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ENDPOINT ESTIMATES FOR MULTILINEAR COMMUTATOR OF MARCINKIEWICZ OPERATOR

Yu Wenxin and Liu Lanzhe

Department of Mathematics,
Changsha University of Science and Technology,
Changsha, 410076, P.R.of China
(e-mail: lanzheliu@163.com)

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Abstract. In this paper, we prove the endpoint estimates for the multilinear commutator of Marcinkiewicz operator.

1. INTRODUCTION and PRELIMINARIES

As the development of singular integral operators, their commutators have been well studied. Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator, the commutator $[b, T]$ generated by b and T is defined by

$$[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x).$$

A classical result of Coifman, Rochberb and Weiss (see [3]) proved that the commutator $[b, T]$ is bounded on $L^p(R^n)$, $(1 < p < \infty)$. In [2, 5], the boundedness properties of the commutators for the extreme values of p are obtained. And note that $[b, T]$

is not bounded for the end point boundedness (that is $p = 1$ and $p = \infty$). In this paper, we will introduce the multilinear commutator of Marcinkiewicz operator and prove the boundedness properties of the operator for the extreme cases.

First let us introduce some notations (see [1, 4, 7, 8]). In this paper, $Q = Q(x, r)$ will denote a cube of R^n with sides parallel to the axes and center at x and edge is r. For a cube Q and a locally integrable function f, let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and $f(Q) = \int_Q f(x)dx$, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.$$

f is said to belong to $BMO(R^n)$ if $f^{\#} \in L^\infty(R^n)$ and define $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. We nave $|f_{2Q} - f_Q| \le C||f||_{BMO}$ and $||f - f_{2^kQ}||_{BMO} \le Ck||f||_{BMO}$ for $k \ge 1$ (see [4, 8]). We also define the central BMO space by $CMO(R^n)$, which is the space of those functions $f \in L_{loc}(R^n)$ such that

$$||f||_{CMO} = \sup_{r>1} |Q(0, r)|^{-1} \int_Q |f(y) - f_Q| dy < \infty.$$

It is well-known that

$$||f||_{CMO} \approx \sup_{r>1} \inf_{c\in C} |Q(0, r)|^{-1} \int_Q |f(x) - c| dx.$$

Let M be the Hardy-Littlewood maximal operator, that is

$$M(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

Definition 1. *A function a is called a $H^1(R^n)-$atom, if there exists a cube Q, such that*

1°) *supp* $a \subset Q = Q(x_0, r)$,

2°) $||a||_{L^\infty} \le |Q|^{-1}$,

3°) $\int_{R^n} a(x)dx = 0$.

It is well known that the Hardy space $H^1(R^n)$ has the atomic decomposition characterization (see [4], [8]).

The A_p weight is defined by (see [4])

$$A_p = \left\{ w : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)}dx \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,$$

and

$$A_1 = \{w : M(w)(x) \leq Cw(x), a.e.\}.$$

Definition 2. *Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B_p^\delta(R^n)$ the space of those functions f on R^n such that*

$$||f||_{B_p^\delta} = \sup_{r>1} r^{-n(1/p-\delta/n)} ||f\chi_{Q(0,r)}||_{L^p} < \infty.$$

Definition 3. *Let $0 < \delta < n$, $0 < \gamma \leq 1$ and Ω be homogeneous of degree zero on R^n such that $\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0$. Assume that $\Omega \in Lip_\gamma(S^{n-1})$, that is there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x - y|^\gamma$. Let m be the positive integer and b_j be the locally integrable functions on R^n ($j = 1, \cdots, m$). The Marcinkiewicz multilinear commutator is defined by*

$$\mu_\delta^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \left[\prod_{j=1}^m (b_j(x) - b_j(y)) \right] f(y)dy.$$

Set

$$F_t(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y)dy,$$

we also define that

$$\mu_\delta(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

which is the Marcinkiewicz operator (see [6, 10]).

Let H be the space $H = \left\{ h : ||h|| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty \right\}$. Then, it is clear that

$$\mu_\delta(f)(x) = ||F_t(f)(x)|| \text{ and } \mu_\delta^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \ldots = b_m$, $\mu_\delta^{\vec{b}}$ is just the m order commutator, and, when $m = 1$ and $\delta = 0$, $\mu_\delta^{\vec{b}}$ is just the commutator generated by Marcinkiewicz operator and b (see [9, 10]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1, 2, 3, 5, 6, 7]). Our main purpose is to establish the boundedness properties of the operator for the extreme cases.

Given a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \ldots, \sigma(j)\}$ of $\{1, \ldots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \ldots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \ldots, b_m)$ and $\sigma = \{\sigma(1), \ldots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \ldots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \ldots ||b_{\sigma(j)}||_{BMO}$.

