*Kragujevac J. Math.* 31 (2008) 53–57.

# AN INEQUALITY FOR THE LEBESGUE MEASURE AND ITS FURTHER APPLICATIONS

**Ivan D. Aranđelović[1] and Dojčin S. Petković[2]**

[1]*Faculty of Mechanical Engineering,*
*Kraljice Marije 16, 11000 Beograd, Serbia*
(e-mail: iarandjelovic@mas.bg.ac.yu)

[2]*Faculty of Science and Mathematics,*
*Knjaza Miloša 7, 28220 Kosovska Mitrovica, Serbia*

*(Received May 26, 2008)*

**Abstract.** In [*Univ. Beograd Publ. Elektrotehn. Fak. Ser. Math.* **15** (2004), 85–86], the first author of this paper proved a new inequality for the Lebesgue measure and gave some applications. Here, we present a new application of this inequality.

## 1. INTRODUCTION

If $\lambda$ is the Lebesgue measure on the set of real numbers $\mathbf{R}$ and $\{A_n\}$ sequences of Lebesgue measurable sets in $\mathcal{R}$, then we have the following inequality:

$$\lambda(\underline{\lim} A_n) \leq \underline{\lim} \lambda(A_n).$$

But for the inequality

$$\overline{\lim} \lambda(A_n) \leq \lambda(\overline{\lim} A_n)$$

we must suppose that $\lambda(\cup_{i=n}^{\infty} A_n) < \infty$ for at least one value of $n$ (see [7], p 40.). Example: for a family of intervals $I_n = [n, n+1)$, $n = 0, 1, \ldots$, we have: $\overline{\lim}\lambda(A_n) = 1$ and $\lambda(\overline{\lim} A_n) = 0$.

In [3] first author present the following inequality, and as its applications short and simple proofs of two famous Steinhaus' results.

**Proposition 1.** *Let $A$ be a measurable set of a positive measure and $\{x_n\}$ a bounded sequence of real numbers. Then*

$$\lambda(A) \leq \lambda(\overline{\lim}(x_n + A)).$$

Further applications of this inequality was given in [4].

## 2. RESULTS

Now we prove the following additive form of the Aljančić -Aranđelović [1] Uniform convergence theorem, which is one the fundamental results in the theory of $\mathcal{O}$ - regularly varying functions (see also [2], [5] or [6]).

**Proposition 2. (S. Aljančić and D. Aranđelović [1] )** *Let $f, g : \mathbf{R} \to \mathbf{R}$ be a measurable functions such that*

$$\overline{\lim_{s\to\infty}}(f(t+s) - f(s)) = g(t) < +\infty,$$

*for all $t \in \mathbf{R}$. Then*

$$\overline{\lim_{s\to\infty}} \sup_{t\in[a,b]} (f(t+s) - f(s)) < +\infty,$$

*for any $a, b \in \mathbf{R}$, $(a < b)$.*

**Proof.** Let

$$g_r(t) = \sup_{x\geq r}(f(t+x) - f(x)).$$

Hence

$$\lim_{r\to\infty} g_r(t) = g(t) < +\infty$$

for all $t \in \mathbf{R}$. By Egoroff's theorem follows that for any $a, b \in \mathbf{R}$ $(a < b)$, there exists measurable set $A \subseteq [a, b]$ such that $\lambda(A) > 0$ and convergence is uniform on $A$. So

$$\overline{\lim_{s\to\infty}} \sup_{t\in A}(f(t+s) - f(s)) < +\infty.$$

Assume now that convergence is not uniform on $[a, b]$. Then there exists $\{x_n\} \subseteq [a, b]$ and $\{y_n\} \subseteq \mathbf{R}$ such that $\lim y_n = \infty$ and

$$\lim(f(x_n + y_n) - f(y_n)) = \infty.$$

By Proposition 1, it follows that

$$\lambda(\overline{\lim}(A - x_n)) \geq \lambda(A) > 0,$$

which implies that there exists $t \in \mathbf{R}$ and subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ such that $\{t + x_{n_j}\} \subseteq A$. Then

$$|f(x_{n_j} + y_{n_j}) - f(y_{n_j})| \leq |f(x_{n_j} + t + y_{n_j} - t) - f(y_{n_j} - t)| + |f(y_{n_j} - t) - f(y_{n_j})|.$$

Now

$$\overline{\lim}|f(x_{n_j} + t + y_{n_j} - t) - f(y_{n_j} - t)| < \infty,$$

because $\{t + x_{n_j}\} \subseteq A$ and $\lim(y_{n_j} - t) = \infty$. From

$$\overline{\lim}(f(y_{n_j} - t) - f(y_{n_j})) < \infty$$

follows

$$\underline{\lim}(f(x_n + y_n) - f(y_n)) < \infty,$$

which is a contradiction. □

## 3. ADDITIONAL COMMENTS

Condition $g$ is measurable can not be omitted because it apply Egoroff's theorem to function $g$ which introduce assumption that $g$ is measurable.

**Proposition 3.** *There exists measurable function $f : \mathbf{R} \to \mathbf{R}$ such that*

$$\overline{\lim}_{s\to\infty}(f(t+s) - f(s)) = g(t) < +\infty$$

*and $g$ is non measurable.*

**Proof.** Let $A \subseteq \mathbf{R}$ be a measurable set such that:

1°) $A = -A$;

2°) $A + 4 = A$;

3°) $A - A$ is the non measurable set;

4°) $A \cap (A + 2) = \emptyset$.

For existence of such set see [8].

Let $\varphi_B$ denote characteristic function of set $B$. The function $f : \mathbf{R} \to \mathbf{R}$ defined by

$$f(x) = \varphi_A(x) - \varphi_{A+2}(x) = \begin{cases} -1, & x \in (A+2); \\ 0, & x \notin A \cup (A+2); \\ 1, & x \in A, \end{cases}$$

is measurable. But function $g$ defined by

$$g(t) = \overline{\lim_{x\to\infty}}(f(x+t) - f(x)),$$

is non measurable because such is the set

$$\{t \mid g(t) = 2\} = A - (A+2) = A - A - 2.$$

□

# References


[1] S. Aljančić and D. Aranđelović, $\mathcal{O}$ − *Regularly varying functions*, Publ. Inst. Math. (Beograd) **22 (36)** (1977), 5-22.

[2] D. Aranđelović, $\mathcal{O}$ − *Regularly variation and uniform convergence*, Publ. Inst. Math. (Beograd) **48 (62)** (1990), 25-40.

[3] I. Aranđelović, *An inequality for the Lebesgue measure*, Univ. Beog. Publ. Elek. Fak. ser. Math. **15** (2004) 85-86.

[4] I. Aranđelović and D. Petković, *An inequality for the Lebesgue measure and its applications*, Facta Universitatis **22** (2007) 11 - 14.

[5] N. H. Bingham, C. M. Goldie and J. L. Teugels, *Regular Variation*, Cambridge Univ. Press 1987.

[6] J. L. Geluk and L. de Haan, *Regular variation, extensions and Tauberian theorems*, CWI Tract 40, Amsterdam 1987.

[7] P. Halmos, *Measure theory*, Van Nonstrand, Princeton 1950.

[8] L. A. Rubel, *A pathological Lebesgue-measurable function*, J. London Math. Soc. **38** (1963) 1-4.