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SOLUTION OF THE DIRICHLET PROBLEM WITH L^p BOUNDARY CONDITION

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Abstract. The solution of the Dirichlet problem for the Laplace equation is looked for in the form of the sum of a single layer and a double layer potentials with the same density f. The original problem is reduced to the solving of the integral equation with an unknown density f. The solution f of this integral equation is given by the Neumann series.

1. INTRODUCTION

This paper is devoted to the Dirichlet problem for the Laplace equation on a Lipschitz domain $G \subset \mathbf{R}^m$ with a boundary condition $g \in L^p(\partial G)$, where $m > 2$ and $2 \leq p < \infty$. This problem has been studied for years. B. J. E. Dahlberg proved in 1979 that there is a Perron-Wiener-Brelot solution u of this problem, the nontangential maximal function of u is in $L^p(\partial G)$ and $g(x)$ is the nontangential limit of u for almost all $x \in \partial G$ (see [2]). Such solutions have been studied by integral equations method. It was shown for G bounded with connected boundary that for each $g \in L^p(\partial G)$ there is $f \in L^p(\partial G)$ such that the double layer potential $\mathcal{D}f$ with density f is a solution of the Dirichlet problem with the boundary condition g (see [11, 6]). This result does not hold for a general G. If G is unbounded or if the boundary of G is not connected then for each $p \in \langle 2, \infty)$ there is $g \in L^p(\partial G)$ such that the solution of the Dirichlet problem with the boundary condition g has not a form of a double layer potential with a density from $L^p(\partial G)$. We look for a solution in another form. Denote by $\mathcal{S}f$ the single layer potential with density f. We have proved that for every $g \in L^p(\partial G)$ there is $f \in L^p(\partial G)$ such that $\mathcal{D}f + \mathcal{S}f$ is a solution of the Dirichlet problem with the boundary condition g.

We look for a solution of the Dirichlet problem in the form $\mathcal{D}f + \mathcal{S}f$. The original problem is reduced to the solving of the integral equation $Tf = g$ (see §4). If we look for a solution of the Neumann problem with the boundary condition g in the form $\mathcal{S}f$ we get the integral equation $\tau f = g$. For G bounded and convex and $p = 2$ Fabes, Sand and Seo (see [4]) proved that

$$f = -2 \sum_{j=0}^{\infty} (2\tau + I)^j g$$

is a solution of the problem $\tau f = g$. If we look for a solution of the Robin problem $\Delta u = 0$ in G, $\partial u / \partial n + hu = g$ in the form of a single layer potential $\mathcal{S}f$ we get the integral equation $\tilde{\tau} f = g$. The following result was proved in [9]: Let ∂G is locally a C^1-deformation of a boundary of a convex set (i.e. for each $x \in \partial G$ there are a convex domain $D(x)$ in $\mathbf{R}^m$, a neighbourhood $U(x)$ of x, a coordinate system centred at x and Lipschitz functions Ψ_1, Ψ_2 defined on $\{y \in \mathbf{R}^{m-1}; |y| < r\}$, $r > 0$ such that $\Psi_1 - \Psi_2$ is a function of class C^1, $(\Psi_1 - \Psi_2)(0, \dots, 0) = 0$, $\partial_j (\Psi_1 - \Psi_2)(0, \dots, 0) = 0$ for $j = 1, \dots, m-1$ and $U(x) \cap \partial G = \{[y', s]; y' \in \mathbf{R}^{m-1}, |y'| < r, s = \Psi_1(y')\}$, $U(x) \cap \partial D(x) = \{[y', s]; y' \in \mathbf{R}^{m-1}, |y'| < r, s = \Psi_2(y')\}$), $1 < p \leq 2$, $\alpha > \alpha_0$ and $g \in L^p(\partial G)$. Then

$$f = \alpha^{-1} \sum_{j=0}^{\infty} (I - \alpha^{-1} \tilde{\tau})^j g$$

is a solution of the equation $\tilde{\tau} f = g$. (Here α_0 depends on h.) Using this result we prove that for G with boundary which is locally C^1-deformation of a boundary of a convex domain, $2 \leq p < \infty$ and $g \in L^p(\partial G)$ the solution f of the equation $Tf = g$, corresponding to the Dirichlet problem with the boundary condition g, is given by

$$f = \alpha^{-1} \sum_{j=0}^{\infty} (I - \alpha^{-1} T)^j g.$$

Here

$$\alpha > \frac{1}{2} + \frac{1}{2} \|\mathcal{S} \chi_{\partial G}\|_{L^\infty(\partial G)}$$

and $\chi_{\partial G}$ is the characteristic function of ∂G.

