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STABILITY OF SOLUTIONS OF DIFFERENTIAL-OPERATOR AND OPERATOR-DIFFERENCE EQUATIONS WITH RESPECT TO PERTURBATION OF OPERATORS[1]

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Abstract. It is obtained estimates of stability with respect to perturbation of operator for solution of the first- and second-order differential-operator equations. For two- and three-level operator-difference schemes with weights similar estimates holds. Using the obtained results we construct the estimates of coefficient stability for one-dimensional parabolic and hyperbolic equations as well as for the difference schemes approximating the corresponding differential problems.

1. INTRODUCTION

Abstract problem $\mathcal{A}u = f$ is called *stable* if its solution u continuously depends on input data f, i.e., there exists value $\varrho > 0$ independent of solution and input data such that for all f and $\tilde{f}$ form certain admissible set the following inequality holds

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$$\|\tilde{u} - u\|_{(0)} \le \varrho \|\tilde{f} - f\|_{(1)},$$

where $\tilde{u}$ is the solution of similar problem with perturbed input data $\mathcal{A}\tilde{u} = \tilde{f}$; $\|\cdot\|_0$ and $\|\cdot\|_1$ are certain norms.

When we define a problem we set not only right-hand side but operator $\mathcal{A}$ as well. If, e.g., $\mathcal{A}$ is differential or difference operator then the coefficients of corresponding equation have to be defined. Naturally, the solution of abstract problem should be continuously depended on perturbations of both right-hand side and operator $\mathcal{A}$ (e.g., coefficients of differential or difference equation), i.e., the following estimate should holds

$$\|\tilde{u} - u\|_{(0)} \le \varrho_1 \|\tilde{f} - f\|_{(1)} + \varrho_2 \|\tilde{A} - A\|_{(2)},$$

where $\tilde{u}$ is the solution of the problem with perturbed operator and right-hand side $\tilde{\mathcal{A}}\tilde{u} = \tilde{f}$; $\|\cdot\|_{(2)}$ is certain operator norm. In this case the problem becomes nonlinear even if operators $\mathcal{A}$, $\tilde{\mathcal{A}}$ are linear, and values ϱ_k depend on $\tilde{u}$, u.

In [13, 16, 18, 20] it is obtained *a priori* estimates, expressing continuous dependence of the solution of stationary problems (first-kind operator equations) with respect to perturbation of right-hand side and operator, Similar results for time-dependent problems with unbounded operators are derived in [9, 12, 11, 15, 17, 19, 23, 24].

In this paper we survey and develop some results concerning stability with respect to perturbation of operator obtained in [1, 2, 5, 6, 7].

The paper is organized as follows. In Section 1 the results concerning stability of solution of the first-order differential-operator equation with respect to perturbation of operator. In Section 2 similar estimates are obtained for two-level operator-difference schemes. In Sections 3 and 4 the estimates of stability with respect to perturbation of operator for solutions of the second-order differential-operator equations and three-level operator difference schemes, respectively.

2. STABILITY OF SOLUTIONS OF THE FIRST-ORDER DIFFERENTIAL-OPERATOR EQUATIONS WITH RESPECT TO PERTURBATION OF OPERATOR

This section is devoted to analysis of stability of solution of the first-order differential-operator equation with respect to perturbation of operator. The estimates of perturbation of the solution are obtained in energy Hilbert spaces as well as in the integral with respect to time norms.

2.1. PROBLEM DEFINITION

Consider the Cauchy problem for the first-order differential-operator equation

$$\frac{du}{dt} + \mathcal{A}u = f(t), \quad t > 0, \quad u(0) = u_0, \tag{1}$$

where $\mathcal{A}$ is linear operator (generally it is unbounded) acting form normalized space $\mathscr{H}$ to normalized space $\mathscr{G}$, $u(t)$ is unknown function taking interval $(0, \infty)$ (or $(0, T)$) to $\mathscr{H}$, $f(t)$ is the given function taking interval $(0, \infty)$ (or $(0, T)$) to $\mathscr{G}$, u_0 is the given element of the space $\mathscr{H}$.

Together with (1) we consider the problem with perturbed operator

$$\frac{d\tilde{u}}{dt} + \tilde{\mathcal{A}}\tilde{u} = f(t), \quad t > 0, \quad \tilde{u}(0) = u_0, \tag{2}$$

where $\tilde{\mathcal{A}} : \mathscr{H} \to \mathscr{G}$ is linear operator. We say that the solution of the Cauchy problem (1) is *stable with respect to perturbation of operator* if the following estimate holds:

$$\|\tilde{u}(t) - u(t)\|_{\mathscr{H}} \leq M\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}\to\mathscr{G}}. \tag{3}$$

Below we shall assume that $\mathscr{H} = \mathscr{G}$ is separable Hilbert space with the inner product $(\cdot, \cdot)$ and the norm $\|\cdot\|$, $\mathcal{A}$ is self-adjoint positive definite operator with everywhere dense in $\mathscr{H}$ domain $\mathscr{D}(\mathcal{A})$, i.e.,

$$(\mathcal{A}u, u) \geq m\|u\|^2 \qquad u \in \mathscr{D}(\mathcal{A}), \quad > . \tag{4}$$

