Kragujevac J. Math. 30 (2007) 201–210.

CONVOLUTED C-OPERATOR FAMILIES AND ABSTRACT CAUCHY PROBLEMS

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(Received December 20, 2006)

Abstract. We present the basic structural properties of convoluted C-cosine functions and semigroups in a Banach space setting and consider the corresponding abstract Cauchy problems.

1. INTRODUCTION AND PRELIMINARIES

Local convoluted C-semigroups were introduced and studied in the papers of I. Ciorănescu and G. Lumer [3]-[5] as a generalization of local integrated C-semigroups (cf. [1], [12], [13], [18], [22], [23]). The first comprehensive look at global convoluted C-semigroups and cosine functions was obtained in [10] and [11] where we also discussed the basic properties of introduced class of analytic convoluted C-semigroups. For example, the poliharmonic operator $(-\Delta)^{2^n}$, $n \in \mathbf{N}$, acting on $L^2[0,\pi]$, with appropriate boundary conditions, generates an exponentially bounded K_n-convoluted cosine function, and consequently, an exponentially bounded, analytic K_{n+1}-convoluted semigroup of angle $\frac{\pi}{2}$, for suitable exponentially bounded kernels K_n and K_{n+1} ([11]).

Notation. By E and $L(E)$ are denoted a complex Banach space and Banach algebra of bounded linear operators on E. For a closed linear operator A on E, $D(A)$, Kern(A), $R(A)$, $\rho(A)$ denote its domain, kernel, range and resolvent set, respectively. Put $D_\infty(A) := \bigcap\limits_{n\in\mathbf{N}_0} D(A^n)$. By $[D(A)]$ is denoted the Banach space $D(A)$ endowed with the graph norm. In this paper, $C \in L(E)$ is an injective operator satisfying $CA \subset AC$. The C-resolvent set of A, denoted by $\rho_C(A)$, is defined by $\rho_C(A) := \{\lambda \in \mathbf{C} : R(C) \subset R(\lambda - A)$ and $\lambda - A$ is injective$\}$. Further on, in some statements which are to follow we use that a complex valued function $K \in L^1_{loc}([0,\tau)),\ 0 < \tau \leq \infty$ is a kernel which means that for every $\phi \in C([0,\tau))$, the assumption $\int\limits_0^t K(t-s)\phi(s)ds = 0$, for every $t \in [0,\tau)$, implies $\phi(t) = 0,\ t \in [0,\tau)$.

Recall, K-convoluted C-semigroups and functions are important tools in the study of the following abstract Cauchy problems discussed in Section 2:

$$(\Theta C) : \begin{cases} u \in C([0,\tau) : [D(A)]) \cap C^1([0,\tau) : E), \\ u'(t) = Au(t) + \Theta(t)Cx,\ t \in [0,\tau), \\ u(0) = 0, \end{cases}$$

$$(ACP_2)_\Theta : \begin{cases} u \in C([0,\tau) : [D(A)]) \cap C^2([0,\tau) : E), \\ u''(t) = Au(t) + \Theta(t)Cx + \int\limits_0^t \Theta(s)Cyds,\ t \in [0,\tau), \\ u(0) = 0,\ u'(0) = 0. \end{cases}$$

It is said that (ΘC), resp., $(ACP_2)_\Theta$, is well-posed if for every $x,\ y \in E$ there exists a unique solution of (ΘC), resp., $(ACP_2)_\Theta$. The existence of a unique solution of $(ACP_2)_\Theta$ is closely connected with the existence of a unique K-convoluted mild solution of the problem (ACP_2), where

$$(ACP_2) : \begin{cases} u \in C([0,\tau) : [D(A)]) \cap C^2([0,\tau) : E), \\ u''(t) = Au(t),\ t \in [0,\tau), \\ u(0) = x,\ u'(0) = y. \end{cases}$$

The notion of such kind of mild solutions is introduced by S. W. Wang and Z. Huang in [20] in the particular case: $\tau = \infty$ and $K(t) = \frac{t^{n-1}}{(n-1)!},\ t \geq 0,\ n \in \mathbf{N}$.

Definition 1. *Let A be a closed operator and $K \in L^1_{loc}([0,\tau)),\ 0 < \tau \leq \infty$. If there exists a strongly continuous operator family $(C_K(t))_{t\in[0,\tau)}$ such that:*

(i) $C_K(t)A \subset AC_K(t)$, $t \in [0, \tau)$,

(ii) $C_K(t)C = CC_K(t)$, $t \in [0, \tau)$ *and*

(iii) for all $x \in E$ *and* $t \in [0, \tau)$: $\int_0^t (t-s)C_K(s)xds \in D(A)$ *and*

$$A\int_0^t (t-s)C_K(s)xds = C_K(t)x - \Theta(t)Cx, \text{ where } \Theta(t) := \int_0^t K(s)ds, \qquad (1)$$

then it is said that A *is a subgenerator of a* K*-convoluted* C*-cosine function* $(C_K(t))_{t\in[0,\tau)}$. *If* $\tau = \infty$, *then we say that* $(C_K(t))_{t\geq 0}$ *is an exponentially bounded,* K*-convoluted* C*-cosine function with a subgenerator* A *if, additionally, there exist* $M > 0$ *and* $\omega \in \mathbf{R}$ *such that* $||C_K(t)|| \leq Me^{\omega t}$, $t \geq 0$.