2. FORMULATION OF THE PROBLEM

Let a domain $G \subset \mathbf{R}^m$, $m > 2$, have a compact nonempty boundary ∂G, which is locally a graph of a Lipschitz function, and $\partial G = \partial(\mathbf{R}^m \setminus \operatorname{cl} G)$. Here $\operatorname{cl} G$ denotes the closure of G. It means that for each $x \in \partial G$ there is a coordinate system centred at x and a Lipschitz function Φ in $\mathbf{R}^{m-1}$ such that $\Phi(0, \ldots, 0) = 0$ and in some neighbourhood of x the set G lies under the graph of Φ and $\mathbf{R}^m \setminus \operatorname{cl} G$ lies above the graph of Φ. (We do not suppose that ∂G is connected.) Then the outward unit normal $n(x)$ to G exists at almost any point x of ∂G.

If $x \in \partial G$, $\alpha > 0$, denote the non-tangential approach region of opening α at the point x

$$\Gamma_\alpha(x) = \{y \in G; |x - y| < (1 + \alpha) \operatorname{dist}(y, \partial G)\},$$

where $\operatorname{dist}(y, \partial G)$ is the distance of y from ∂G. If u is a function on G we denote on ∂G the non-tangential maximal function of u

$$N_\alpha(u)(x) = \sup\{u(y); y \in \Gamma_\alpha(x)\}.$$

If

$$c = \lim_{y \to x, y \in \Gamma_\alpha(x)} u(y)$$

for each $\alpha > \alpha_0$, we say that c is the nontangential limit of u at x.

Since G is a Lipschitz domain there is $\alpha_0 > 0$ such that $x \in \operatorname{cl} \Gamma_\alpha(x)$ for each $x \in \partial G$, $\alpha > \alpha_0$.

If $g \in L^p(\partial G)$, $1 < p < \infty$, we define L^p-solution of the Dirichlet problem

$$\Delta u = 0 \qquad \text{in G,} \tag{1}$$

$$u = g \qquad \text{on } \partial\text{G} \tag{2}$$

as follows:

Find a function u harmonic in G, such that $N_\alpha(u) \in L^p(\partial G)$ for each $\alpha > \alpha_0$, u has the nontangential limit $u(x)$ for almost all $x \in \partial G$ and $u(x) = g(x)$ for almost all $x \in \partial G$. If G is unbounded require moreover that $u(x) \to 0$ as $|x| \to \infty$.

We will suppose to the end of the paragraph that G is bounded. Let f be a function defined on ∂G. Denote by Φf the set of all hyperharmonic and bounded below functions u on G such that

$$\liminf_{y \to x, y \in G} u(y) \geq f(x)$$

for all $x \in \partial G$. Denote by Ψf the set of all hypoharmonic and bounded above functions u on G such that

$$\limsup_{y \to x, y \in G} u(y) \leq f(x)$$

for all $x \in \partial G$. Put $\overline{H}f(x) = \inf\{u(x); u \in \Phi f\}$, $\underline{H}f(x) = \sup\{u(x); u \in \Psi f\}$. Then $\underline{H}f \leq \overline{H}f$ (see [1], Theorem 6.2.5). If $\underline{H}f = \overline{H}f$ we write $Hf = \overline{H}f$. If $\underline{H}f = \overline{H}f$ then $Hf \equiv +\infty$ or $Hf \equiv -\infty$ or Hf is a harmonic function in G (see [1], Theorem 6.2.5). A function f is called resolutive if $\underline{H}f$ and $\overline{H}f$ are equal and finite-valued. If f is resolutive then Hf is called the PWB-solution (Perron-Wiener-Brelot solution) of the Dirichlet problem with the boundary condition f. If $x \in G$ then there is a unique probabilistic measure μ_x supported on ∂G such that

$$Hf(x) = \int_{\partial G} f \, d\mu_x$$

for each resolutive function f (see [1], §6.4). The measure μ_x is called the harmonic measure.