In the standard way we introduce the inner product $(u, v)_{\mathcal{A}} = (\mathcal{A}u, v)$ and corresponding energy space $\mathscr{H}_{\mathcal{A}} \subset \mathscr{H}$. At the same time $\mathscr{H}_{\mathcal{A}^-} = \mathscr{H}_{\mathcal{A}}^*$ is adjoint space to $\mathscr{H}_{\mathcal{A}}$. The inner product (u, v) can be continuously extended on $\mathscr{H}_{\mathcal{A}^-} \times \mathscr{H}_{\mathcal{A}}$ and operator $\mathcal{A}$ can be extended to the map $\mathcal{A} : \mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}$ [22]. Naturally, operators $\mathcal{A} : \mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}$, $\mathcal{A} : \mathscr{H}_{\mathcal{A}} \to \mathscr{H}$ or $\mathcal{A} : \mathscr{H} \to \mathscr{H}_{\mathcal{A}^-}$ are bounded. Actually, for $k = 0, 1, 2$ we have

$$\|\mathcal{A}u\|_{\mathcal{A}^{k-2}} = \|\mathcal{A}^{k/2-1}\mathcal{A}u\| = \|\mathcal{A}^{k/2}u\| = \|u\|_{\mathcal{A}^k},$$

whence

$$\|\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}^k} \to \mathscr{H}_{\mathcal{A}^{k-2}}} = \sup_{u \in \mathscr{H}_{\mathcal{A}^k},\, u \neq 0} \frac{\|\mathcal{A}u\|_{\mathcal{A}^{k-2}}}{\|u\|_{\mathcal{A}^k}} = 1.$$

Let us introduce Lebesgue space $L_2((a, b); \mathscr{H})$ (see [10]) of functions $u = u(t)$, mapping interval $(a, b) \subset \mathbb{R}$ to $\mathscr{H}$, with the following inner product and norm:

$$(u,\, v)_{L_2((a,b);\mathscr{H})} = \int_a^b (u(t),\, v(t))\, dt, \quad \|u\|_{L_2((a,b);\mathscr{H})} = (u,\, u)^{1/2}_{L_2((a,b);\mathscr{H})}.$$

Also, we shall use the spaces $L_2((a, b); \mu; \mathscr{H})$ and $L_p((a, b); \mathscr{H})$ with corresponding norms and semi-norms [21, 10].

2.2. STABILITY IN THE CASE OF SELF-ADJOINT OPERATOR

Let in problem (1) $\mathcal{A}$ be self-adjoint positive-definite linear operator in $\mathscr{H}$, $\mathscr{D}(\mathcal{A})$ be domain everywhere dense in $\mathscr{H}$. Also, let operator of perturbed problem (2) $\tilde{\mathcal{A}}$ satisfies the same conditions and, moreover, $\mathscr{D}(\mathcal{A}) = \mathscr{D}(\mathcal{A})$.

Taking inner product of (1) with $2u$, we get

$$\frac{d(\|u(t)\|^2)}{dt} + 2\|u(t)\|^2_{\mathcal{A}} = 2(f(t), u(t)). \tag{5}$$

Taking into account (4) and trivial relation $2(f(t), u(t)) \leq \|f(t)\|^2_{\mathcal{A}^{-1}} + \|u(t)\|^2_{\mathcal{A}}$, from (5) we get the following inequality:

$$\frac{d(\|u(t)\|^2)}{dt} + m\|u(t)\|^2 \leq \|f(t)\|^2_{\mathcal{A}^{-1}}.$$

From here and the Gronwall inequality for the solution of problem (1) we obtain the estimate

$$\|u(t)\|^2 \le e^{-mt}\left(\|u_0\|^2 + \int_0^t e^{ms}\|f(s)\|^2_{\mathcal{A}^{-1}}ds\right). \tag{6}$$

Similar estimate holds for the solution $\tilde{u}$ of problem (2) with perturbed operator. Acting on problem (2) by the operator $\tilde{\mathcal{A}}^{1/2}$ and using inequality (6), we get

$$\|\tilde{u}(t)\|^2_{\tilde{\mathcal{A}}} \le e^{-mt}\left(\|u_0\|^2_{\tilde{\mathcal{A}}} + \int_0^t e^{ms}\|f(s)\|^2 ds\right). \tag{7}$$

We obtain also the estimate of the solution of problem (1) in the norm of the space $L_2((0,t); e^{ms}; \mathscr{H}_{\mathcal{A}})$. For that, we take inner product of (1) with $2e^{mt}u$. Taking into account the relation

$$2\left(e^{mt}u, \frac{du}{dt}\right) = \frac{d}{dt}\left(e^{mt}u, u\right) - me^{mt}(u,u),$$

we obtain the following energy identity:

$$\frac{d}{dt}\left(e^{mt}u, u\right) - me^{mt}\|u\|^2 + 2e^{mt}\|u\|^2_{\mathcal{A}} = 2e^{mt}(u,f).$$

Hence, using Cauchy—Schwartz inequality, ε-inequality and (4), we get

$$\frac{d}{dt}\left(e^{mt}\|u\|^2\right) + \frac{\varepsilon - 1}{\varepsilon}e^{mt}\|u\|^2_{\mathcal{A}} \le \varepsilon e^{mt}\|f\|^2_{\mathcal{A}^{-1}} \qquad \forall \varepsilon > 0.$$

Setting $\varepsilon = 2$ and integrating the last inequality with respect to t, we obtain the estimate

$$\int_0^t e^{ms}\|u(s)\|^2_{\mathcal{A}}ds \le 4\left(\|u_0\|^2 + \int_0^t e^{ms}\|f(s)\|^2_{\mathcal{A}^{-1}}ds\right). \tag{8}$$

note that similar estimate in space $L_2((0,t); e^{ms}; \mathscr{H}_{\mathcal{A}})$ holds for the solution $\tilde{u}$ of problem (2) with perturbed operator. Acting on problem (1) by the operator $\tilde{\mathcal{A}}^{1/2}$ and using inequality (8), we get the following estimate:

$$\int_0^t e^{ms}\|\tilde{\mathcal{A}}\tilde{u}(s)\|^2 ds \le 4\left(\|u_0\|^2_{\tilde{\mathcal{A}}} + \int_0^t e^{ms}\|f(s)\|^2 ds\right). \tag{9}$$

The obtained estimates we shall use to prove the following theorem.

