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ANOTHER WEIGHTED OSTROWSKI-GRÜSS TYPE INEQUALITY FOR TWICE DIFFERENTIABLE MAPPINGS

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Abstract. We establish another weighted Ostrowski-Grüss type inequality for twice differentiable mappings in terms of the upper and lower bounds of second derivative. The new established inequality is then applied to numerical integration.

1. INTRODUCTION

We assume that the weight function $w : (a, b) \longrightarrow [0, \infty)$, is integrable, non-negative and

$$\int_a^b w(t)dt < \infty.$$

The domain of w may be finite or infinite and w may vanish at the boundary points. We denote the moments to be m, M, N and notations μ and σ as:

$$\begin{aligned} m(a,b) &= \int_a^b w(t)dt, \quad M(a,b) = \int_a^b tw(t)dt, \quad N(a,b) = \int_a^b t^2 w(t)dt, \\ \mu(a,b) &= \frac{M(a,b)}{m(a,b)} \\ \text{and} \quad \sigma^2(a,b) &= \frac{N(a,b)}{m(a,b)} - \mu^2(a,b). \end{aligned}$$

The following integral inequality is well known in the literature as weighted Grüss inequality [3].

Theorem 1. *Let f, g : $[a,b] \to \mathbb{R}$ be two integrable functions such that $\varphi \le f(x) \le \Phi$ and $\gamma \le g(x) \le \Gamma$ for all $x \in [a,b]$ and φ, Φ, γ and Γ are constants. Then we have*

$$\begin{aligned} &\left| \frac{1}{m(a,b)} \int_a^b f(x)g(x)w(x)dx - \frac{1}{m(a,b)} \int_a^b f(x)w(x)dx \times \frac{1}{m(a,b)} \int_a^b g(x)w(x)dx \right| \\ \le& \frac{1}{4} (\Phi - \varphi)(\Gamma - \gamma). \end{aligned} \tag{1.1}$$

The constant $\frac{1}{4}$ is sharp. □

In [5] Dragomir et al, pointed out Ostrowski-Grüss type inequality for single differentiable mappings in terms of the upper and lower bounds of first derivative. The inequality then applied to numerical integration and for special means and given in the form of the following theorem:

Theorem 2. *Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuous on $[a,b]$ and differentiable function on (a,b), whose first derivative satisfies the condition:*

$$\gamma \le f'(x) \le \Gamma,$$

for all $x \in (a,b)$.

Then, we have the inequality:

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{1}{4} (b-a) (\Gamma - \gamma), \qquad (1.2)$$

for all $x \in (a, b)$. □

In $[1, 2]$ Barnett et al, and also Cerone et al, established Ostrowski-Grüss type inequality for twice differentiable mappings in terms of the upper and lower bounds of second derivative in the form of the following theorem:

Theorem 3. *Let* $f : [a, b] \longrightarrow \mathbb{R}$ *be continuous on* $[a, b]$ *and twice differentiable function on* (a, b), *whose second derivative* $f'' : (a, b) \longrightarrow \mathbb{R}$ *satisfies the condition:*

$$\varphi \leq f''(x) \leq \Phi,$$

for all $x \in (a, b)$.

Then, we have the following inequality:

$$\left| f(x) - \left(x - \frac{a+b}{2} \right) f'(x) + \left[\frac{(b-a)^2}{24} + \frac{1}{2} \left(x - \frac{a+b}{2} \right)^2 \right] \frac{f'(b) - f'(a)}{b-a} \right.$$
$$\left. - \frac{1}{b-a} \int_a^b f(t)dt \right|$$
$$\leq \frac{1}{8} (\Phi - \varphi) \left[\frac{1}{2} (b-a) + \left| x - \frac{a+b}{2} \right| \right]^2, \qquad (1.3)$$

for all $x \in [a, b]$. □

The main aim of this paper is to point out new estimation of (1.3) and to apply it in numerical integration. It turns out that these new estimations can give much better results than estimations based on (1.3). Some closely related new results are also given.

2. MAIN RESULTS

We now give our main theorem:

Theorem 4. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b) and whose second derivative $f'':(a,b)\to\mathbb{R}$, satisfies the condition $\varphi\le f''(x)\le\Phi$ for all $x\in(a,b)$. Then we have*

$$\left|f(x)-(x-\mu(a,b))f'(x)+\left[(x-\mu(a,b))^2+\sigma^2(a,b)\right]f''(x)\right.$$
$$\left.-\frac{1}{m(a,b)}\int_a^b f(t)w(t)dt\right|$$
$$\le\frac{1}{8}(\Phi-\varphi)\left[x-\mu(a,b)+\frac{1}{m(a,b)}\left|\int_a^b sgn(t-x)(x-t)w(t)dt\right|\right],\qquad(2.1)$$

for all $x\in[a,b]$.