Let $1 < p < \infty$. If G has not boundary of class C^1 suppose $2 \leq p < \infty$. Let $g \in L^p(\partial G)$ and u be a harmonic function in G. Then u is a PWB-solution of the

Dirichlet problem with the boundary condition g if and only if u is an L^p-solution of the problem (1)-(2) (see Theorem 4.2).

For $1 < p < \infty$ and $0 < s < 1$ the Sobolev space L^p_s is defined by

$$L^p_s = \{(I - \Delta)^{-s/2} g; g \in L^p(\mathbf{R}^m)\}.$$

Define

$$S_s f(x) = \left(\int_0^\infty \left(\int_{|y|<1} |f(x+ry) - f(x)| dy \right)^2 \frac{dr}{r^{1+2s}} \right)^{1/2}.$$

Remark that a function f belongs to L^p_s if and only if $f \in L^p(\mathbf{R}^m)$ and $S_s f \in L^p(\mathbf{R}^m)$ (see [5], Theorem 3.4). Define $L^p_s(G)$ as the space of restrictions of functions in L^p_s to G.

For $0 < s < 1$, $1 < p, q < \infty$ let us introduce Besov spaces

$$B^{p,q}_s \equiv \left\{ f \in L^p(\mathbf{R}^m); \int \frac{1}{|y|^{m+ps}} \left[\int |f(x) - f(x+y)|^p dx \right]^{q/p} dy < \infty \right\}.$$

Define $B^{p,q}_s(G)$ as the space of restrictions of functions in $B^{p,q}_s$ to G.

Remark 2.1. *Let $2 \leq p < \infty$, G be bounded, $g \in L^p(\partial G)$. If u is an L^p-solution of the Dirichlet problem (1), (2) then $u \in L^p_{1/p}(G) \cap B^{p,p}_{1/p}(G)$.*

Proof. According to [5], Theorem 5.15 there is $v \in L^p_{1/p}(G)$ which is an L^p-solution of the problem (1), (2). The uniqueness of an L^p-solution of the Dirichlet problem (see [6], Corollary 2.1.6 or [5], Theorem 5.3) gives that $u = v \in L^p_{1/p}(G)$. Since $u \in L^p_{1/p}(G)$ we get $u \in B^{p,p}_{1/p}(G)$ by [5], Theorem 4.1 and [5], Theorem 4.2. □

Remark 2.2. *Let G be a bounded domain with boundary of class C^1. Let $1 < p \leq 2$, $g \in L^p(\partial G)$. If u is an L^p-solution of the Dirichlet problem (1), (2) then $u \in B^{p,2}_{1/p}(G)$.*

Proof. According to [5], Theorem 5.15 there is $v \in B^{p,2}_{1/p}(G)$ which is an L^p-solution of the problem (1), (2). The uniqueness of an L^p-solution of the Dirichlet problem (see [5], Theorem 5.3) gives that $u = v \in B^{p,2}_{1/p}(G)$. □

3. POTENTIALS

The solution of the Dirichlet problem has been looked for in the form of a double layer potential.

Denote by $\Omega_r(x)$ the open ball with the center x and the radius r and by $\mathcal{H}_k$ the k-dimensional Hausdorff measure normalized so that $\mathcal{H}_k$ is the Lebesque measure in $\mathbf{R}^k$.

