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ON IMBALANCES IN DIGRAPHS

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Abstract. The imbalance of a vertex in a digraph is defined as b_{v_i}(or simply b_i) $= d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i. The set of imbalances of the vertices in a digraph is called its imbalance set. In this paper, we obtain an inequality for imbalances in simple directed graphs. Also, we give the existence of an oriented graph with a given imbalance set.

1. INTRODUCTION

An oriented graph is a digraph with no symmetric pairs of directed arcs and without loops. Define a_{v_i}(or simply a_i) $= n - 1 + d^+_{v_i} - d^-_{v_i}$, the score of a vertex v_i in an oriented graph D, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i and n is the number of vertices in D. The score sequence of an oriented graph is formed by listing the vertex scores in non-decreasing order.

The following result, due to Avery [1], is the characterization of score sequences in oriented graphs, and a new proof can be found in [5].

Theorem 1. *A sequence $A = [a_1, a_2, \cdots, a_n]$ of non-negative integers in non-decreasing order is the score sequence of some oriented graph if and only if $\sum_{i=1}^{k} a_i \geq k(k-1)$, for $1 \leq k < n$ with equality when $k = n$.*

The set of distinct scores of the vertices in an oriented graph is called its score set. Pirzada and Naikoo [4] obtained various results for score sets in oriented graphs.

A digraph without loops and without multi-arcs is called a simple digraph. Dhruv et al. [2] defined the imbalance of a vertex v_i in a digraph as b_{v_i}(or simply b_i) $= d^+_{v_i} - d^-_{v_i}$, where $d^+_{v_i}$ and $d^-_{v_i}$ are respectively the outdegree and indegree of v_i. The imbalance sequence of a simple digraph is formed by listing the vertex imbalances in non-increasing order. A sequence of integers $F = [f_1, f_2, \cdots, f_n]$ with $f_1 \geq f_2 \geq \cdots \geq f_n$ is feasible if it has sum zero and satisfies $\sum_{i=1}^{k} f_i \leq k(n-k)$, for $1 \leq k < n$.

The next result [2] provides necessary and sufficient conditions for a sequence of integers to be an imbalance sequence of a simple directed graph.

Theorem 2. *A sequence is realizable as an imbalance sequence if and only if it is feasible.*

The above result is equivalent to saying that a sequence of integers $B = [b_1, b_2, \cdots, b_n]$ with $b_1 \geq b_2 \geq \cdots \geq b_n$ is an imbalance sequence of a simple directed graph if and only if

$$\sum_{i=1}^{k} b_i \leq k(n-k), \quad \text{for} \quad 1 \leq k < n \tag{1}$$

with equality when $k = n$.

On arranging the imbalance sequence in non-decreasing order, we have the following result [3].

Corollary 3. *A sequence of integers $B = [b_1, b_2, \cdots, b_n]$ with $b_1 \leq b_2 \leq \cdots \leq b_n$ is an imbalance sequence of a simple directed graph if and only if $\sum_{i=1}^{k} b_i \geq k(k-n)$, for $1 \leq k < n$ with equality when $k = n$.*

Various results for imbalances in oriented graphs can be found in [3].

2. MAIN RESULTS

Now, we obtain the following inequalities for imbalances in simple directed graphs.

Theorem 4. *If $B = [b_1, b_2, \cdots, b_n]$ is an imbalance sequence of a simple directed graph with $b_1 \geq b_2 \geq \cdots \geq b_n$, then $\sum_{i=1}^{k} b_i^2 \leq \sum_{i=1}^{k}(2n - 2k - b_i)^2$, for $1 \leq k < n$ with equality when $k = n$.*