Proof. Let us define the mapping $P(.,.):[a,b]^2\longrightarrow\mathbb{R}$, given by

$$P(x,t)=\begin{cases}\int_a^t(t-u)w(u)du \text{ if } t\in[a,x],\\ \int_b^t(t-u)w(u)du \text{ if } t\in(x,b].\end{cases}\qquad(2.2)$$

Using integrating by parts techniques, the moment's and notations, Dragomir et al proved the following identity in [4]:

$$\int_a^b P(x,t)f''(t)dt=m(a,b)(x-\mu(a,b))f'(x)-m(a,b)f(x)+\int_a^b f(t)w(t)dt.\qquad(2.3)$$

From (2.2), it can be easily seen that

$$\int_a^b P(x,t)dt=m(a,b)\left[(x-\mu(a,b))^2+\sigma^2(a,b)\right].\qquad(2.4)$$

Now, let us observe that the Kernel P satisfies the estimation,

$$
\begin{aligned}
0 &\leq P(x,t) \leq \max \begin{cases} \int\limits_x^b (x-u)w(u)du \ \text{ if } x \in [a, \frac{a+b}{2}] \\ \int\limits_a^x (x-u)w(u)du \ \text{ if } \ x \in (\frac{a+b}{2}, b] \end{cases} \\
&= \max \left(\int\limits_x^b (x-u)w(u)du \ , \int\limits_a^x (x-u)w(u)du \right) \\
&= \frac{1}{2} \left(\int\limits_a^b (x-t)w(t)dt + \left| \int\limits_a^b sgn(t-x)(x-t)w(t)dt \right| \right). \qquad (2.5)
\end{aligned}
$$

Applying weighted Grüss integral inequality (1.1) for the mappings $f(\cdot) = f''(\cdot)$, $g(\cdot) = \frac{P(x,\cdot)}{w(x)}$ and using (2.5), we get

$$
\begin{aligned}
&\left| \frac{1}{m(a,b)} \int\limits_a^b P(x,t)f''(t)dt - \frac{1}{m(a,b)} \int\limits_a^b f''(t)w(t)dt \frac{1}{m(a,b)} \int\limits_a^b P(x,t)dt \right| \\
&\leq \frac{1}{8}(\Phi - \varphi) \left(\int\limits_a^b (x-t)w(t)dt + \left| \int\limits_a^b sgn(t-x)(x-t)w(t)dt \right| \right),
\end{aligned}
$$

implies

$$
\begin{aligned}
&\left| \frac{1}{m(a,b)} \int\limits_a^b P(x,t)f''(t)dt - \frac{f''(x)}{m(a,b)} \int\limits_a^b P(x,t)dt \right| \\
&\leq \frac{1}{8}(\Phi - \varphi) \left(\int\limits_a^b (x-t)w(t)dt + \left| \int\limits_a^b sgn(t-x)(x-t)w(t)dt \right| \right). \qquad (2.6)
\end{aligned}
$$

Using (2.3) and (2.4) in (2.6), we obtain the desired inequality (2.1). □

Remark 1. If we put $w(t) = 1$ in (2.1), we get (1.3). It shows that (1.3) is a special case of (2.1).

Corollary 1. *Under the assumptions of theorem 4 and putting $x = \frac{a+b}{2}$ in (2.1), we have the mid -point like inequality:*

$$\left| f\left(\frac{a+b}{2}\right) - \left(\frac{a+b}{2} - \mu(a,b)\right) f'\left(\frac{a+b}{2}\right) \right.$$
$$\left. + \left[\left(\frac{a+b}{2} - \mu(a,b)\right)^2 + \sigma^2(a,b)\right] f''\left(\frac{a+b}{2}\right) - \frac{1}{m(a,b)} \int\limits_a^b f(t)w(t)dt \right|$$
$$\leq \frac{1}{8}(\Phi - \varphi) \times \left[\frac{a+b}{2} - \mu(a,b) + \frac{1}{m(a,b)} \left| \int\limits_a^b sgn\left(t - \frac{a+b}{2}\right)\left(\frac{a+b}{2} - t\right) w(t)dt \right| \right]. \tag{2.7}$$

□

Corollary 2. *Under the assumptions of theorem 4, we have the following trapezoidal like inequality:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{2}\left[af'(a) + bf'(b) - \mu(a,b)\left(f'(b)+f'(a)\right)\right] \right.$$
$$+\frac{1}{2}\left[(a-\mu(a,b))^2 f''(a) + (b-\mu(a,b))^2 f''(b) + \sigma^2(a,b)\left(f''(b)+f''(a)\right)\right]$$
$$\left. - \frac{1}{m(a,b)} \int\limits_a^b f(t)w(t)dt \right|$$
$$\leq \frac{\Phi - \varphi}{8}\left[b - \mu(a,b)\right]. \tag{2.8}$$

Proof. The inequality (2.8) can be drawn from (2.1) with $x = a$ and $x = b$, adding the results, using the triangular inequality of modulus and dividing by 2. □