Theorem 1. *Let $u_0 \in \mathscr{H}$, $f \in L_2((0,t); e^{ms}; \mathscr{H}_{\mathcal{A}^-})$. Then the solution of problem (1) is stable with respect to perturbation of operator and the following estimate holds:*

$$\|\tilde{u}(t) - u(t)\|^2 \le M_1 e^{-mt}\|\tilde{\mathcal{A}} - \mathcal{A}\|^2_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}}, \tag{10}$$

where $M_1 = \|\tilde u\|^2_{L_2((0,t);e^{ms};\mathscr{H}_{\mathcal{A}})} \le 4\left(\|u_0\|^2 + \int_0^t e^{ms}\|f(s)\|^2_{\tilde{\mathcal{A}}^{-1}} ds\right)$.

Let $u_0 \in \mathscr{H}_{\mathcal{A}}$, $f \in L_2((0,t);e^{ms};\mathscr{H})$. *Then the solution of problem (1) is stable with respect to perturbation of operator and the following estimate holds:*

$$\|\tilde u(t) - u(t)\|^2_{\mathcal{A}} \le M_2 e^{-mt}\|\tilde{\mathcal{A}} - \mathcal{A}\|^2_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}}, \tag{11}$$

where $M_2 = \|\tilde u\|^2_{L_2((0,t);e^{ms};\mathscr{H}_{\mathcal{A}})} \le 4\left(\|u_0\|^2_{\tilde{\mathcal{A}}} + \int_0^t e^{ms}\|f(s)\|^2 ds\right)$.

Proof. For perturbation $\delta u = \tilde u - u$ we have the Cauchy problem

$$\frac{d\,\delta u}{dt} + \mathcal{A}\,\delta u = (\tilde{\mathcal{A}} - \mathcal{A})\tilde u, \quad t > 0, \quad \delta u(0) = 0. \tag{12}$$

For the solution of problem (12) it is fulfilled estimate (6), whence

$$\|\delta u(t)\|^2 \le e^{-mt}\int_0^t e^{ms}\|(\tilde{\mathcal{A}}-\mathcal{A})\tilde u(s)\|^2_{A^{-1}} ds \le e^{-mt}\|\tilde{\mathcal{A}}-\mathcal{A}\|^2_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}}\int_0^t e^{ms}\|\tilde u(s)\|^2_{\tilde{\mathcal{A}}} ds.$$

Hence, taking into account inequality (8), we obtain desired estimate (10).

For the solution of problem (12) the estimate of the form (7) holds:

$$\|\delta u(t)\|^2_{\mathcal{A}} \le e^{-mt}\int_0^t e^{ms}\|(\tilde{\mathcal{A}} - \mathcal{A})\tilde u(s)\|^2 ds \le e^{-mt}\|\tilde{\mathcal{A}} - \mathcal{A}\|^2_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}}\int_0^t e^{ms}\|\tilde{\mathcal{A}}\tilde u(s)\|^2 ds.$$

From the last inequality using (8), we get desired estimate (11).

Let us obtain the estimates of stability if the solution of equation (1) with respect to perturbation of operator in time-integral norms. For that we need corresponding estimates of stability with respect to initial data and right-hand side. Taking into account the equation $f = du/dt + Au$, for the right-hand side of identity (5) we have

$$2(f,u) = 2(\mathcal{A}^{-1}f, \mathcal{A}u) = 2\left(\mathcal{A}^{-1}\frac{du}{dt}, \mathcal{A}u\right) + 2\|u\|^2_{\mathcal{A}} =$$

$$= \left\|\mathcal{A}^{-1/2}\left(\frac{du}{dt} + \mathcal{A}u\right)\right\|^2 + \|u\|^2_{\mathcal{A}} - \left\|\frac{du}{dt}\right\|^2_{\mathcal{A}^{-1}} = \|f\|^2_{\mathcal{A}^{-1}} + \|u\|^2_{\mathcal{A}} - \left\|\frac{du}{dt}\right\|^2_{\mathcal{A}^{-1}}.$$

Substituting this in (5) and integrating the result with respect to t, we obtain the following energy identity

$$\|u(t)\|^2 + \int_0^t\left(\|u(s)\|^2_{\mathcal{A}} + \left\|\frac{du(s)}{dt}\right\|^2_{\mathcal{A}^{-1}}\right)ds = \|u_0\|^2 + \int_0^t \|f(s)\|^2_{\mathcal{A}^{-1}} ds.$$

From this identity it follows, in particular, the known Hadamard inequality [22, . 403]

$$\int_0^t \left(\|u(s)\|_{\mathcal{A}}^2 + \left\| \frac{du(s)}{dt} \right\|_{\mathcal{A}^{-1}}^2 \right) ds \le \|u_0\|^2 + \int_0^t \|f(s)\|_{\mathcal{A}^{-1}}^2 ds. \qquad (13)$$

Acting on problem (1) by the operators $\mathcal{A}^{1/2}$, $\mathcal{A}^{-1/2}$ and using inequality (13), we get the corresponding estimates:

$$\int_0^t \left(\|\mathcal{A}u(s)\|^2 + \left\| \frac{du(s)}{dt} \right\|^2 \right) ds \le \|u_0\|_{\mathcal{A}}^2 + \int_0^t \|f(s)\|^2 ds, \qquad (14)$$

$$\int_0^t \left(\|u(s)\|^2 + \left\| \mathcal{A}^{-1} \frac{du(s)}{dt} \right\|^2 \right) ds \le \|u_0\|_{\mathcal{A}^{-1}}^2 + \int_0^t \|\mathcal{A}^{-1} f(s)\|^2 ds. \qquad (15)$$

These estimates are also well known.