2. THEOREMS AND PROOFS

We begin with some preliminaries lemmas.

Lemma 1. *Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \ldots, k$ and $k \in N$. Then, we have*

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^k ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^k |b_j(y) - (b_j)_Q|^r dy\right)^{1/r} \le C \prod_{j=1}^k ||b_j||_{BMO}.$$

Proof. In fact, we just need to choose $p_j > 1$ and $q_j > 1$, where $1 \le j \le k$, such that $1/p_1 + \ldots + 1/p_k = 1$ and $r/q_1 + \ldots + r/q_k = 1$. After that, using the Hölder's inequality with exponent $1/p_1 + \ldots + 1/p_k = 1$ and $r/q_1 + \ldots + r/q_k = 1$ respectively, we may get the results by [4] and [8]. □

Lemma 2. [10] *Let $w \in A_1$, $0 < \delta < n$, $1 < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then μ_δ is bounded from $L^p(w)$ to $L^q(w)$.*

Lemma 3. *Let $w \in A_p$, $1 < p < \infty$, then $w\chi_{Q'} \in A_p$ for any cube Q', where $\chi_{Q'}$ denotes the characteristic function of the cube Q'.*

Proof. By definition, we have

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty,$$

thus

$$\begin{aligned} &\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\chi_{Q'}(x)dx\right)\left(\frac{1}{|Q|}\int_Q (w(x)\chi_{Q'}(x))^{-1/(p-1)}dx\right)^{p-1} \\ \leq\ &\sup_Q \left(\frac{1}{|Q|}\int_Q w(x)dx\right)\left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} < \infty, \end{aligned}$$

that is $w\chi_{Q'} \in A_p$. □

Theorem 1. *Let $0 < \delta < n$ and $\vec{b} = (b_1, \ldots, b_m)$ with $b_j \in BMO(R^n)$ for $1 \leq j \leq m$. Then $\mu_\delta^{\vec{b}}$ is bounded from $L^{n/\delta}$ to $BMO(R^n)$.*

Proof. It is only to prove that there exist a constant C_Q such that

$$\frac{1}{|Q|}\int_Q |\mu_\delta^{\vec{b}}(f)(x) - C_Q|dx \leq C||f||_{L^{n/\delta}}.$$

Fix a cube Q, $Q = Q(x_0, r)$, we decompose f into $f = f_1 + f_2$ with $f_1 = f\chi_{2Q},\ f_2 = f\chi_{(R^n\setminus 2Q)}$.

When $m = 1$, set $(b_1)_Q = |Q|^{-1}\int_Q b_1(y)dy$, we have

$$F_t^{b_1}(f)(x) = (b_1(x) - (b_1)_Q)F_t(f)(x) - F_t((b_1 - (b_1)_Q)f_1)(x) - F_t((b_1 - (b_1)_Q)f_2)(x),$$

so

$$\begin{aligned} &|\mu_\delta^{b_1}(f)(x) - \mu_\delta(((b_1)_Q - b_1)f_2)(x_0)| \\ =\ &\Big|||F_t^{b_1}(f)(x)|| - ||F_t(((b_1)_Q - b_1)f_2)(x_0)||\Big| \\ \leq\ &||F_t^{b_1}(f)(x) - F_t(((b_1)_Q - b_1)f_2)(x_0)|| \\ \leq\ &||(b_1(x) - (b_1)_Q)F_t(f)(x)|| + ||F_t((b_1 - (b_1)_Q)f_1)(x)|| \\ &+||F_t((b_1 - (b_1)_Q)f_2)(x) - F_t((b_1 - (b_1)_Q)f_2)(x_0)|| \\ =\ &A(x) + B(x) + C(x). \end{aligned}$$

For $A(x)$, set $1 < p < n/\delta$, $1/q = 1/p - \delta/n$ and $1/q + 1/q' = 1$, by the Hölder's inequality and Lemma 2.3, we get

$$\begin{aligned}
\frac{1}{Q}\int_Q |A(x)|dx &\le \left(\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_Q|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_{R^n} |\mu_\delta(f)(x)|^q\chi_Q(x)dx\right)^{1/q}\\
&\le C||b_1||_{BMO}\frac{1}{|Q|^q}\left(\int_{R^n}|f(x)|^p\chi_Q(x)dx\right)^{1/p}\\
&\le C||b_1||_{BMO}\frac{1}{|Q|^q}||f||_{L^{n/\delta}}|Q|^{(1-(\delta p/n))/p}\\
&\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

For $B(x)$, taking $1<r<n/\delta$ and $1/s=1/r-\delta/n$, by the Hölder's inequality, we have