Fix $f \in L^p(\partial G)$, $1 < p < \infty$. Define

$$\mathcal{D}f(x) = \frac{1}{\mathcal{H}_{m-1}(\partial\Omega_1(0))} \int_{\partial G} f(y) \frac{n(y) \cdot (x-y)}{|x-y|^m} \, d\mathcal{H}_{m-1}(y)$$

the double layer potential with density f and

$$\mathcal{S}f(x) = \frac{1}{(m-2)\mathcal{H}_{m-1}(\partial\Omega_1(0))} \int_{\partial G} f(y)|x-y|^{2-m} \, d\mathcal{H}_{m-1}(y)$$

the single layer potential with density f whenever these integrals have a sense.

The potentials $\mathcal{D}f, \mathcal{S}f$ are harmonic functions in G, $N_\alpha(\mathcal{D}f) \in L^p(\partial G)$, $N_\alpha(\mathcal{S}f) \in L^p(\partial G)$ and $\mathcal{S}f(x)$ is the nontangential limit of $\mathcal{S}f$ at x for almost all $x \in \partial G$ (see [6], Theorem 2.2.13 and [11], Lemma 2.18).

For $\epsilon > 0$, $x \in \partial G$ define

$$K_\epsilon f(x) = \frac{1}{\mathcal{H}_{m-1}(\Omega_1(0))} \int_{\partial G \setminus \Omega_\epsilon(x)} \frac{n(y) \cdot (x-y)}{|y-x|^m} f(y) \, d\mathcal{H}_{m-1}(y),$$

$$K_\epsilon^* f(x) = \frac{1}{\mathcal{H}_{m-1}(\Omega_1(0))} \int_{\partial G \setminus \Omega_\epsilon(x)} \frac{n(x) \cdot (y-x)}{|y-x|^m} f(y) \, d\mathcal{H}_{m-1}(y).$$

Then for almost all $x \in \partial G$ there are

$$Kf(x) = \lim_{\epsilon \to 0_+} K_\epsilon f(x), \qquad K^* f(x) = \lim_{\epsilon \to 0_+} K_\epsilon^* f(x).$$

Moreover, $\frac{1}{2} f(x) + Kf(x)$ is the nontangential limit of $\mathcal{D}f$ at x for almost all $x \in \partial G$ (see [6], Theorem 2.2.13). The operators K, K^* are bounded operators in $L^p(\partial G)$ (see [6], Theorem 2.2.13). The operator K in $L^p(\partial G)$ and the operator K^* in $L^{p/(p-1)}(\partial G)$ are adjoint operators.

4. SOLVABILITY OF THE PROBLEM

We will look for an L^p-solution of the Dirichlet problem (1), (2) in a form

$$u = \mathcal{D}f + \mathcal{S}f \tag{3}$$

with $f \in L^p(\partial G)$. Then u is an L^p-solution of the problem (1), (2) if and only if

$$Tf = g \tag{4}$$

where

$$Tf = \frac{1}{2}f + Kf + \mathcal{S}f. \tag{5}$$

Denote by T^* the adjoint operator of T. Then $T^*f = \frac{1}{2}f + K^*f + \mathcal{S}f$ for $f \in L^{p/(p-1)}(\partial G)$.

Lemma 4.1. *Let $1 < p < \infty$. If G has not boundary of class C^1 suppose that $p \geq 2$. Then T is a continuously invertible operator in $L^p(\partial G)$.*

Proof. The operator T^* is continuously invertible in $L^{p/(p-1)}(\partial G)$ by [9], Theorem 5.2, [9], Theorem 5.3 and [9], Theorem 6.3. Therefore T is a continuously invertible operator in $L^p(\partial G)$ (see [12], Chapter VIII, §6, Theorem 2). □

Theorem 4.2. *Let $1 < p < \infty$. If G has not boundary of class C^1 suppose that $p \geq 2$. If $g \in L^p(\partial G)$ then $\mathcal{D}(T^{-1}g) + \mathcal{S}(T^{-1}g)$ is a unique L^p-solution of the Dirichlet problem for the Laplace equation with the boundary condition g. If G is bounded then $\mathcal{D}(T^{-1}g) + \mathcal{S}(T^{-1}g)$ is a PWB-solution of the Dirichlet problem for the Laplace equation with the boundary condition g.*

Proof. Since there is $T^{-1}g$ by Lemma 4.1 the function $\mathcal{D}(T^{-1}g) + \mathcal{S}(T^{-1}g)$ is an L^p-solution of the problem.