To construct the estimate in integral semi-norm of the fraction order with respect to time variable t we use (similarly to [5]) Fourier series by cosine and sine for the function $u(t) : [0, T] \to \mathscr{H}$: $u(t) = a_0/2 + \sum_{j=1}^{\infty} a_j \cos(j\pi t/T)$ $u(t) = \sum_{j=1}^{\infty} b_j \sin(j\pi t/T)$, where $a_j = a_j[u] = (2/T) \int_0^T u(s) \cos(j\pi s/T) ds$, $b_j = b_j[u] = (2/T) \int_0^T u(s) \sin(j\pi s/T) ds$, and integrals are Bokhner integrals [22, . 384]. It can easily be checked that

$$\int_0^T \|u(s)\|^2 ds = 0.5T \left(0.5\|a_0[u]\|^2 + \sum_{j=1}^{\infty} \|a_j[u]\|^2 \right) = 0.5T \sum_{j=1}^{\infty} \|b_j[u]\|^2. \qquad (16)$$

Similar results holds when we substitute the energy space $\mathscr{H}_{\mathcal{A}}$ for $\mathscr{H}$.

Multiply both sides of equation (1) by $\sin(k\pi t/T)$ and integrate the result with respect to t from 0 to T. Using expansion $du(t)/dt = -\sum_{j=1}^{\infty} a_j[u](j\pi/T) \sin(j\pi t/T)$ and orthogonality of sines on $(0, T)$, we get the equality $(k\pi/T) a_k[u] = Ab_k[u] - b_k[f]$. Taking the inner product of the last identity with $a_k[u]$ and summing the result with respect to k, we have

$$\frac{\pi}{T} \sum_{k=1}^{\infty} k\|a_k[u]\|^2 = \sum_{k=1}^{\infty} (Ab_k[u], a_k[u]) - \sum_{k=1}^{\infty} (b_k[u], a_k[u]) \le$$

$$\le 0.5 \sum_{k=1}^{\infty} (\|b_k[u]\|_{\mathcal{A}}^2 + 2\|a_k[u]\|_{\mathcal{A}}^2 + \|b_k[f]\|_{\mathcal{A}^{-1}}^2).$$

Using (16), we obtain inequality

$$\sum_{k=1}^{\infty} k\|a_k[u]\|^2 \leq \frac{1}{\pi}\int_0^T (3\|u(t)\|^2_{\mathcal{A}} + \|f(t)\|^2_{\mathcal{A}^{-1}})dt. \tag{17}$$

Consider expression $J_1 = \int_{-T}^{T}\int_{-T}^{T} \|u(t) - u(t-s)\|^2 dsdt$. Suppose that $u(t)$ has even extension outside of interval $[0, T]$: $u(t) = u(-t)$, $t \in [-T, 0]$, $u(t) = u(2T - t)$, $t \in [T, 2T]$ etc. Using periodicity of $u(t)$ and cosine expansion, we obtain

$$J_1 = \int_{-T}^{T}\left[\int_{-T}^{T}(u(t), -u(t+s) + 2u(t) - u(t-s))dt\right]\frac{ds}{s^2} =$$

$$= \int_{-T}^{T}\int_{-T}^{T}\left(\frac{a_0[u]}{2} + \sum_{j=1}^{\infty} a_j[u]\sin\frac{j\pi t}{T}, \sum_{k=1}^{\infty} a_k[u]\left(-\cos\frac{k\pi(t+s)}{T} + 2\cos\frac{k\pi t}{T} - \right.\right.$$

$$\left.\left. - \cos\frac{k\pi(t-s)}{T}\right)\right) dt\,\frac{ds}{s^2} =$$

$$= \int_{-T}^{T}\int_{-T}^{T}\left(\frac{a_0[u]}{2} + \sum_{j=1}^{\infty} a_j[u]\sin\frac{j\pi t}{T}, \sum_{k=1}^{\infty} 4a_k[u]\sin^2\frac{k\pi s}{2T}\cos\frac{k\pi t}{T}\right) dt\,\frac{ds}{s^2} =$$

$$= 4T\sum_{k=1} \infty\|a_k[u]\|^2 \int_{-T}^{T}\sin^2\frac{k\pi s}{2T}\frac{ds}{s^2}.$$

Taking into account

$$\int_{-T}^{T}\sin^2\frac{k\pi s}{2T}\frac{ds}{s^2} = 2\int_0^T \sin^2\frac{k\pi s}{2T}\frac{ds}{s^2} = \frac{k\pi}{T}\int_0^T\frac{\sin^2\theta}{\theta^2}d\theta \leq \frac{k\pi}{T}\int_0^\infty\frac{\sin^2\theta}{\theta^2}d\theta = \frac{k\pi^2}{2T},$$

we get

$$J_1 \leq 2\pi^2\sum_{k=1}^{\infty} k\|a_k[u]\|^2. \tag{18}$$

Using (17), (18) and trivial inequality

$$J_2 \equiv \int_0^T\int_0^T \frac{\|u(t) - u(t')\|^2}{|t - t'|^2}dt\,dt' \leq 0.5J_1,$$

we obtain the estimate $J_2 \leq \pi\int_0^T (3\|u(t)\|^2_{\mathcal{A}} + \|f(t)\|^2_{\mathcal{A}^{-1}})dt$. Hence, taking into account (13), we get

$$J_2 \leq M\left(\|u_0\|^2 + \int_0^T \|f(t)\|_{\mathcal{A}^{-1}}dt\right). \tag{19}$$

Note that similar estimates derived in [10, . 2, . 83–90] by means of Fourier transform.