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |B(x)|dx &\le \left(\frac{1}{|Q|}\int_{R^n}(\mu_\delta((b_1(x)-(b_1)_Q)f_1)(x))^s dx\right)^{1/s}\\
&\le C|Q|^{-1/s}||(b_1(x)-(b_1)_Q)f\chi_{2Q}||_{L^r}\\
&\le C\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_Q|^s dx\right)^{1/s}||f||_{L^{n/\delta}}\\
&\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}$$

For $C(x)$, note that $r\le|x-y|$ and $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$, we have

$$\begin{aligned}
C(x) &= \left(\int_0^\infty\left|\int_{|x-y|\le t}\frac{\Omega(x-y)f_2(y)}{|x-y|^{n-1-\delta}}(b_1(y)-(b_1)_Q)dy\right.\right.\\
&\quad \left.\left.-\int_{|x_0-y|\le t}\frac{\Omega(x_0-y)f_2(y)}{|x_0-y|^{n-1-\delta}}(b_1(y)-(b_1)_Q)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\\
&\le \left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}}|b_1(y)-(b_1)_Q|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\le t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}}\left|b_1(y)-(b_1)_Q\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1-\delta}}\right|\right.\right.\\
&\quad\left.\left.\times|b_1(y)-(b_1)_Q||f_2(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\equiv I_1+I_2+I_3,
\end{aligned}$$

thus

$$
\begin{aligned}
I_1 &\le C\int_{(2Q)^c} |(b_1(y)-(b_1)_Q)|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left(\int_{|x-y|\le t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy\\
&\le C\int_{(2Q)^c} |b_1(y)-(b_1)_Q|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left|\frac{1}{|x-y|^2}-\frac{1}{|x_0-y|^2}\right|^{1/2}dy\\
&\le C\int_{(2Q)^c} |b_1(y)-(b_1)_Q|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\frac{|x_0-x|^{1/2}}{|x-y|^{3/2}}dy\\
&\le C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_Q|\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2-\delta}}dy\\
&\le C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|b_1(y)-(b_1)_Q|^{n/(n-\delta)}dy\right)^{(n-\delta)/n}\\
&\quad\times\left(\int_{2^{k+1}Q}|f(y)|^{n/\delta}dy\right)^{\delta/n}\\
&\le C\sum_{k=1}^{\infty}k2^{-k/2}||b_1||_{BMO}||f||_{L^{n/\delta}}\\
&\le C||b_1||_{BMO}||f||_{L^{n/\delta}};
\end{aligned}
$$

similarly, we have $I_2 \le C||b_1||_{BMO}||f||_{L^{n/\delta}}$.

We now estimate I_3. By the following inequality:

$$
\left|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x_0-y)}{|x_0-y|^{n-1-\delta}}\right|\le C\left(\frac{|x-x_0|}{|x_0-y|^{n-\delta}}+\frac{|x-x_0|^{\gamma}}{|x_0-y|^{n-1-\delta+\gamma}}\right),
$$

we gain

$$
\begin{aligned}
I_3 &\le C\int_{(2Q)^c}|b_1(y)-(b_1)_{2Q}||f(y)|\left(\frac{|x-x_0|}{|x_0-y|^{n-\delta}}+\frac{|x-x_0|^{\gamma}}{|x_0-y|^{n-1-\delta+\gamma}}\right)\\
&\quad\times\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy\\
&\le C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}|b_1(y)-(b_1)_{2Q}|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy\\
&\le C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}|b_1(y)-(b_1)_{2Q}||f(y)|dy\\
&\le C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}|(b_1(y)-(b_1)_{2Q})|^{n/(n-\delta)}dy\right)^{(n-\delta)/n}\\
&\quad\times\left(\int_{2^{k+1}Q}|f(y)|^{n/\delta}dy\right)^{\delta/n}\\
&\le C||b_1||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

This completes the proof of the case $m = 1$.

When $m > 1$, set $\vec{b}_Q = ((b_1)_Q, \ldots, (b_m)_Q) \in R^n$, where $(b_j)_Q = |Q|^{-1} \int_Q b_j(y)dy$, $1 \leq j \leq m$, we have

$$\begin{aligned} F_t^{\vec{b}}(f)(x) &= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q) F_t(f)(x) \\ &\quad + (-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f)(x) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (\vec{b}(x) - \vec{b}_Q)_\sigma \int_{R^n} (\vec{b}(y) - \vec{b}_Q)_{\sigma^c} \psi_t(x-y) f(y) dy \\ &= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q) F_t(f)(x) \\ &\quad + (-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_1)(x) \\ &\quad + (-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_2)(x) \\ &\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} (-1)^{m-j} (\vec{b}(x) - \vec{b}_Q)_\sigma F_t((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x), \end{aligned}$$

thus

$$\begin{aligned} & |\mu_\delta^{\vec{b}}(f)(x) - \mu_\delta(((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m) f_2)(x_0)| \\ \leq\ & ||(b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q) F_t(f)(x)|| \\ & + \sum_{j=1}^{m-1} \sum_{\sigma \in C_j^m} ||(\vec{b}(x) - \vec{b}_Q)_\sigma F_t((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|| \\ & + ||F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_1)(x)|| \\ & + ||F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_2)(x) - F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q) f_2)(x_0)|| \\ =\ & S_1(x) + S_2(x) + S_3(x) + S_4(x). \end{aligned}$$