Proof. By conditions (1), we have

- $k(n-k) \geq \sum_{i=1}^{k} b_i$ for $1 \leq k < n$ with equality when $k = n$, or
- $\sum_{i=1}^{k} b_i^2 + 2(2n-2k)k(n-k) \geq \sum_{i=1}^{k} b_i^2 + 2(2n-2k)\sum_{i=1}^{k} b_i$, for $1 \leq k < n$ with equality when $k = n$, or
- $\sum_{i=1}^{k} b_i^2 + k(2n-2k)^2 - 2(2n-2k)\sum_{i=1}^{k} b_i \geq \sum_{i=1}^{k} b_i^2$, for $1 \leq k < n$ with equality when $k = n$, or
- $b_1^2 + b_2^2 + \cdots + b_k^2 + \underbrace{(2n-2k)^2 + (2n-2k)^2 \cdots + (2n-2k)^2} - 2(2n-2k)b_1 - 2(2n-2k)b_2 - \cdots - 2(2n-2k)b_k \geq \sum_{i=1}^{k} b_i^2$, for $1 \leq k < n$ with equality when $k = n$, or
- $(2n-2k-b_1)^2 + (2n-2k-b_2)^2 + \cdots + (2n-2k-b_k)^2$, for $1 \leq k < n$ with equality when $k = n$, or
- $\sum_{i=1}^{k}(2n-2k-b_k)^2 \geq \sum_{i=1}^{k} b_i^2$, for $1 \leq k < n$ with equality when $k = n$. $\square$

The set of distinct imbalances of the vertices in an oriented graph is called its imbalance set.

Let D be an oriented graph with vertex set V, and let $X, Y \subseteq V$. If there is an arc from each vertex of X to every vertex of Y, then we denote it by $X \to Y$.

The following result gives the existence of an oriented graph with a given imbalance set.

Proposition 5. *Let $P = \{p_1, p_2, \cdots, p_m\}$ and $Q = \{-q_1, -q_2, \cdots, -q_n\}$ where $p_1, p_2, \cdots, p_m, q_1, q_2, \cdots, q_n$ are positive integers such that $p_1 < p_2 < \cdots < p_m$ and $q_1 < q_2 < \cdots < q_n$. Then there exists an oriented graph with imbalance set $P \cup Q$.*

Proof. Construct an oriented graph D with vertex set V as follows. Let $V = X_1^1 \cup X_2^1 \cup \cdots \cup X_m^1 \cup X_1^2 \cup X_1^3 \cup \cdots \cup X_1^n \cup Y_1^1 \cup Y_2^1 \cup \cdots \cup Y_m^1 \cup Y_1^2 \cup Y_1^3 \cup \cdots \cup Y_1^n$ with $X_i^j \cap X_k^l = \phi$, $Y_i^j \cap Y_k^l = \phi$, $X_i^j \cap Y_k^l = \phi$, $|X_i^1| = q_1$ for all $1 \leq i \leq m$, $|X_1^i| = q_i$ for all $2 \leq i \leq n$, $|Y_i^1| = p_i$ for all $1 \leq i \leq m$ and $|Y_1^i| = p_1$ for all $2 \leq i \leq n$. Let $X_i^1 \to Y_i^1$ for all $1 \leq i \leq m$ and $X_1^i \to Y_1^i$ for all $2 \leq i \leq n$, so that we obtain the oriented graph D with the imbalances of vertices as follows:

- for $1 \leq i \leq m$, $b_{x_i^1} = |Y_i^1| - 0 = p_i$, for all $x_i^1 \in X_i^1$;
- for $2 \leq i \leq n$, $b_{x_1^i} = |Y_1^i| - 0 = p_1$, for all $x_1^i \in X_1^i$;
- for all $1 \leq i \leq m$, $b_{y_i^1} = 0 - |X_i^1| = -q_1$, for all $y_i^1 \in Y_i^1$;
- and for $2 \leq i \leq n$, $b_{y_1^i} = 0 - |X_1^i| = -q_i$, for all $y_1^i \in Y_1^i$.

Therefore, imbalance set of D is $P \cup Q$. $\square$

References

[1] P. Avery, *Score sequences of oriented graphs*, J. Graph Theory, **15, 3** (1991) 251-157.

[2] M. Dhruv, T. G. Will and D.B. West, *Realizing degree imbalances in directed graphs*, Discrete Mathematics, **239**(2001) 147-153.

[3] S. Madhukar, *Some properties of oriented graphs*, Ph.D.Thesis, AMU Aligarh(2006).

[4] S. Pirzada and T. A. Naikoo, *Score sets in oriented graphs*, Applicable Analysis and Discrete Mathematics, to appear.

[5] S. Pirzada, T. A. Naikoo and N. A. Shah, *Score sequences in oriented graphs*, J. Applied Mathematics and Computing, **23, 1-2** (2007) 257-268.