Suppose now that G is bounded. Let u be an L^p-solution of the problem (1), (2). According to [2], Theorem 3 there is $f \in L^p(\partial G)$ such that

$$\lim_{y \to x} |Hf(y) - u(y)| = 0$$

for each $x \in \partial G$, where Hf is the PWB-solution of the Dirichlet problem with the boundary condition f. Then the nontangential limit of Hf at x is $g(x)$ at almost all $x \in \partial G$. Let $\{f_n\}$ be a sequence of functions from $C(\partial G)$ such that $f_n \to f$ in $L^p(\partial G)$ as $n \to \infty$. We can suppose that $f_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in \partial G$ (compare [13], Theorem 1.6.1). Let $\alpha > 0$ be such that $x \in \mathrm{cl}\,\Gamma_\alpha(x)$ for each $x \in \partial G$. According to [2], Theorem 2 there is a constant C_α such that

$$\int_{\partial G} [N_\alpha(Hf - Hf_n)(x)]^p \, d\mathcal{H}_{m-1}(x) \leq C_\alpha \int_{\partial G} |f(x) - f_n(x)|^p \, d\mathcal{H}_{m-1}(x). \tag{6}$$

We can suppose that

$$\int_{\partial G} |f(x) - f_n(x)|^p \, d\mathcal{H}_{m-1}(x) \leq n^{-2p}. \tag{7}$$

Denote $K_n = \{x \in \partial G; N_\alpha(Hf - Hf_n)(x)] \geq 1/n\}$. According to (6), (7) we have $\mathcal{H}_{m-1}(K_n) \leq C_\alpha n^{-p}$. Put

$$K = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} K_n.$$

Then $\mathcal{H}_{m-1}(K) = 0$. Fix now $x \in \partial G \setminus K$ such that $f_n(x) \to f(x)$ as $n \to \infty$. Fix $\epsilon > 0$. Fix $n_0 \geq 3/\epsilon$ such that $|f(x) - f_n(x)| < \epsilon/3$ for each $n \geq n_0$. Fix now $n \geq n_0$ such that $x \notin K_n$. Since $f_n \in C(\partial G)$ there is $\delta > 0$ such that $|Hf_n(y) - f_n(y)| < \epsilon/3$ for each $y \in G$, $|y - x| < \delta$ (see [1], Theorem 6.6.15). If $y \in \Gamma_\alpha(x)$, $|x - y| < \delta$ then $|Hf(y) - f(x)| \leq |Hf(y) - Hf_n(y)| + |Hf_n(y) - f_n(x)| + |f_n(x) - f(x)| < \epsilon$. This gives

$$\lim_{y \to x, y \in \Gamma_\alpha(x)} Hf(y) = f(x). \tag{8}$$

Since $g(x)$ is the nontangential limit of Hf at x and (8) holds for almost all $x \in \partial G$ we deduce that $f = g$ almost everywhere in ∂G. Since the harmonic measure for G is absolutely continuous with respect to the surface measure on ∂G (see [3], Theorem 1) we have that $u = Hf = Hg$.

Let now $g \equiv 0$ and u be an L^p-solution of the problem (1), (2). If G is bounded then u is a PWB-solution of the Dirichlet problem with zero boundary condition. Since $0 \leq \underline{H}g = u = \overline{H}g \leq 0$ we deduce $u \equiv 0$. Suppose now that G is unbounded. Fix $R > 0$ such that $\partial G \subset \Omega_R(0)$. Put $G_R = G \cap \Omega_R(0)$, $g_R = 0$ on ∂G, $g_R = u$

on $\partial\Omega_R(0)$. Since the set G_R is regular (see [1], Theorem 6.6.15) there is a classical solution of the Dirichlet problem for G_R with the boundary condition g_R. Since u is an L^p-solution of the Dirichlet problem for G_R with the boundary condition g_R we deduce from the uniqueness of an L^p-solution of the Dirichlet problem that u is a classical solution of the Dirichlet problem, i.e. that $u \in C(\mathrm{cl}\, G_R)$. Since $u \in C(\mathrm{cl}\, G)$, $u = 0$ on ∂G and $u(x) \to 0$ as $|x| \to \infty$ we get from the maximum principle that $u \equiv 0$. $\square$