As a consequence of (i) and (iii), we have $CA \subset AC$. Indeed, if $x \in D(A)$, choose a $t \in [0, \tau)$ with $\Theta(t) \neq 0$. Then (i) and (iii) implies $C_K(t)Ax - \Theta(t)CAx = A\int_0^t (t-s)C_K(s)Axds = A^2\int_0^t (t-s)C_K(s)xds = A[C_K(t)x - \Theta(t)Cx]$. Since $C_K(t)x \in D(A)$, we obtain $Cx \in D(A)$ and $CAx = ACx$.

The integral generator of $(C_K(t))_{t\in[0,\tau)}$ is defined by

$$\{(x, y) \in E^2 : C_K(t)x - \Theta(t)Cx = \int_0^t (t-s)C_K(s)yds, \ t \in [0, \tau)\}.$$

The integral generator of $(C_K(t))_{t\in[0,\tau)}$ is a closed linear operator which is an extension of any subgenerator of $(C_K(t))_{t\in[0,\tau)}$. Even if $(C(t))_{t\geq 0}$ is a global, exponentially bounded C-cosine function, the set of all subgenerators of $(C(t))_{t\geq 0}$ need not be monomial and, furthermore, the set of all subgenerators of a K-convoluted C-cosine function can be consisted of infinitely many elements. In order to illustrate this fact (see also [19, Example 2.14]), we would like to present a simple example appearing in [11]. Choose an arbitrary $K \in L^1_{loc}([0, \infty))$. Put $E := l_\infty$, $C\langle x_n\rangle := \langle 0, x_1, 0, x_2, 0, x_3, \ldots\rangle$ and $C_K(t)\langle x_n\rangle := \Theta(t)C\langle x_n\rangle$, $t \geq 0$, $\langle x_n\rangle \in E$. If $I \subset 2\mathbf{N}+1$, define $E_I := \{\langle x_n\rangle \in E : x_i = 0, \text{ for all } i \in (2\mathbf{N}+1) \setminus I\}$. It is clear that E_I is a closed subspace of E which contains $R(C)$ and that $E_{I_1} \neq E_{I_2}$, if $I_1 \neq I_2$. Define a closed linear operator A_I on E by: $D(A_I) = E_I$ and $A_I\langle x_n\rangle = 0$, $\langle x_n\rangle \in D(A_I)$.

It is straightforward to see that every subgenerator of $(C_K(t))_{t\geq 0}$ is of the form A_I, for some $I \subset 2\mathbf{N}+1$, and consequently, there exist the continuum many subgenerators of $(C_K(t))_{t\geq 0}$.

Definition 2. *Let A be a closed operator and K be a locally integrable function on $[0,\tau)$, $0 < \tau \leq \infty$. If there exists a strongly continuous operator family $(S_K(t))_{t\in[0,\tau)}$ such that, for $t \in [0,\tau)$, $S_K(t)C = CS_K(t)$, $S_K(t)A \subset AS_K(t)$, $\int\limits_0^t S_K(s)xds \in D(A)$, $x \in E$ and*

$$A\int\limits_0^t S_K(s)xds = S_K(t)x - \Theta(t)Cx,\ x \in E, \tag{2}$$

then $(S_K(t))_{t\in[0,\tau)}$ is called a (local) K-convoluted C-semigroup having A as a subgenerator. If $\tau = \infty$, then it is said that $(S_K(t))_{t\geq 0}$ is an exponentially bounded, K-convoluted C-semigroup with a subgenerator A if, in addition, there are constants $M > 0$ and $\omega \in \mathbf{R}$ such that $||S_K(t)|| \leq Me^{\omega t}$, $t \geq 0$.

The integral generator of $(S_K(t))_{t\in[0,\tau)}$ is defined by

$$\{(x,y) \in E^2 : S_K(t)x - \Theta(t)Cx = \int\limits_0^t S_K(s)yds,\ t \in [0,\tau)\}.$$

It is straightforward to see that the integral generator of $(S_K(t))_{t\in[0,\tau)}$ is an extension of any subgenerator of $(S_K(t))_{t\in[0,\tau)}$. Furthermore, the subgenerators of a (local) convoluted C-semigroup equipped with appropriate algebraic operations form a lattice which, in general, need not be trivial.