From inequalities (13) and (19) it follows that

$$\int_0^T \|u(t)\|_{\mathcal{A}}^2 dt + J_2 \le M \left(\|u_0\|^2 + \int_0^T \|f(t)\|_{\mathcal{A}^{-1}} dt \right). \tag{20}$$

From (15) we obviously have

$$\int_0^T \|u(t)\|^2 dt \le \|u_0\|_{\mathcal{A}^{-1}}^2 + \int_0^T \|\mathcal{A}^{-1} f(t)\| dt. \tag{21}$$

We need the obtained estimates to prove the following theorem concerning stability of the solution of the differential-operator equation (1) with respect to perturbation of operator.

Theorem 2. *Let $u_0 \in \mathscr{H}_{\mathcal{A}}$ and $f \in L_2((0,t);\mathscr{H})$. Then the solution of problem (1) is stable with respect to perturbation of operator and the following estimate holds:*

$$\int_0^t \left(\|\mathcal{A}\,\delta u(s)\|^2 + \left\| \frac{d\,\delta u(s)}{ds} \right\|^2 \right) ds \le M_3 \|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}}, \tag{22}$$

where $M_3 = \|\tilde{u}\|_{L_2((0,t);\mathscr{H}_{\mathcal{A}})} \le \|u_0\|_{\tilde{\mathcal{A}}}^2 + \int_0^t \|f(s)\|^2 ds$.

Let $u_0 \in \mathscr{H}$ and $f \in L_2((0,t);\mathscr{H}_{\mathcal{A}^-})$. Then the solution of problem (1) is stable with respect to perturbation of operator and the following estimate holds:

$$\int_0^t \|\delta u(s)\|_{\mathcal{A}}^2 ds + \int_0^t \int_0^t \frac{\|\delta u(s) - \delta u(s')\|^2}{|s - s'|^2} ds\, ds' \le M_4 \|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}}^-}, \tag{23}$$

where $M_4 = M \|\tilde{u}\|_{L_2((0,t);\mathscr{H}_{\mathcal{A}})} \le M \left(\|u_0\|^2 + \int_0^t \|f(s)\|_{\tilde{\mathcal{A}}^{-1}}^2 ds \right)$.

Let $u_0 \in \mathscr{H}_{\mathcal{A}^-}$ and $f \in L_2((0,t);\mathscr{H}_{\mathcal{A}^-})$. Then the solution of problem (1) is stable with respect to perturbation of operator and the following estimate holds:

$$\int_0^t \|\delta u(s)\|^2 ds \le M_5 \|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H} \to \mathscr{H}_{\mathcal{A}^-}}, \tag{24}$$

where $M_5 = \|\tilde{u}\|_{L_2((0,t);\mathscr{H})} \le \|u_0\|_{\tilde{\mathcal{A}}^{-1}}^2 + \int_0^t \|f(s)\|_{\tilde{\mathcal{A}}^{-2}}^2 ds$.

Proof. Applying estimates (14), (20) and (21) to the solution of the Cauchy problem for perturbation (12) we prove the theorem.

Sometimes, operator norms in (10), (11), (22)–(24) can be replaced by more simple norms. Thus, for example, if $\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}} \le \delta < 1$ then the corresponding norms $\|u\|_{\mathcal{A}}$ and $\|u\|_{\tilde{\mathcal{A}}}$, $\|u\|_{\mathcal{A}^{-1}}$ and $\|u\|_{\tilde{\mathcal{A}}^{-1}}$, $\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}}$, $\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}}$ and $\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}}$ are equivalent and the following inequalities holds:

$$\sqrt{1-\delta}\|u\|_{\mathcal{A}} \le \|u\|_{\tilde{\mathcal{A}}} \le \sqrt{1+\delta}\|u\|_{\mathcal{A}}, \quad \frac{1}{\sqrt{1+\delta}}\|u\|_{\mathcal{A}^{-1}} \le \|u\|_{\tilde{\mathcal{A}}^{-1}} \le \frac{1}{\sqrt{1-\delta}}\|u\|_{\mathcal{A}^{-1}},$$

$$\frac{1}{\sqrt{1+\delta}}\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}} \le \|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}} \le \frac{1}{\sqrt{1-\delta}}\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}}$$
$$\frac{1}{1+\delta}\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}}, \le \|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}} \le \frac{1}{1-\delta}\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}_{\mathcal{A}^-}}.$$

Also, if $\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}} \le \delta < 1$ then the following inequalities holds

$$(1-\delta)\|u\|_{\mathcal{A}^2} \le \|u\|_{\tilde{\mathcal{A}}^2} \le (1+\delta)\|u\|_{\mathcal{A}^2}, \quad \frac{1}{1+\delta}\|u\|_{\mathcal{A}^{-2}} \le \|u\|_{\tilde{\mathcal{A}}^{-2}} \le \frac{1}{1-\delta}\|u\|_{\mathcal{A}^{-2}},$$

$$(1-\delta)\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}} \le \|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}} \le (1+\delta)\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}_{\mathcal{A}}\to\mathscr{H}}$$

$$(1-\delta)\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}\to\mathscr{H}_{\mathcal{A}^-}}, \le \|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}\to\mathscr{H}_{\mathcal{A}^-}} \le (1+\delta)\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}\to\mathscr{H}_{\mathcal{A}^-}}.$$

The same inequalities is fulfilled under the assumption $\|\tilde{\mathcal{A}}-\mathcal{A}\|_{\mathscr{H}\to\mathscr{H}_{\mathcal{A}^-}} \le \delta < 1$.