5. SOLUTION OF THE PROBLEM

Definition 5.1. *For a real vector space Y denote by* $\mathrm{compl}\, Y = \{x + iy; x, y \in Y\}$ *its complexification. If R is a linear operator in Y define $R(x + iy) = Rx + iRy$ its extension onto* $\mathrm{compl}\, Y$. *Let Q be a bounded linear operator on the complex Banach space X. The operator Q is called Fredholm if $\alpha(Q)$, the dimension of the kernel of Q, is finite, the range of Q is a closed subspace of X and $\beta(Q)$, the dimension of the kernel of the adjoint of Q, is finite. The number $i(Q) = \alpha(Q) - \beta(Q)$ is the index of Q. Denote by $r_e(Q) = \sup\{|\lambda|; \lambda I - Q$ is not a Fredholm operator with index $0\}$ the essential spectral radius of Q.*

Proposition 5.2. *Let $1 < p < \infty$, ∂G is of class C^1. Then $r_e(T - \frac{1}{2}I) = 0$ in* $\mathrm{compl}\, L^p(\partial G)$.

Proof. Since $T - \frac{1}{2}I$ is a compact operator by [6], Corollary 2.2.14 and [9], Lemma 3.1 we obtain $r_e(T - \frac{1}{2}I) = 0$ in $\mathrm{compl}\, L^p(\partial G)$ (see [10], Theorem 4.12). $\square$

Proposition 5.3. *Let $2 \le p < \infty$. Suppose that for each $x \in \partial G$ there are a convex domain $D(x)$ in $\mathbf{R}^m$, a neighbourhood $U(x)$ of x, a coordinate system centred at x and Lipschitz functions Ψ_1, Ψ_2 defined on $\{y \in \mathbf{R}^{m-1}; |y| < r\}$, $r > 0$ such that $\Psi_1 - \Psi_2$ is a function of class C^1, $(\Psi_1 - \Psi_2)(0, \dots, 0) = 0$, $\partial_j(\Psi_1 - \Psi_2)(0, \dots, 0) = 0$ for $j = 1, \dots, m-1$ and $U(x) \cap \partial G = \{[y', s]; y' \in \mathbf{R}^{m-1}, |y'| < r, s = \Psi_1(y')\}$, $U(x) \cap \partial D(x) = \{[y', s]; y' \in \mathbf{R}^{m-1}, |y'| < r, s = \Psi_2(y')\}$. Then $r_e(T - \frac{1}{2}I) < \frac{1}{2}$ in* $\mathrm{compl}\, L^p(\partial G)$.

Proof. $r_e(T^* - \frac{1}{2}I) < \frac{1}{2}$ in $\operatorname{compl} L^{p/(p-1)}(\partial G)$ by [9], Theorem 7.8. Using argument for adjoint operators (see [10], Theorem 7.19 and [10], Theorem 7.22) we get $r_e(T - \frac{1}{2}I) < \frac{1}{2}$ in $\operatorname{compl} L^p(\partial G)$. □

Theorem 5.4. *Let* $1 < p < \infty$ *and* $r_e(T - \frac{1}{2}I) < \frac{1}{2}$ *in* $\operatorname{compl} L^p(\partial G)$. *Put*

$$\alpha_0 = \frac{1}{2} + \frac{1}{2} \sup_{x \in \partial G} \mathcal{S}\chi_{\partial G}(x) = \frac{1}{2} + \sup_{x \in \partial G} \int_{\partial G} \frac{|x-y|^{2-m}}{2(m-2)\mathcal{H}_{m-1}(\partial\Omega_1(0))} \, d\mathcal{H}_{m-1}(y), \tag{9}$$

