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ON THE OSCILLATIONS OF THE SOLUTIONS OF THE EQUATION $y'' + a(x)y = f(x)$

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Abstract. It is known that the equation (2) determines non-elemental linear oscillations of the second order, which are the first generalization of ordinary harmonical oscillations. In the paper, there is firstly a resume of some our previous results which more precisely determine nature of oscillatory solutions, Sturm's theorems on locations of zeroes and extremes are given next, and finally there is discussion on oscillatoriness and general solution for arbitrary integration constants C_1 and C_2
Furthermore, for the non-homogeneous equation (2), the isue of oscillatorines of the general solution is discussed. Problems of amplitudes, oscillatoriness, resonance and stability of possible solutions are approached.
A momentum and intiative is given that the same is to be done for the most general non-homogeneous linear equations, primarily of the second order, but of higher orders as well.

1. INTRODUCTION

Very important question of entire engineering related to the theory of oscillations is the question of oscillations and zeroes of solution of the equation.

$$y'' + a(x)y = f(x) \tag{1}$$

All the questions of practical usefulness of the equation and especially the most important - the stability, depend on it.

The appropriate homogeneous equation

$$y'' + a(x)y = 0, \tag{2}$$

with the characteristic of the oscillatoriness of solutions: $a(x)$ is continuouos on $[0, +\infty)$, $a(x) > 0$, and $\int\limits_0^{+\infty} a(x)dx$ diverges, has the linearly independent particular fundamental solutions in the form of series-iterations ([2], [3])

$$\begin{aligned} y_1 = \sin_a x = x - \int\limits_0^x \int\limits_0^x xa(x)dx^2 + \int\limits_0^x \int\limits_0^x a(x) \int\limits_0^x \int\limits_0^x xa(x)dx^4 \\ - \int\limits_0^x \int\limits_0^x a(x) \int\limits_0^x \int\limits_0^x a(x) \int\limits_0^x \int\limits_0^x xa(x)dx^6 + ... \end{aligned} \tag{3}$$

$$\begin{aligned} y_2 = \cos_a x = 1 - \int\limits_0^x \int\limits_0^x a(x)dx^2 + \int\limits_0^x \int\limits_0^x a(x) \int\limits_0^x \int\limits_0^x a(x)dx^4 - \\ - \int\limits_0^x \int\limits_0^x a(x) \int\limits_0^x \int\limits_0^x a(x) \int\limits_0^x \int\limits_0^x a(x)dx^6 + ... \end{aligned} \tag{4}$$

It has been proved ([2], [3]) for the solutions that those are approximate to ordinary functions $\sin x$ and $\cos x$.

Besides, it has been proved that in the same time y_1 has zeroes and y_2 has extremes in the points of the solutions of the equation

$$x\sqrt{a(x)} = n\pi,\ n = 0, 1, 2, 3, ... \tag{5}$$

It has also been proved that in the same time y_1 has extremes and y_2 has zeroes which are approximatively points of the solutions of the equation

$$x\sqrt{a(x)} = (2n - 1)\frac{\pi}{2},\ n = 0, 1, 2, 3, ... \tag{6}$$

Sturm's theorems are confirmed by this, additionaly being supplemented and broadened by achievement of more precise locations of zeroes and extremes.

An approximative representation of non-elementary functions (3) and (4) by means of elementary functions ($\sin x$ and $\cos x$) of complex argument has been found. The formulae are

$$y_1 = \sin_{a(x)} x \approx \frac{\sin\left(x\sqrt{a(x)}\right)}{\sqrt{a(x)}}, \tag{7}$$

$$y_2 = \cos_{a(x)} x \approx \cos\left(x\sqrt{a(x)}\right). \tag{8}$$

Those are, taking into consideration (5) and (6), obviously oscillatory. From the form of the general solution

$$y = C_1 \cos_a x + C_2 \sin_a x \tag{9}$$

it is concluded that the solution is compulsorily oscillatory, since the equation $y(x) = 0$ gives

$$C_1 \cos_a x + C_2 \sin_a x = 0,$$

wherefrom

$$\frac{\sin_a x}{\cos_a x} = -\frac{C_1}{C_2}.$$

By introducing non-elementary function $\tan_{a(x)} x$, it is easily proved that it has the same characteristics as ordinary $\tan x$, and there is the equation