2.3. STABILITY IN THE CASE OF NONSELF-ADJOINT OPERATOR

We construct the estimates when $\mathcal{A}$ and $\tilde{\mathcal{A}}$ are nonself-adjoint positive -definite operators. We represent the operator $\mathcal{A}$ in the form of $\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1$, where $\mathcal{A}_0 = 0.5(\mathcal{A}+\mathcal{A}^*)$ is symmetric part of the operator $\mathcal{A}$ and $\mathcal{A}_1 = 0.5(\mathcal{A}-\mathcal{A}^*)$ is skew-symmetrical part of the operator $\mathcal{A}$. It follows that $\mathcal{A}_0 = \mathcal{A}_0^*$, $\mathcal{A}_1 = -\mathcal{A}_1^*$, and, consequently, $(\mathcal{A}_1 u, u) = 0$ for all $u \in \mathscr{H}$ (recall that the space $\mathscr{H}$ is real). Similar presentation holds for the operator $\tilde{\mathcal{A}}$.

Suppose $\mathcal{A}_0$ and $\tilde{\mathcal{A}}_0$ are positive definite operators, i.e., they satisfy condition (4).

Taking inner product of (1) with $2u$ and taking into account that $(\mathcal{A}u, u) = (\mathcal{A}_0 u, u)$, we obtain the following energy identity:

$$\frac{d(\|u(t)\|^2)}{dt} + 2\|u(t)\|^2_{\mathcal{A}_0} = 2(f(t), u(t)) \le \|f(t)\|^2_{\mathcal{A}_0^{-1}} + \|u(t)\|^2_{\mathcal{A}_0}. \tag{25}$$

Hence, using (4) and Gronwall inequality, for the solution of problem (1) we obtain

$$\|u(t)\|^2 \le e^{-mt}\left(\|u_0\|^2 + \int_0^t e^{ms}\|f(s)\|^2_{\mathcal{A}_0^{-1}} ds\right). \tag{26}$$

Similarly to derivation of (8) we can obtain also the estimate of the solution of problem (1) in the norm of the space $L_2((0,t);e^{ms};\mathscr{H}_{\mathcal{A}})$

$$\int_0^t e^{ms}\|u(s)\|^2_{\mathcal{A}_0} ds \leq 4\left(\|u_0\|^2 + \int_0^t e^{ms}\|f(s)\|^2_{\mathcal{A}_0^{-1}} ds\right). \tag{27}$$

The following theorem concerning stability with respect to perturbation of operator is fulfilled.

Theorem 3. *Let $u_0 \in \mathscr{H}$ and $f \in L_2((0,t);e^{ms};\mathscr{H}_{\mathcal{A}^-})$. Then the solution of problem (1) is stable with respect to perturbation of operator and the following estimate holds:*

$$\|\tilde{u}(t) - u(t)\|^2 \leq M_1 e^{-mt}\|\tilde{\mathcal{A}} - \mathcal{A}\|^2_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}}, \tag{28}$$

where $M_1 = \|\tilde{u}\|^2_{L_2((0,t);e^{ms};\mathscr{H}_{\mathcal{A}})} \leq 4\left(\|u_0\|^2 + \int_0^t e^{ms}\|f(s)\|^2_{\tilde{\mathcal{A}}_0^{-1}} ds\right)$.

Proof. For the solution of problem (12) with nonself-adjoint positive definite operator $\mathcal{A}$ it is fulfilled estimate (26), whence, we have

$$\|\delta u(t)\|^2 \leq e^{-mt}\int_0^t e^{ms}\|(\tilde{\mathcal{A}} - \mathcal{A})\tilde{u}(s)\|^2_{\mathcal{A}_0^{-1}} ds \leq e^{-mt}\|\tilde{\mathcal{A}} - \mathcal{A}\|^2_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}}\int_0^t e^{ms}\|\tilde{u}(s)\|^2_{\tilde{\mathcal{A}}_0} ds.$$

Combining this with inequality (27) we get the desired estimate (28).

Similarly to estimate (13), taking into account (25), we obtain the following energy inequality:

$$\int_0^t \left(\|u(s)\|^2_{\mathcal{A}_0} + \left\|\frac{du(s)}{dt}\right\|^2_{\mathcal{A}_0^{-1}}\right) ds \leq \|u_0\|^2 + \int_0^t \|f(s)\|^2_{\mathcal{A}_0^{-1}} ds. \tag{29}$$

Also, as in derivation of (20), we obtain the estimate of integral semi-norm of fraction order in time variable t for nonself-adjoint opeator $\mathcal{A}$

$$\int_0^t \|u(s)\|^2_{\mathcal{A}_0} ds + \int_0^t\int_0^t \frac{\|u(s) - u(s')\|^2}{|s-s'|^2} ds\, ds' \leq M\left(\|u_0\|^2 + \int_0^t \|f(s)\|_{\mathcal{A}_0^{-1}} ds\right). \tag{30}$$

Using estimates (29) and (30) for the solution of problem for perturbation (12), we prove the following statement.