where $\chi_{\partial G}$ *is the characteristic function of the set* ∂G. *Then* $\alpha_0 < \infty$. *Fix* $\alpha \in (\alpha_0, \infty)$. *Then there are constants* $d \in \langle 1, \infty)$, $q \in (0,1)$ *such that for each natural number* n

$$\|(I - \alpha^{-1}T)^n\|_{L^p(\partial G)} \leq dq^n \tag{10}$$

and

$$T^{-1} = \alpha^{-1} \sum_{n=0}^{\infty} (I - \alpha^{-1}T)^n \tag{11}$$

in $L^p(\partial G)$. *If* $g \in L^p(\partial G)$ *then* $u = \mathcal{D}(T^{-1}g) + \mathcal{S}(T^{-1}g)$ *is an* L^p*-solution of the Dirichlet problem (1), (2) with the boundary condition* g.

Proof. Since $r_e(T - \frac{1}{2}I) < \frac{1}{2}$ in $\operatorname{compl} L^p(\partial G)$ we get by [10], Theorem 7.19 and [10], Theorem 7.22 that $r_e(T^* - \frac{1}{2}I) < \frac{1}{2}$ in $\operatorname{compl} L^{p/(p-1)}(\partial G)$. According to [9], Theorem 8.2 there are constants $d \in \langle 1, \infty)$, $q \in (0,1)$ such that for each natural number n

$$\|(I - \alpha^{-1}T^*)^n\|_{L^{p/(p-1)}(\partial G)} \leq dq^n.$$

Since

$$\|(I - \alpha^{-1}T)^n\|_{L^p(\partial G)} = \|(I - \alpha^{-1}T^*)^n\|_{L^{p/(p-1)}(\partial G)} \leq dq^n$$

by [10],Theorem 3.3 we get (10). Easy calculation gives (11). The rest is a consequence of the paragraph 4. □

6. SUCCESSIVE APPROXIMATION METHOD

Let $1 < p < \infty$ be such that $r_e(T - \frac{1}{2}I) < \frac{1}{2}$ in $\operatorname{compl} L^p(\partial G)$. (This is true if G is a bounded convex domain and $p \geq 2$.) Let $g \in L^p(\partial G)$. Put $\varphi = T^{-1}g$ (see

Theorem 5.4). Then $\mathcal{D}\varphi + \mathcal{S}\varphi$ is an L^p-solution of the Dirichlet problem (1), (2). We construct φ by the successive approximation method.

Fix $\alpha > \alpha_0$ where α_0 is given by (9). We can rewrite the equation $T\varphi = g$ as $\varphi = (I - \alpha^{-1}T)\varphi + \alpha^{-1}g$. Put

$$\varphi_0 = \alpha^{-1}g,$$

$$\varphi_{n+1} = (I - \alpha^{-1}T)\varphi_n + \alpha^{-1}g$$

for nonnegative integer n. Then

$$\varphi_{n+1} = \alpha^{-1}\sum_{k=0}^{n}(I - \alpha^{-1}T)^k g$$

and $\lim_{n\to\infty}\varphi_n = \varphi$ by Theorem 5.4. Since

$$\varphi - \varphi_n = \alpha^{-1}\sum_{k=n+1}^{\infty}(I - \alpha^{-1}T)^k g$$

there are constants $q \in (0,1)$ and $d \in \langle 1, \infty)$ depending only on G, p and α such that

$$\|\varphi - \varphi_n\|_{L^p(\partial G)} \le dq^n \|g\|_{L^p(\partial G)}.$$

We need to estimate α_0. The following lemma might help us.