$$\tan_a x = -\frac{C_1}{C_2}. \tag{10}$$

Regardless the constants C_1 and C_2, taking into consideration zeroes and extremes of the functions $\sin x$ and $\cos x$, it is easily concluded from the form of the graph $\tan_a x$ (characteristical monotone growth in the continuity intervals) that the curves $y = \tan_a x$ and horizontal line $y = -\frac{C_1}{C_2}$ always have intersections. It means that the equation (10) always has solutions. This implies that from (9) $y(x) = 0$ always has solutions. This means that the general solution of the equaton (2) is oscillatory (see the Figure 1). The following theorem is the resume of the discussion.

Theorem 1. *If $a(x)$ is continuouos on $[0, +\infty)$, $a(x) > 0$, and $\int\limits_0^{+\infty} a(x)dx$ diverges, the general solution of the equation (2) is oscillatory for arbitraty constants C_1 and C_2.*

Let's consider more general non-homogeneous equation (1). The Lagrange method of variation of constants yields that particular integral of the equation (1) is

$$Y_p = y_1 \int \frac{y_2}{W} f(x)dx - y_2 \int \frac{y_1}{W} f(x)dx, \tag{11}$$

where y_1 and y_2 are given with (3) and (4). Wronskian of the equation (2), as it has no factor with the first derivative, is

$$W(x) = W(y_1, y_2) = y_1' y_2 - y_1 y_2' = 1.$$

There is the implication

$$Y_p = y_1 \int y_2 f(x)dx - y_2 \int y_1 f(x)dx, \tag{12}$$

and the general solution of the equation (1) has the form

$$y = C_1 y_1 + C_2 y_2 + y_1 \int y_2 f(x)dx - y_2 \int y_1 f(x)dx. \tag{13}$$

The question is: when is the general solution of (13) also oscillatory, i.e. what should the functions $a(x)$ and $f(x)$ be like, in order the general solution be oscillatory?

The condition is that the equation

$$C_1 \cos_a x + C_2 \sin_a x + Y_p = 0$$

has infinite number of isolated solutions. The last transcedent equation depends on four elements: the functions $a(x)$ and $f(x)$ and arbitrary constants C_1 and C_2; all the elements have the power od continuum of real numbers. If the above is rewritten

$$C_1 \cos_a x + C_2 \sin_a x = \sin_a x \int f \cos_a x dx - \cos_a x \int f \sin_a x dx$$

then the question is when the equation has the solution for arbitrary C_1 and C_2, and for which functions $a(x)$ and $f(x)$?

After division of the last equation by $\cos_a x$ there is equation which is almost equivalent with (10)

$$C_1 + C_2 \tan_a x = \tan_a x \int f \cos_a x dx - \int f \sin_a x dx$$

If the equation is gruped, then there is

$$\tan_a x = \frac{\int f(x)\sin_a x dx + C_1}{\int f(x)\cos_a x dx - C_2}, \tag{14}$$

or $L_1 = L_2$. The curve L_1, just as any tangent, monotonously grows in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and periodically repeats in every interval from $-\infty$ to $+\infty$.

There is the problem with the right side of the equation, i.e. the curve L_2. It is obvious important that L_2 is either continuous from $x = 0$ to $+\infty$, or, if it is not continuous, that the discontinuities do not match the doscontinuities of $\tan_a x$ (since the curves might not have intersection then, i.e. somehow remain with paralel branches). That's why it is important to research the integral

$$\int f(x)\sin_a x dx$$

and its oscillatoriness in L_2.

Let $f(x)$ be continuous in $[0, +\infty)$, with or without zeroes. It is known that $\sin_a x$ is continuous, after the way of its construction, by means of series-iterations (the basis is Picard's theorem on on succesive approximations, which implies continuity of solutions of oscillatory equation (2)). The product $f(x)\sin_a x$ is then continuous, as well as the integral $\int f(x)\sin_a x dx$. It implies that the numerator in L_2: $\int f(x)\sin_a x dx + C_2$ is continuous. As regards denominator, it is, for same reasons, continuous. If the denominator has zeroes, L_2 is discontinuous.