Theorem 4. *Let $u_0 \in \mathscr{H}$ and $f \in L_2((0,t);\mathscr{H}_{\mathcal{A}^-})$. Then the solution of problem (1) is stable with respect to perturbation of operator and the following estimate holds:*

$$\int_0^t \|\delta u(s)\|^2_{\tilde{\mathcal{A}}_0} ds + \int_0^t\int_0^t \frac{\|\delta u(s) - \delta u(s')\|^2}{|s-s'|^2} ds\, ds' \leq M_2\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}}, \tag{31}$$

where $M_2 = \|\tilde{u}\|_{L_2((0,t);\mathscr{H}_\mathcal{A})} \leq M \left(\|u_0\|^2 + \int_0^t \|f(s)\|^2_{\tilde{\mathcal{A}}_0^{-1}} ds \right)$.

2.4. EXAMPLE

Consider the initial-boundary value problem for the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + f(x,t), \quad 0 < x < 1, \quad t > 0, \tag{32}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{33}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1. \tag{34}$$

We can write problem (32)–(34) in the form of (1) assuming $\mathscr{H} = \mathscr{L}(,)$, $\mathcal{A}u = -\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)$. If $k(x)$ is continuously differentiable function ($k(x) \in C^1[0,1]$) satisfying conditions $0 < m_1 \leq k(x) \leq m_2$, $|k'(x)| \leq m_3$, then operator $\mathcal{A}$ mapping the set $\mathscr{D}(\mathcal{A}) = \overset{\mathscr{H}}{\circ}(,) \cap \mathscr{H}(,)$ to $L_2(0,1)$. It can easily be checked that the inverse operator $\mathcal{A}^{-1}$ is defined by

$$(\mathcal{A}^{-1}v)(x) = -\int_0^x \frac{1}{k(x')}\int_0^{x'} v(x'')dx''\,dx' +$$

$$+ \left(\int_0^1 \frac{dx'}{k(x')}\right)^{-1}\left(\int_0^x \frac{dx'}{k(x')}\right)\int_0^1 \frac{1}{k(x')}\int_0^{x'} v(x'')dx''\,dx'.$$

The following inequalities holds (see [8])

$$m_0\|v\|^2_{H^1(0,1)} \leq (\mathcal{A}v, v) = \int_0^1 k(x)|v'(x)|^2 dx \leq m_2\|v\|^2_{H^1(0,1)}, \quad v \in H^1(0,1), \tag{35}$$

$$m_4\|v\|^2_{H^2(0,1)} \leq \|\mathcal{A}v\|^2_{L_2(0,1)} \leq m_5\|v\|^2_{H^2(0,1)}, \quad v \in \overset{H^1}{\circ}(0,1) \cap H^2(0,1), \tag{36}$$

where $m_0 = m_1\pi^2/(1+\pi^2)$, and constants m_4 and m_5 are independent of m_1, m_2 and m_3. Thus $\mathscr{H}_\mathcal{A} = \overset{\mathscr{H}}{\circ}(,)$, $\mathscr{H}_{\mathcal{A}^-} = \overset{\mathscr{H}^-}{\circ}(,)$ and $\mathscr{H}_\mathcal{A} = \overset{\mathscr{H}}{\circ}(,) \cap \mathscr{H}(,)$.

Together with (32)–(34) we consider perturbed initial-boundary value problem

$$\frac{\partial \tilde{u}}{\partial t} = \frac{\partial}{\partial x}\left(\tilde{k}(x)\frac{\partial \tilde{u}}{\partial x}\right) + f(x,t), \quad 0 < x < 1, \quad t > 0, \tag{37}$$

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \quad t > 0, \tag{38}$$

$$\tilde{u}(x,0) = u_0(x), \quad 0 < x < 1. \tag{39}$$

Suppose that coefficients of the perturbed problem satisfy the following conditions: $\tilde{k}(x) \in C^1[0,1]$, $0 < \tilde{m}_1 \leq \tilde{k}(x) \leq \tilde{m}_2$, $|\tilde{k}'(x)| \leq \tilde{m}_3$. Similarly to non-perturbed problem we define $\tilde{\mathcal{A}}\tilde{u} = -\frac{\partial}{\partial x}\left(\tilde{k}(x)\frac{\partial \tilde{u}}{\partial x}\right)$.

The following inequalities holds:

$$\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}} \leq M\|\tilde{k} - k\|_{C[0,1]}, \tag{40}$$

$$\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}} \leq M\|\tilde{k} - k\|_{C^1[0,1]}, \tag{41}$$

$$\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H} \to \mathscr{H}_{\mathcal{A}^-}} \leq M\|\tilde{k} - k\|_{C^1[0,1]}, \tag{42}$$

where M is a certain constant.

Indeed, inequality (40) follows from the relations

$$\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}} = \sup_{v \in \mathscr{H}_{\mathcal{A}}} \frac{\|(\tilde{\mathcal{A}} - \mathcal{A})v\|_{\mathcal{A}^{-1}}}{\|v\|_{\mathcal{A}}} = \sup_{v \in \mathscr{H}_{\mathcal{A}}} \frac{|((\tilde{\mathcal{A}} - \mathcal{A})v, v)|}{(\mathcal{A}v, v)} =$$

$$= \sup_{v \in \overset{\circ}{H}{}^1(0,1)} \frac{\left|\int_0^1 (\tilde{k}(x) - k(x))|v'(x)|^2 dx\right|}{\int_0^1 k(x)|v'(x)|^2 dx}$$

and the properties of coefficients $k(x)$, $\tilde{k}(x)$.