3. APPLICATIONS IN NUMERICAL INTEGRATION

Let $I_n : a = x_0 < x_1 < x_2 < .... < x_{n-1} < x_n = b$ be the division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$, $i = 1, 2, ..., n-1$. We have the following quadrature formula:

Theorem 5. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b) and $f'' : (a,b) \to \mathbb{R}$, satisfies the condition $\varphi \leq f''(x) \leq \Phi$ for all $x \in (a,b)$.*

Then, we have the following perturbed Riemann's type quadrature formula:

$$\int_a^b f(t)w(t)dt = A(f, f', f'', \xi, I_n) + R(f, f', f'', \xi, I_n), \tag{3.1}$$

where

$$\begin{aligned} & A(f, f', f'', \xi, I_n) \\ = & \sum_{i=0}^{n-1} m(x_i, x_{i+1}) f(\xi_i) - \sum_{i=0}^{n-1} m(x_i, x_{i+1})(\xi_i - \mu(x_i, x_{i+1})) f'(\xi_i) \\ & + \sum_{i=0}^{n-1} m(x_i, x_{i+1}) \left[(\xi_i - \mu(x_i, x_{i+1}))^2 + \sigma^2(x_i, x_{i+1}) \right] f''(\xi_i), \end{aligned} \tag{3.2}$$

and the remainder term satisfies the estimation:

$$\begin{aligned} & |R(f, f', f'', \xi, I_n)| \\ \leq & \frac{1}{8}(\Phi - \phi) \sum_{i=0}^{n-1} \left(m(x_i, x_{i+1})(\xi_i - \mu(x_i, x_{i+1})) + \left| \int_{x_i}^{x_{i+1}} sgn(t - \xi_i)(\xi_i - t)w(t)dt \right| \right), \end{aligned} \tag{3.3}$$

for all $\xi_i \in [x_i, x_{i+1}]$ *where* $h_i = x_{i+1} - x_i$ *for* $i = 1, 2, \ ..., \ n-1$.

Proof. Apply theorem 4 on the interval $[x_i, x_{i+1}]$, $\xi_i \in [x_i, x_{i+1}]$ where $h_i = x_{i+1} - x_i$ for $i = 1, 2, \ ..., \ n-1$, to get

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} f(t)w(t)dt - m(x_i, x_{i+1}) f(\xi_i) + m(x_i, x_{i+1}) (\xi_i - \mu(x_i, x_{i+1})) f'(\xi_i) \right. \\ & \left. - m(x_i, x_{i+1}) \left[(\xi_i - \mu(x_i, x_{i+1}))^2 + \sigma^2(x_i, x_{i+1}) \right] f''(\xi_i) \right| \\ \leq & \frac{1}{8}(\Phi - \phi) \times \left(m(x_i, x_{i+1})(\xi_i - \mu(x_i, x_{i+1})) + \left| \int_{x_i}^{x_{i+1}} sgn(t - \xi_i)(\xi_i - t)w(t)dt \right| \right). \end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalized triangular inequality, we deduce the desired estimation (3.3). □

Corollary 3. *Under the assumptions of theorem 4, by choosing* $\xi_i = \frac{x_i + x_{i+1}}{2}$ *in the above theorem, we recapture the midpoint like quadrature formula:*

$$\int_{x_i}^{x_{i+1}} f(t)w(t)dt = A_M(f, f', f'', \xi, I_n) + R_M(f, f', f'', \xi, I_n),$$

where

$$\begin{aligned}
& A_M(f, f', f'', \xi, I_n) \\
= & \sum_{i=0}^{n-1} m(x_i, x_{i+1}) f\left(\frac{x_i + x_{i+1}}{2}\right) \\
& - \sum_{i=0}^{n-1} m(x_i, x_{i+1}) \left(\frac{x_i + x_{i+1}}{2} - \mu(x_i, x_{i+1})\right) f'\left(\frac{x_i + x_{i+1}}{2}\right) \\
& + \sum_{i=0}^{n-1} m(x_i, x_{i+1}) \left[\left(\frac{x_i + x_{i+1}}{2} - \mu(x_i, x_{i+1})\right)^2 + \sigma^2(x_i, x_{i+1})\right] f''\left(\frac{x_i + x_{i+1}}{2}\right),
\end{aligned}$$

and the remainder term satisfies the estimation:

$$\begin{aligned}
R_M(f, f', f'', \xi, I_n) \leq & \frac{1}{8}(\Phi - \phi) \sum_{i=0}^{n-1} \left[m(x_i, x_{i+1}) \left(\frac{x_i + x_{i+1}}{2} - \mu(x_i, x_{i+1})\right) \right. \\
& \left. + \left| \int_{x_i}^{x_{i+1}} sgn\left(t - \frac{x_i + x_{i+1}}{2}\right)\left(\frac{x_i + x_{i+1}}{2} - t\right) w(t) \right| \right].
\end{aligned}$$

□

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