Lemma 6.1. *Let $G_1, \ldots, G_k$ be bounded convex domains with the diameters $R_1, \ldots, R_k$. If $\partial G \subset \partial G_1 \cup \ldots \partial G_k$ then*

$$\sup_{x\in\partial G} \mathcal{S}\chi_{\partial G} \le \frac{m(m-1)}{m-2}(R_1 + \ldots + R_k).$$

Proof. Let H be a bounded convex domain with the diameter R. We estimate $\mathcal{H}_{m-1}(\partial H)$. Put

$$P_i(H) = \sup\Big\{\int_H \partial_i v; v \in C^\infty(\mathbf{R}^m), |v| \le 1\Big\}$$

for $i = 1, \ldots, m$. Since

$$\mathcal{H}_{m-1}(\partial H)) = \sup\{\int_H \sum_{i=1}^{m} \partial_i v_i; v_1, \ldots, v_m \in C^\infty(\mathbf{R}^m), \sum_{i=1}^{m} v_i^2 \le 1\}$$

by [7], p. 355 we obtain

$$P_i(H) \le \mathcal{H}_{m-1}(\partial H) \le P_1(H) + \ldots + P_m(H)$$

for $i = 1, \ldots, m$. If $y \in \mathbf{R}^{m-1}$ and $G \cap \{[y,t]; t \in \mathbf{R}^1\} \neq \emptyset$ then $G \cap \{[y,t]; t \in \mathbf{R}^1\} = \{[y,t]; t_1(y) < t < t_2(y)\}$. Since the diameter of H is R

$$P_m(H) = \sup\Big\{ \int_{\{y \in \mathbf{R}^{m-1}; \{[y,t]; t \in \mathbf{R}^1\} \cap G \neq \emptyset\}} [v(y, t_1(y)) - v(y, t_2(y))] \, d\mathcal{H}_{m-1}(y);$$

$$v \in C^\infty(\mathbf{R}^m), |v| \le 1 \Big\} = 2\mathcal{H}_{m-1}(\{y \in \mathbf{R}^{m-1}, \{[y,t]; t \in \mathbf{R}^1\} \cap G \neq \emptyset\})$$

$$\le 2\mathcal{H}_{m-1}(\{[y,0]; y \in \mathbf{R}^{m-1}, |y| < R\}) = P_m(\Omega_R(0)) \le \mathcal{H}_{m-1}(\partial \Omega_R(0)).$$

Similarly $P_i(H) \le R^{m-1}\mathcal{H}_{m-1}(\partial\Omega_1(0))$ for $i = 1, \ldots, m$ and thus

$$\mathcal{H}_{m-1}(\partial H) \le m R^{m-1} \mathcal{H}_{m-1}(\partial \Omega_1(0)). \tag{12}$$

Put $c = (m-2)^{-1}(\mathcal{H}_{m-1}(\partial\Omega_1(0)))^{-1}$, $d = m\mathcal{H}_{m-1}(\partial\Omega_1(0))$,

$$u_j(x) = c \int_{\partial G_j} |x-y|^{2-m} \, d\mathcal{H}_{m-1}(y)$$

for $x \in \mathbf{R}^m$, $j = 1, \ldots, k$. If $x \in \partial G_j$ we get using [13], Lemma 1.5.1 and (12)

$$u_j(x) = \int_0^\infty \mathcal{H}_{m-1}(\{y \in \partial G_j; c|x-y|^{2-m} > t\}) \, dt$$

$$= cR_j^{2-m}\mathcal{H}_{m-1}(\partial G_j) + \int_{cR_j^{2-m}}^\infty \mathcal{H}_{m-1}(\partial G_j \cap \{y; |x-y| < c^{1/(m-2)} t^{2-m}\}) \, dt$$

$$\le dcR_j + \int_{cR_j^{2-m}}^\infty dc^{(m-1)/(m-2)} t^{-(m-1)/(m-2)} \, dt = \frac{m(m-1)}{m-2} R_j.$$

Since u_j is a harmonic function in $\mathbf{R}^m \setminus \partial G_j$, continuous in $\mathbf{R}^m$ (see [8], Corollary 2.17 and [8], Lemma 2.18) and $u_j(x) \to 0$ as $|x| \to \infty$ the maximum principle gives that $u_j \le R_j m(m-1)/(m-2)$ in $\mathbf{R}^m$. Hence

$$\sup_{x \in \partial G} \mathcal{S}\chi_{\partial G}(x) \le \sup_{x \in \partial G} (u_1(x) + \ldots u_k(x)) \le \frac{m(m-1)}{m-2}(R_1 + \ldots + R_k).$$

$\square$

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