For $S_1(x)$, taking $1 < p < n/\delta$, and $1/q = 1/p - \delta/n$, by the Hölder's inequality and Lemma 1,2,3, we have

$$\begin{aligned} & \frac{1}{|Q|} \int_Q S_1(x) dx \\ \leq\ & \left(\frac{1}{|Q|} \int_Q |\prod_{j=1}^m (b_j(x) - (b_j)_Q)|^{q'} dx \right)^{1/q'} \left(\frac{1}{|Q|} \int_Q |\mu_\delta(f)(x)|^q dx \right)^{1/q} \\ \leq\ & C ||\vec{b}||_{BMO} |Q|^{-1/q} \left(\int_Q |f(x)|^p dx \right)^{1/p} |Q|^{(1-(\delta p/n))/p} \\ \leq\ & C ||\vec{b}||_{BMO} ||f||_{L^{n/\delta}}. \end{aligned}$$

For $S_2(x)$, taking $1<p<n/\delta$ and $1/q=1/p-\delta/n$, then

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q S_2(x)dx\\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_\sigma|^{q'}dx\right)^{1/q'}\left(\frac{1}{|Q|}\int_Q|\mu_\delta((\vec{b}-\vec{b}_Q)_{\sigma^c})f)(x)|^q dx\right)^{1/q}\\
\leq\ & C\sum_{j=1}^{m-1}||\vec{b}_\sigma||_{BMO}|Q|^{1/q}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c})f(x)|^p\chi_Q(x)dx\right)^{1/p}\\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^q dx\right)^{1/q}||f||_{L^{n/\delta}}\\
\leq\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}||f||_{L^{n/\delta}}\\
\leq\ & C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

For $S_3(x)$, taking $1<p<n/\delta$ and $1/q=1/p-\delta/n$, we get

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q S_3(x)dx\\
\leq\ & \left(\frac{1}{|Q|}\int_Q|\mu_\delta((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^q dx\right)^{1/q}\\
\leq\ & C|Q|^{-1/q}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f_1(x)||_{L^p}\\
\leq\ & C\left(\frac{1}{|2Q|}\int_{2Q}|(b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)|^q dx\right)^{1/q}||f||_{L^{n/\delta}}\\
\leq\ & C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

For $S_4(x)$, we have

$$
\begin{aligned}
&S_4(x)\\
\leq\ & \left(\int_0^\infty\left[\int_{|x-y|\leq t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\leq t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&+\left(\int_0^\infty\left[\int_{|x-y|\leq t,|x_0-y|\leq t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1-\delta}}\right|\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times \left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right| |f_2(y)|dy\Bigg]^2 \frac{dt}{t^3}\Bigg)^{1/2} \\
\equiv\ & V_1+V_2+V_3,
\end{aligned}
$$

thus

$$
\begin{aligned}
V_1 \ &\le\ C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)|}{|x-y|^{n-1-\delta}}\left(\int_{|x-y|\le t<|x_0-y|}\frac{dt}{t^3}\right)^{1/2}dy \\
&\le\ C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2-\delta}}dy \\
&\le\ C\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right||f(y)|dy \\
&\le\ C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|^{n/(n-\delta)}dy\right)^{(n-\delta)/n}||f||_{L^{n/\delta}} \\
&\le\ C||\vec{b}||_{BMO}||f||_{L^{n/\delta}};
\end{aligned}
$$

similarly, we have $V_2\le C||\vec{b}||_{BMO}||f||_{L^{\delta/n}}$.

We now estimate V_3. Similar to the case m=1, we gain

$$
\begin{aligned}
V_3 \ &\le\ C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)||x-x_0|}{|x_0-y|^{n-\delta}}\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy \\
&\quad +C\int_{(2Q)^c}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\frac{|f(y)||x-x_0|^{\gamma}}{|x_0-y|^{n-1-\delta+\gamma}}\left(\int_{|x_0-y|\le t,|x-y|\le t}\frac{dt}{t^3}\right)^{1/2}dy \\
&\le\ C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy \\
&\le\ C\sum_{k=1}^{\infty}k^m(2^{-k}+2^{-k\gamma})||\vec{b}||_{BMO}||f||_{L^{n/\delta}} \\
&\le\ C||\vec{b}||_{BMO}||f||_{L^{n/\delta}}.
\end{aligned}
$$

