \right)^{1/q}.$$

for all $x \in [a, b]$.

Proof. From (20), we have:

$$N = \left(\int_a^b |m(x)|^q \, dx \right)^{1/q}$$
$$= \left(\int_a^{\frac{5a+b}{6}} m^q \left(x, \frac{5a+b}{6} \right) dx + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} m^q \left(\frac{5a+b}{6}, x \right) dx \right.$$
$$\left. + \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} m^q \left(x, \frac{a+5b}{6} \right) dt + \int_{\frac{a+5b}{6}}^{b} m^q \left(\frac{a+5b}{6}, x \right) dt \right)^{1/q}. \tag{31}$$

Using (22) and the properties of modulus with Hölder's inequality, we have:

$$|S_w(f, g)| \le \frac{\|f'\|_p \|g'\|_p}{m^2(a, b)} \left(\int_a^b |m(x)|^q \, dx \right)^{1/q}. \tag{32}$$

From (31) and (32), we get (29).

Using the properties of modulus with Hölder's inequality in (24), we have:

$$|H_w(f, g)| \le \frac{1}{m^2(a, b)} \int_a^b w(x) \left(|g(x)| \int_a^b |m(x)| \, |f'(x)| \, dx \right.$$
$$\left. + |f(x)| \int_a^b |m(x)| \, |g'(x)| \, dx \right) dx$$
$$\le \frac{1}{m^2(a, b)} \int_a^b \left(|g(x)| \, \|f'\|_p + |f(x)| \, \|g'\|_p \right) w(x) \left(\int_a^b |m(x)|^q \, dx \right)^{1/q} dx. \tag{33}$$

From (31) and (33), this proves (30).

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