For the proof of the statements (a), (c), (d) and (e) of the following theorem one can proceed exactly in the same way as in [9] while the proof of (b) follows similarly as in the proof of [12, Proposition 1.3].

Proposition 1. *Suppose A is a subgenerator of a (local) K-convoluted C-semigroup $(S(t))_{t\in[0,\tau}$. Let B be the integral generator of $(S(t))_{t\in[0,\tau)}$. Then:*

(a)

$$S(t)S(s) = \left[\int\limits_0^{t+s} - \int\limits_0^{t} - \int\limits_0^{s}\right] K(t+s-r)S(r)Cdr,\quad 0 \leq t,\ s,\ t+s < \tau.$$

(b) $B = C^{-1}BC$.

If K is a kernel, then the next conditions are satisfied:

(c) (ΘC) *is well posed.*

(d) $B = C^{-1}AC$.

(e) For every $\lambda \in \rho_C(A)$ *:* $(\lambda - A)^{-1}CS(t) = S(t)(\lambda - A)^{-1}C,\ t \in [0, \tau)$.

2. RELATIONS WITH THE ABSTRACT CAUCHY PROBLEMS

In this section we relate convoluted C-cosine functions to the corresponding Cauchy problem $(ACP_2)_\Theta$. We investigate the well-posedness of this problem through the existence of K-convoluted mild solutions of (ACP_2).

Definition 3. *The abstract Cauchy problem* $(ACP_2)_\Theta$ *is well-posed if for every* $x,\ y \in E$ *there exists a unique solution* u *of it. If, additionally, for every* $x,\ y \in E$, *the solution* u *satisfies* $||u(t)|| \leq Me^{\omega t},\ 0 \leq t < \tau$, *for some* $M > 0$ *and* $\omega \in \mathbf{R}$, *then we say that* $(ACP_2)_\Theta$ *is exponentially well-posed. Further on, a function* $v \in C([0, \tau) : E)$ *is a* K*-convoluted mild solution of* (ACP_2) *at* $(x, y) \in E^2$ *if, for all* $t \in [0, \tau)$, $\int_0^t (t - s)v(s)ds \in D(A)$ *and*

$$A \int_0^t (t - s)v(s)ds = v(t) - \Theta(t)x - \int_0^t \Theta(s)yds,\ t \in [0, \tau).$$

Let $C = I$. It is clear that $u \in C^2([0, \tau) : E) \cap C([0, \tau) : [D(A)])$ is a (unique) solution of $(ACP_2)_\Theta$ on $[0, \tau)$ if and only if $v = u'' \in C([0, \tau) : E)$ is a (unique) K-convoluted mild solution of (ACP_2) on $[0, \tau)$.

Let A be a subgenerator of a K-convoluted C-cosine function $(C_K(t))_{t \in [0, \tau)}$, $0 < \tau \leq \infty$ and $x,\ y \in E$. Then it is straightforward to see that $v(t) = C_K(t)x + \int_0^t C_K(s)yds$, $t \in [0, \tau)$, is a K-convoluted mild solution of (ACP_2) at (Cx, Cy) and that $u(t) = \int_0^t (t-s)v(s)ds,\ t \in [0,\tau)$, is a solution of $(ACP_2)_\Theta$. If K is a kernel, then we would like to point out that v is a *unique* K-convoluted mild solution of (ACP_2) and u is a unique solution of $(ACP_2)_\Theta$, see [9, Proposition 4.2] and [20, Theorem 1.5].

Proposition 2. *Assume that for each $x \in R(C)$ there exists a unique K-convoluted mild solution of (ACP_2) at $(Cx, 0)$, $0 < \tau \le \infty$. Then A is a subgenerator of a K-convoluted C-cosine function on $[0,\tau)$.*

Proof. Let $t \in [0,\tau)$ and $x \in E$. Define $C_K(t)x := v(t)$, where v is the K-convoluted mild solution of (ACP_2) at $(Cx, 0)$. The uniqueness of mild solutions implies that $(C_K(t))_{t\in[0,\tau)}$ is a strongly continuous family of linear operators satisfying (iii) of Definition 1. The proof of (i) and (ii) of Definition 1 can be obtain similarly as in the proof of [20, Theorem 1.5]. For the sake of completeness, we will prove only (i). Fix an $x \in D(A)$ and define

$$\overline{C}_K(t)x := \int_0^t (t-s)C_K(s)Axds + \Theta(t)Cx,\ t \in [0,\tau).$$