This completes the total proof of Theorem 1. □

Theorem 2. *Let $0<\delta<n$, $1<p<n/\delta$ and $\vec{b}=(b_1,\cdots,b_m)$ with $b_j\in BMO(R^n)$ for $1\le j\le m$. Then $\mu_\delta^{\vec{b}}$ is bounded from $B_p^\delta(R^n)$ to $CMO(R^n)$.*

Proof. It suffices to prove that there exist constant C_Q, such that

$$\frac{1}{|Q|}\int_Q |\mu_\delta^{\vec{b}}(f)(x) - C_Q|dx \le C||f||_{B_p^\delta}$$

holds for any cube $Q = Q(0,d)$ with $d > 1$. Fix a cube $Q = Q(0,d)$ with $d > 1$. Set $f_1 = f\chi_{2Q}$, $f_2 = f\chi_{R^n\setminus 2Q}$ and $\vec{b}_Q = ((b_1)_Q, \dots, (b_m)_Q)$, where $(b_j)_Q = |Q|^{-1}\int_Q |b_j(y)|dy$, $1 \le j \le m$, we have

$$\begin{aligned}
&|\mu_\delta^{\vec{b}}(f)(x) - \mu_\delta(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)| \\
\le\ & ||(b_1(x) - (b_1)_Q)\cdots(b_m(x) - (b_m)_Q)F_t(f)(x)|| \\
&+ \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m} ||(\vec{b}(x) - \vec{b}_Q)_\sigma F_t((\vec{b} - \vec{b}_Q)_{\sigma^c} f)(x)|| \\
&+ ||F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_1)(x)|| \\
&+ ||F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_2)(x) - F_t((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_2)(x_0)|| \\
=\ & H_1(x) + H_2(x) + H_3(x) + H_4(x).
\end{aligned}$$

Taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$, by the Hölder's inequality and Lemma 1,2,3, we have

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q H_1(x)dx \\
\le\ & \left(\frac{1}{|Q|}\int_Q |\prod_{j=1}^m (b_j(x) - (b_j)_Q)|^{q'} dx\right)^{1/q'} \left(\frac{1}{|Q|}\int_Q |\mu_\delta(f)(x)|^q dx\right)^{1/q} \\
\le\ & C||\vec{b}||_{BMO}|Q|^{-1/q}\left(\int_Q |f(x)|^p dx\right)^{1/p} \\
\le\ & C||\vec{b}||_{BMO} d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p} \\
\le\ & C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}$$

For $H_2(x)$, taking $1 < p < n/\delta$, $1/s = 1/r - \delta/n$, and $1/s' + 1/s = 1$, then

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q H_2(x)dx \\
\le\ & C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{s'} dx\right)^{1/s'} \left(\frac{1}{|Q|}\int_Q |\mu_\delta((\vec{b} - \vec{b}_Q)_{\sigma^c})f)(x)|^s dx\right)^{1/s}
\end{aligned}$$

$$\le C\sum_{j=1}^{m-1}||\vec{b}_\sigma||_{BMO}|Q|^{-1/s}\left(\int_{R^n}|(b(\vec{x})-\vec{b}_Q)_{\sigma^c})f(x)|^r\chi_Q(x)dx\right)^{1/r}$$
$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}\left(\frac{1}{|Q|}\int_Q|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{pr/(p-r)}dx\right)^{(p-r)/pr}|Q|^{(\delta/n-1/p)}||f\chi_Q||_{L^p}$$
$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}||\vec{b}_{\sigma^c}||_{BMO}d^{-n(1/p-\delta/n)}||f\chi_Q||_{L^p}$$
$$\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.$$

For $H_3(x)$, taking $1<p<n/\delta$, $1/s=1/r-\delta/n$ and $1/s'+1/s=1$, we get

$$\frac{1}{|Q|}\int_Q H_3(x)dx$$
$$\le\left(\frac{1}{|Q|}\int_Q|\mu_\delta((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|^s dx\right)^{1/s}$$
$$\le C|Q|^{-1/s}||((b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)f\chi_{2Q}||_{L^r}$$
$$\le C\left(\frac{1}{|2Q|}\int_{2Q}|(b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)|^{pr/(p-r)}dx\right)^{(p-r)/pr}d^{-n(1/p-\delta/n)}||f\chi_{2Q}||_{L^p}$$
$$\le C||\vec{b}||_{BMO}||f||_{B_p^\delta}.$$