If the primitive non-elementary function of non-elementary function $f(x)\cos_a x$ is denoted as $F(x)$

$$\int f(x)\cos_a x dx = F(x) + C_3, \tag{15}$$

where C_3 is a given arbitrary constant, then the discontinuities of the right side in (14) will occur in the points ξ which are solutions of the equation

$$F(x) = -C_3.$$

If those points match singular points of $\tan x$, which is the left side of the equation (14), i.e. for $\cos_a x = 0$, approximatively

$$\cos\left(x\sqrt{a(x)}\right) = 0$$

or

$$\xi_k \sqrt{a\left(\xi_k\right)} = (2k-1)\,\frac{\pi}{2},\; k = 1, 2, 3, ...$$

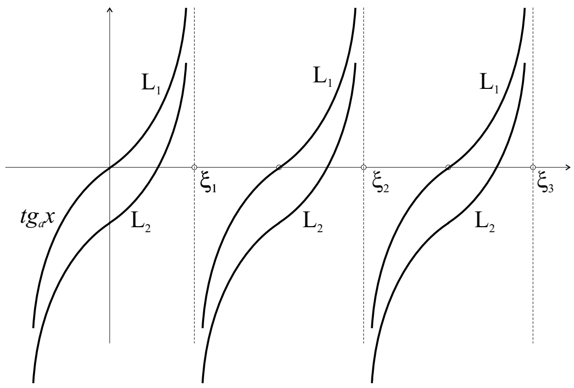

Figure 1:

then vertical asymptotes of the curves L_1 and L_2 match and those could have no intersections, i.e. the equation (14) could have no zeroes. In all the other cases (14) has solutions, i.e. the general solution of (13) has zeroes and the general solution is oscillatory. There implies

Theorem 2. *If all of the solutions of the equation (2) are oscillatory, then all of the solutions of the equation (1) are oscillatory as well, except if the solutions of the equation* $F(x) = -C_3$ *where* $F(x)$ *is primitive function given with (15), match zeroes of one solution of the homogeneous equation (2).ntegral of square of an entire and analytical periodical function*

$$\int p^2(x)dx$$

cannot be periodical.

Geometry of the oscillatoriness and locating of zeroes is given in the Figure 2.

With the problems of oscillatoriness of the solutions of the equations (2) and (3), the following problem obviosly becomes very important:

- if $f(x)$ is a continuous and $g(x)$ is an oscillatory function, is the primitive function $F(x) = \int f(x)g(x)dx$ oscillatory, and what is it like?

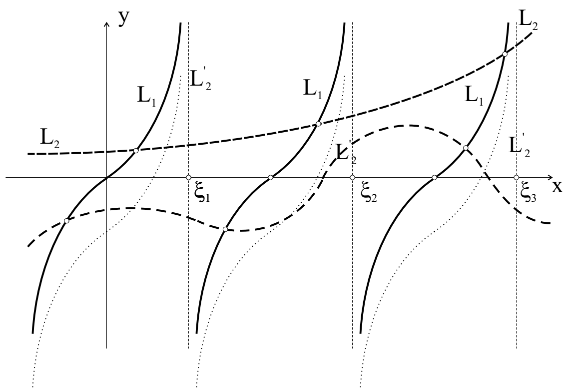

Figure 2:

APPLICATIONS

1. ELEMENTARY SOLUTIONS, RESONANCE

Very important and known result [1] is when $a(x) = Const = \omega^2$ and $f(x)$ is a periodical function in the equation (1). This is the most frequent case of oscillations in engineering (mechanical and electrical). Those are the equations of the following form

$$y'' + \omega^2 y = A \sin \omega_1 x.$$

From the solution of the homogeneous equation

$$y = C_1 \sin \omega x + C_2 \cos \omega x,$$

it is well-known that Lagrange's method of variation of constants gives general solution

$$y = C_1 \sin \omega x + C_2 \cos \omega x - \frac{A}{2\omega}\left(\frac{\cos(\omega+\omega_1)x}{\omega+\omega_1} + \frac{\cos(\omega_1-\omega)x}{\omega_1-\omega}\right)\sin \omega x -$$
$$-\frac{A}{2\omega}\left(\frac{\cos(\omega+\omega_1)x}{\omega+\omega_1} + \frac{\cos(\omega_1-\omega)x}{\omega_1-\omega}\right)\cos \omega x,$$

where the particular integral of non-homogeneous equation is given as

$$Y_p = -\frac{A}{2\omega}(\sin \omega x + \cos \omega x)\left(\frac{\cos(\omega+\omega_1)x}{\omega+\omega_1} + \frac{\cos(\omega_1-\omega)x}{\omega_1-\omega}\right).$$