Clearly, the mapping $t \mapsto \overline{C}_K(t)x$ belongs to $C([0,\tau) : E)$. Furthermore, for every $t \in [0,\tau)$,

$$\begin{aligned} A\int_0^t (t-s)\overline{C}_K(s)xds &= A\int_0^t (t-s)[\int_0^s (s-r)C_K(r)Axdr + \Theta(s)Cx]ds \\ &= \int_0^t (t-s)A\int_0^s (s-r)C_K(r)Axdrds + \int_0^t (t-s)\Theta(s)ACxds \\ &= \int_0^t (t-s)[C_K(s)Ax - \Theta(s)CAx]ds + \int_0^t (t-s)\Theta(s)ACxds \\ &= \overline{C}_K(t)x - \Theta(t)Cx. \end{aligned}$$

The uniqueness of K-convoluted mild solutions implies that $\overline{C}_K(t)x = C_K(t)x,\ t \in [0,\tau)$, i.e., $\int_0^t (t-s)C_K(s)Axds = A\int_0^t (t-s)C_K(s)xds$, for all $t \in [0,\tau)$. Differentiate the last equality twice with respect to t to obtain $C_K(t)x \in D(A)$ and $AC_K(t)x = C_K(t)Ax,\ t \in [0,\tau)$. It remains to be shown that

$C_K(t)$, $t \in [0,\tau)$, is a bounded operator. We will follow the proof of [1, Proposition 2.3] with appropriate changes. Consider the mapping $\Phi : E \to C([0,\tau) : [D(A)])$ given by

$$\Phi(x)(t) = \int_0^t (t-s)C_K(s)x\,ds,\ t \in [0,\tau)\,,\ x \in E,$$

where $C([0,\tau) : [D(A)])$ is a Fréchet space with the sequence of seminorms $(p_n)_n$, where

$$p_n(v) := \sup_{t \in \left[0,\tau-\frac{1}{n}\right]} \|v(t)\|_{[D(A)]},\ v \in C([0,\tau) : [D(A)]),\ \text{if } \tau \in (0,\infty),\ \text{resp.}$$

$$p_n(v) := \sup_{t \in [0,n]} \|v(t)\|_{[D(A)]},\ v \in C([0,\tau) : [D(A)]),\ \text{if } \tau = \infty.$$

It can be easily seen that Φ is well defined and that Φ is a linear mapping. Let us show that Φ poses a closed graph. Without loss of generality, we can assume that $\tau \in \mathbf{R}$. Suppose $x_n \to x$, and $\Phi(x_n) \to f,\ n \to \infty$. Choose a $k \in \mathbf{N}$ with $k > \frac{1}{\tau}$. Then $\sup_{t \in \left[0,\tau-\frac{1}{k}\right]} || \int_0^t (t-s)C_K(s)x_n ds - f(t)||_{[D(A)]} \to 0,\ n \to \infty$. Hence, $Af(t) = \lim_{n\to\infty} A\int_0^t (t-s)C_K(s)x_n ds = \lim_{n\to\infty}[C_K(t)x_n - \Theta(t)Cx_n],\ t \in [0,\tau)$, and $\lim_{n\to\infty} C_K(t)x_n = Af(t) + \Theta(t)Cx,\ t \in [0,\tau)$. Using the dominated convergence theorem, we have

$$f(t) = \lim_{n\to\infty} \int_0^t (t-s)C_K(s)x_n ds = \int_0^t (t-s)[Af(s) + \Theta(s)Cx]ds,\ t \in [0,\tau).$$

So, $f(0) = f'(0) = 0,\ f \in C^2([0,\tau) : E)$ and $Af(t) = f''(t) - \Theta(t)Cx,\ t \in [0,\tau)$. Hence, $A\int_0^t (t-s)v(s)ds = v(t) - \Theta(t)Cx,\ t \in [0,\tau)$, where $v = f''$. This implies $v(t) = C_K(t)x$, $t \in [0,\tau)$, and $f = \Phi(x)$. Hence, for all sufficiently large $n \in \mathbf{N}$ there is a $c_n > 0$ such that

$$\left\| A\int_0^t (t-s)C_K(s)x\,ds \right\| \le c_n\|x\|,\ x \in E,\ t \in [0,\tau - \frac{1}{n}).$$

Since $A\int_0^t (t-s)C_K(s)x\,ds = C_K(t)x - \Theta(t)Cx,\ x \in E,\ t \in [0,\tau)$, one can easily conclude that $C_K(t) \in L(E)$, $t \in [0,\tau)$.

The next corollary can be viewed as a collection of structural results related to the abstact Cauchy problems, when K is a kernel.

Corollary 1. *Suppose that K is a kernel and $0 < \tau \leq \infty$. Then the following statements are equivalent.*

(a) $(ACP_2)_\Theta$ *is well-posed.*

(b) For every $x \in E$, there exists a unique K-convoluted mild solution of (ACP_2) at $(Cx, 0)$.

(c) For every $x,\ y \in E$, there exists a unique K-convoluted mild solution of (ACP_2) at (Cx, Cy).

(d) A is a subgenerator of a K-convoluted C-cosine function on $[0, \tau)$.

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