For $H_4(x)$, we have

$$\begin{aligned}H_4(x)\ &\le\ \left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|>t}\frac{|\Omega(x-y)||f_2(y)|}{|x-y|^{n-1-\delta}}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|>t,|x_0-y|\le t}\frac{|\Omega(x_0-y)||f_2(y)|}{|x_0-y|^{n-1-\delta}}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\quad+\left(\int_0^\infty\left[\int_{|x-y|\le t,|x_0-y|\le t}\left|\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}-\frac{|\Omega(x_0-y)|}{|x_0-y|^{n-1-\delta}}\right|\right.\right.\\
&\quad\left.\left.\times\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right||f_2(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\\
&\equiv\ M_1+M_2+M_3,\end{aligned}$$

thus

$$M_1\ \le\ C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})\right|\frac{|Q|^{1/2n}|f(y)|}{|x_0-y|^{n+1/2-\delta}}dy$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right||f(y)|dy\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\left(\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|^{p/(p-1)}dy\right)^{(p-1)/p}\\
&\quad\times\left(\int_{2^{k+1}Q}|f(y)|^p dy\right)^{1/p}\\
&\leq C\sum_{k=1}^{\infty}2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|^{p/(p-1)}dy\right)^{(p-1)/p}\\
&\quad\times|2^{k+1}Q|^{-(1/p-\delta/n)}||f\chi_{2^{k+1}Q}||_{L^p}\\
&\leq C||\vec{b}||_{BMO}||f||_{B_p^\delta};
\end{aligned}
$$

similarly, we have $M_2\leq C||\vec{b}||_{BMO}||f||_{B_p^\delta}$.

We now estimate V_3. We gain

$$
\begin{aligned}
M_3&\leq C\sum_{k=1}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|\left(\frac{|Q|^{1/n}}{|x_0-y|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x_0-y|^{n+\gamma-\delta}}\right)|f(y)|dy\\
&\leq C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})\frac{1}{|2^{k+1}Q|^{1-\delta/n}}\int_{2^{k+1}Q}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right||f(y)|dy\\
&\leq C\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma})\left(\frac{1}{|2^{k+1}Q|}\int_{2^kQ}\left|\prod_{j=1}^{m}(b_j(y)-(b_j)_{2Q})\right|^{p/(p-1)}dy\right)^{(p-1)/p}\\
&\quad\times|2^{k+1}Q|^{-(1/p-\delta/n)}||f\chi_{2^{k+1}Q}||_{L^p}\\
&\leq C||\vec{b}||_{BMO}||f||_{B_p^\delta}.
\end{aligned}
$$

This completes the total proof of Theorem 2. □

Theorem 3. *Let $0<\delta<n$ and $\vec{b}=(b_1,\ldots,b_m)$ with $b_j\in BMO(R^n)$ for $1\leq j\leq m$. If for any $H^1(R^n)-$atom a supported on certain cube Q and $u\in Q$, there is*

$$
\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left(|(b(x)-b_Q)_{\sigma^c}|\left|\left|\int_Q(\vec{b}(y)-\vec{b}_Q)_\sigma a(y)dy\frac{\Omega(x-u)}{|x-u|^{n-1-\delta}}\right|\right|\right)^{n/(n-\delta)}dx\leq C,
$$

then $\mu_\delta^{\vec{b}}$ is bounded from $H^1(R^n)$ to $L^{n/(n-\delta)}(R^n)$.

Proof. Let a be an atom supported in some cube Q. We write

$$\int_{R^n}|\mu_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx=\int_{2Q}|\mu_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx+\int_{(2Q)^c}|\mu_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx=I+II.$$

For I, taking $1<p<n/\delta$ and $1/q=1/p-\delta/n$, we have

$$I\le||\mu_\delta^{\vec{b}}(a)||_{L^q}^{n/(n-\delta)}|2Q|^{1-n/((n-\delta)q)}\le C||a||_{L^p}^{n/(n-\delta)}|Q|^{1-n/((n-\delta)q)}\le C.$$

For II, we first calculate $F_t^{\vec{b}}(a)(x)$, we have

$$\begin{aligned}|F_t^{\vec{b}}(a)(x)| &\le \left|\prod_{j=1}^m(b_j(x)-(b_j)_Q)\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}a(y)dy\right|\\ &\quad+\sum_{j=1}^m\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_{|x-y|\le t}(\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x-u)}{|x-u|^{n-1-\delta}})\\ &\quad\times(\vec{b}(y)-\vec{b}_Q)_\sigma a(y)dy|\\ &\quad+\sum_{j=1}^m\sum_{\sigma\in C_j^m}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\int_{|x-y|\le t}\frac{\Omega(x-u)}{|x-u|^{n-1-\delta}}(\vec{b}(y)-\vec{b}_Q)_\sigma a(y)dy\right|\\ &= \nu_1+\nu_2+\nu_3,\\ \mu_\delta^{\vec{b}}(a)(x) &= ||F_t^{\vec{b}}(a)(x)||\le\left(\int_0^\infty|\nu_1|^2\frac{dt}{t^3}\right)^{1/2}+\left(\int_0^\infty|\nu_2|^2\frac{dt}{t^3}\right)^{1/2}+\left(\int_0^\infty|\nu_3|^2\frac{dt}{t^3}\right)^{1/2}\\ &= A(x)+B(x)+C(x).\end{aligned}$$