The integral implies that if frequency of self-oscillations ω and the frequency of forced oscillations ω_1 get close ($\omega_1 \to \omega$), then Y_p could have very high amplitudes. The very important science on stability of solutions originates from it. In order to more precisely determine the high amplitude of the solution Y_p, a solution of the following form should be looked after in the above equation.

$$Y_p = R\sin\omega_1 x.$$

After derivations

$$Y_p' = R\omega_1\cos\omega_1 x,\ Y_p'' = -R\omega_1^2\sin\omega_1 x$$

and substitution in the equation, there is

$$-R\omega_1^2\sin\omega_1 x + \omega^2 R\sin\omega_1 x = A\sin\omega_1 x;\ R\left(\omega^2 - \omega_1^2\right) = A.$$

Subsequently, there is

$$R = \frac{A}{\omega^2 - \omega_1^2}.$$

Hence, there is one solution with very high amplitudes. The amplitude is

$$Y_p = \frac{A}{\omega^2 - \omega_1^2} \to \infty,\ \omega_1 \to \omega.$$

This is a known, dramatical case of instability in engineering.

2. NON-ELEMENTARY OSCILLATIONS. INSTABILITY

Let there be the equation (1)

$$y'' + a(x)y = f(x)$$

where $a(x)$ is not a constant, but is such that allows for oscillatory solutions of the homogeneous equation $y'' + a(x)y = 0$, being approximatively

$$y_1 = \sin_a x \approx \frac{\sin\left(x\sqrt{a(x)}\right)}{\sqrt{a(x)}}$$

$$y_2 = \cos_a x \approx \cos\left(x\sqrt{a(x)}\right).$$

Let $f(x)$ be a non-elementary oscillation, that is a solution of some other homogeneous equation with some other coefficient $a_1(x)$ providing for oscillatoriness

$$f'' + a_1(x)f = 0,$$

i.e. let the following be correct analogously

$$f_1 = \sin_{a_1(x)} x \approx \frac{\sin\left(x\sqrt{a_1(x)}\right)}{\sqrt{a_1(x)}}$$

$$f_2 = \cos_{a_1(x)} x \approx \cos\left(x\sqrt{a_1(x)}\right).$$

For the equation of non-elementary oscillations

$$y'' + a(x)y = A_1 \sin_{a_1} x, \tag{16}$$

it is also possible then to look for

$$Y_p = R \sin_{a_1} x.$$

From the derivatives

$$Y_p' = R\left(\sin_{a_1} x\right)', \; Y_p'' = R\left(\sin_{a_1} x\right)'' = R\left(-a_1 \sin_{a_1} x\right)$$

and after substitution in (16)

$$-Ra_1 \sin_{a_1} x + aR \sin_{a_1} x = A_1 \sin_{a_1} x$$

the solution for R is found

$$R = \frac{A_1}{a - a_1}.$$

It is obvious that amplitude becomes very high as a approaches a_1.

The conclusion is that there is resonance with such non-elementary oscillations, analogously to elementary oscillations. More complex circumstances, much harder than with constant coefficients and elementary harmonical oscillations, could be observed:

- the functions $a(x)$ and $a_1(x)$ are close to each other for every x;
- the functions are inequal, but have close locations of zeroes.

This all needs a more detailed analysis.

3. THE ISSUE OF INSTABILITY OF GENERAL CASE OF LINEAR OSCILLATIONS

All of the above could be used and applied, by means of refined analysis and technics, to the case of general linear equation of the second order

$$y'' + A(x)y' + B(x)y = F(x)$$

along with met criteria on oscillatoriness of solutions.

It would be an analysis of non-elementary oscillations of the above equations, for various cases of approximation of the function of forced (external) oscillations $F(x)$ with base functions $\sin_{(A,B)} x$ and $\cos_{(A,B)} x$ of corresponding homogeneous equation

$$y'' + A(x)y' + B(x)y = 0$$

In the general case, we would deal with non-elementary functions

$$\sin_{(A,B,F)} x \text{ and } \cos_{(A,B,F)} x.$$

However, this is a comprehensive and special issue.

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