For $A(x)$, denoting $\Gamma(x)=\{(z,t)\in R_+^{n+1}:|x-z|<t\}$ and the characteristic function of $\Gamma(x)$ by $\chi_{\Gamma(x)}$, we have

$$\begin{aligned}&A(x)\\ =&\left(\int_0^\infty\left|\int_{R^n}\left(\frac{\chi_{\Gamma(y)}\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\chi_{\Gamma(u)}\Omega(x-u)}{|x-u|^{n-1-\delta}}\right)a(y)dy\right|^2\frac{dt}{t^3}\right)^{1/2}\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\ \le&\left(\int_0^\infty\left|\int_{R^n}\chi_{\Gamma(y)}\left(\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}a(y)-\frac{\Omega(x-u)}{|x-u|^{n-1-\delta}}\right)a(y)dy\right)^2\frac{dt}{t^3}\right|^{1/2}\\ &\times\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\ &+\left(\int_0^\infty\left|\int_{R^n}\frac{(\chi_{\Gamma(y)}-\chi_{\Gamma(u)})\Omega(x-u)}{|x-u|^{n-1-\delta}}a(y)dy\right)^2\frac{dt}{t^3}\right|^{1/2}\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\ =&A_1(x)+A_2(x),\end{aligned}$$

by the following inequality:

$$\left|\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}-\frac{\Omega(x-u)}{|x-u|^{n-1-\delta}}\right|\le C\left(\frac{|y-u|}{|x-u|^{n-\delta}}+\frac{|y-u|^{\gamma}}{|y-u|^{n-1-\delta+\gamma}}\right),$$

we have

$$\begin{aligned}
&A_1(x)\\
&\le C\left(\int_0^\infty\left[\int_{R^n}\left(\frac{|y-u|}{|x-u|^{n-\delta}}+\frac{|y-u|^{\gamma}}{|x-u|^{n-1-\delta+\gamma}}\right)|a(y)|dy\right]^2\frac{dt}{t^3}\right)^{1/2}\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\
&\le C\int_{R^n}\left(\frac{|y-u|}{|x-u|^{n-\delta}}+\frac{|y-u|^{\gamma}}{|x-u|^{n-1-\delta+\gamma}}\right)|a(y)|\left(\int_0^\infty\frac{dt}{t^3}\right)^{1/2}dy\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\
&\le C\left(\frac{|Q|^{1/n+1}}{|x-u|^{n+1-\delta}}+\frac{|Q|^{\gamma/n+1}}{|x-u|^{n-\delta+\gamma}}\right)||a||_{L^\infty}\prod_{j=1}^m|b_j(x)-(b_j)_Q|,
\end{aligned}$$

thus

$$\begin{aligned}
&\left(\int_{(2Q)^c}(A_1(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}\\
&\le C||a||_{L^\infty}\left[\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}\left(\left(\frac{|Q|^{1/n+1}}{|x-u|^{n+1-\delta}}+\frac{|Q|^{\gamma/n+1}}{|x-u|^{n+\gamma-\delta}}\right)\right.\right.\\
&\quad\left.\left.\times\prod_{j=1}^m|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)}dx\right]^{(n-\delta)/n}\\
&\le C|Q|^{1+1/n}||a||_{L^\infty}\sum_{k=1}^\infty\left(\int_{2^{k+1}Q}\left(\frac{|2^kQ|^{\delta/n}}{|2^kQ|^{(n+1)/n}}\prod_{j=1}^m|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)}dx\right)^{(n-\delta)/n}\\
&\quad+C|Q|^{1+\gamma/n}||a||_{L^\infty}\sum_{k=1}^\infty\left(\int_{2^{k+1}Q}\left(\frac{|2^kQ|^{\delta/n}}{|2^kQ|^{(n+\gamma)/n}}\prod_{j=1}^m|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)}dx\right)^{(n-\delta)/n}\\
&\le C\sum_{k=1}^\infty(2^{-k}+2^{-k\gamma})\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left(\prod_{j=1}^m|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)}dx\right)^{(n-\delta)/n}\\
&\le C||\vec{b}||_{BMO}.
\end{aligned}$$

For $A_2(x)$, we have

$$
\begin{aligned}
&A_2(x)\\
\le\ & C\left(\int_0^\infty\left(\int_{R^n}\frac{|\chi_{\Gamma(y)}-\chi_{\Gamma(u)}|}{|x-u|^{n-1-\delta}}|a(y)|dy\right)^2\frac{dt}{t^3}\right)^{1/2}\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\
\le\ & C\int_{R^n}\left|\int_{|x-z|\le t}\frac{1}{|x-y|^{2n-2-2\delta}}\frac{dt}{t^3}-\int_{|u-z|\le t}\frac{1}{|x-u|^{2n-2-2\delta}}\frac{dt}{t^3}\right|^{1/2}|a(y)|dy\\
&\times\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\
\le\ & C\int_{R^n}\left(\int_{|x|\le t,|x+y-u|\le t}\left|\frac{1}{|x+y-u|^{2n-2-2\delta}}-\frac{1}{|x|^{2n-2-2\delta}}\right|\frac{dt}{t^3}\right)^{1/2}|a(y)|dy\\
&\times\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\
\le\ & C\int_{R^n}\left(\int_{|x|\le t,|x+y-u|\le t}\frac{|y-u|}{|x+y-u|^{2n-1-2\delta}}\frac{dt}{t^3}\right)^{1/2}|a(y)|dy\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\
\le\ & C\int_{R^n}\frac{|y-u|^{1/2}}{|x+y-u|^{n+1/2-\delta}}|a(y)|dy\prod_{j=1}^m|b_j(x)-(b_j)_Q|\\
\le\ & C|Q|^{1/2n}|x-u|^{-(n+1/2-\delta)}\prod_{j=1}^m|b_j(x)-(b_j)_Q|,
\end{aligned}
$$

thus

$$
\begin{aligned}
&\left(\int_{(2Q)^c}(A_2(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}\\
\le\ & C\sum_{k=1}^\infty 2^{-k/2}\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left(\prod_{j=1}^m|b_j(x)-(b_j)_Q|\right)^{n/(n-\delta)}dx\right)^{(n-\delta)/n}\\
\le\ & C||\vec{b}||_{BMO}.
\end{aligned}
$$

For $B(x)$, we have

$$
\begin{aligned}
&\sum_{j=1}^m\sum_{\sigma\in C_j^m}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\left(\int_0^\infty\left(\int_{|x-y|\le t}\frac{\Omega(x-y)}{|x-y|^{n-1-\delta}}a(y)(\vec{b}(y)-\vec{b}_Q)_\sigma dy\right)^2\frac{dt}{t^3}\right)^{1/2}\\
=\ & \sum_{j=1}^m\sum_{\sigma\in C_j^m}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\\
&\times\left(\int_0^\infty\left(\int_{R^n}\chi_{\Gamma(y)}\left(\frac{|\Omega(x-y)|}{|x-y|^{n-1-\delta}}-\frac{|\Omega(x-u)|}{|x-u|^{n-1-\delta}}\right)a(y)(\vec{b}(y)-\vec{b}_Q)_\sigma dy\right)^2\frac{dt}{t^3}\right)^{1/2}
\end{aligned}
$$

$$+ \sum_{j=1}^{m}\sum_{\sigma\in C_j^m}(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}$$
$$\times\left(\int_0^\infty\left(\int_{R^n}\frac{(\chi_{\Gamma(y)}-\chi_{\Gamma(u)})\Omega(x-u)}{|x-u|^{n-1-\delta}}a(y)(\vec{b}(y)-\vec{b}_Q)_\sigma dy\right)^2\frac{dt}{t^3}\right)^{1/2},$$

similarly, we get

$$|B(x)|$$
$$\le C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|\left(\frac{|Q|^{1/n}}{|x-u|^{n+1-\delta}}+\frac{|Q|^{\gamma/n}}{|x-u|^{n-\delta+\gamma}}+\frac{|Q|^{1/2n}}{|x-u|^{n+1/2-\delta}}\right)||\vec{b}_\sigma||_{BMO},$$

thus

$$\begin{aligned}&\left(\int_{(2Q)^c}(B(x))^{n/(n-\delta)}dx\right)^{(n-\delta)/n}\\ \le\ & C\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\sum_{k=1}^{\infty}(2^{-k}+2^{-k\gamma}+2^{-k/2})\\ &\times\left(\frac{1}{|2^{k+1}Q|}\int_{2^{k+1}Q}\left|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}\right|^{n/(n-\delta)}dx\right)^{(n-\delta)/n}||\vec{b}_\sigma||_{BMO}\\ \le\ & C||\vec{b}||_{BMO}.\end{aligned}$$

So, if

$$\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}\int_{(2Q)^c}\left(|(b(x)-b_Q)_{\sigma^c}|\left\|\int_Q(\vec{b}(y)-\vec{b}_Q)_\sigma a(y)dy\frac{\Omega(x-u)}{|x-u|^{n-1-\delta}}\right\|\right)^{n/(n-\delta)}dx\le C,$$

then

$$\int_{R^n}|\mu_\delta^{\vec{b}}(a)(x)|^{n/(n-\delta)}dx\le C.$$

This completes the proof of the Theorem 3. □

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