Inequality (41) follows from (36) and the equalities

$$\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}} = \sup_{v \in \mathscr{H}_{\mathcal{A}}} \frac{\|(\tilde{\mathcal{A}} - \mathcal{A})v\|}{\|\mathcal{A}v\|} = \sup_{v \in \overset{\circ}{H}{}^1(0,1) \cap H^2(0,1)} \frac{\left|\int_0^1 |((\tilde{k}(x) - k(x)v'(x))'|^2 dx\right|}{\int_0^1 |(k(x)v'(x))'|^2 dx}.$$

Inequality (42) follows from definition of the norm of operator

$$\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H} \to \mathscr{H}_{\mathcal{A}^-}} = \sup_{v \in \mathscr{H}} \frac{\|\mathcal{A}^{-1}(\tilde{\mathcal{A}} - \mathcal{A})v\|}{\|v\|} = \sup_{v \in L_2(0,1)} \frac{\|\mathcal{A}^{-1}(\tilde{\mathcal{A}} - \mathcal{A})v\|_{L_2(0,1)}}{\|v\|_{L_2(0,1)}},$$

presentation

$$\mathcal{A}^{-1}(\tilde{\mathcal{A}} - \mathcal{A})v = \frac{\tilde{k}(x') - k(x')}{k(x')}v - \int_0^x \left[\frac{\tilde{k}'(x') - k'(x')}{k(x')} + \frac{k'(x')(k(x') - \tilde{k}(x'))}{k^2(x')}\right] v(x')dx' +$$

$$+\left(\int_0^1 \frac{dx'}{k(x')}\right)^{-1}\left(\int_0^x \frac{dx'}{k(x')}\right)\int_0^1 \left[\frac{\tilde{k}'(x') - k'(x')}{k(x')} + \frac{k'(x')(k(x') - \tilde{k}(x'))}{k^2(x')}\right] v(x')dx'$$

and the properties of coefficients $k(x)$, $\tilde{k}(x)$ and their derivatives.

Thus for the perturbation of the solution of problem (32)–(34), using Theorems 1 and 2 we obtain the following estimates of coefficient stability:

$$\|\delta u(t)\|^2_{L_2(0,1)} \leq M_1^{(0)} e^{-m_1 t} \|\tilde{k} - k\|^2_{C[0,1]},$$

$$\|\delta u(t)\|^2_{\overset{\circ}{H}{}^1(0,1)} \leq M_2^{(0)} e^{-m_1 t} \|\tilde{k} - k\|^2_{C^1[0,1]},$$

$$\int_0^t \left(\|\delta u(s)\|^2_{\overset{\circ}{H}{}^1(0,1) \cap H^2(0,1)} + \left\| \frac{d\, \delta u(s)}{dt} \right\|^2_{L_2(0,1)} \right) ds \leq M_3^{(0)} \|\tilde{k} - k\|^2_{C^1[0,1]}$$

$$\int_0^t \|\delta u(s)\|^2_{\overset{\circ}{H}{}^1(0,1)} ds + \int_0^t \int_0^t \frac{\|\delta u(s) - \delta u(s')\|^2_{L_2(0,1)}}{|s - s'|^2} ds\, ds' \leq M_4^{(0)} \|\tilde{k} - k\|^2_{C[0,1]},$$

$$\int_0^t \|\delta u(s)\|^2_{L_2(0,1)} ds \leq M_5^{(0)} \|\tilde{k} - k\|^2_{C^1[0,1]},$$

where

$$M_1^{(0)} \leq M \left(\|u_0\|^2_{L_2(0,1)} + \int_0^t e^{ms} \|f(s)\|^2_{\overset{\circ}{H}{}^{-1}(0,1)} ds \right),$$

$$M_2^{(0)} \leq M \left(\|u_0\|^2_{\overset{\circ}{H}{}^1(0,1)} + \int_0^t e^{ms} \|f(s)\|^2_{L_2(0,1)} ds \right),$$

$$M_3^{(0)} \leq M \left(\|u_0\|^2_{\overset{\circ}{H}{}^1(0,1)} + \int_0^t \|f(s)\|^2_{L_2(0,1)} ds \right),$$

$$M_4^{(0)} \leq M \left(\|u_0\|^2_{L_2(0,1)} + \int_0^t \|f(s)\|^2_{\overset{\circ}{H}{}^{-1}(0,1)} ds \right),$$

$$M_5^{(0)} \leq M \left(\|u_0\|^2_{\overset{\circ}{H}{}^{-1}(0,1)} + \int_0^t \|f(s)\|^2_{\overset{\circ}{H}{}^{-1}(0,1) \cap H^{-2}(0,1)} ds \right).$$

3. STABILITY OF THE SOLUTIONS OF TWO-LEVEL OPERATOR-DIFFERENCE SCHEMES WITH RESPECT TO PERTURBATION OF OPERATOR

For two-level operator-difference schemes it is fulfilled the results, which is similar to results obtained in Section 1.

3.1 PROBLEM DEFINITION

Let H be a finite-dimensional Hilbert space supplied by inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$, A be linear operator in H. Define uniform grid in time by

$$\bar{\omega}_\tau = \omega_\tau \cup \{T\} = \{t \mid t_n = n\tau, \quad n = 0, 1, \ldots, N_0, \quad N_0\tau = T\}.$$

Consider the following two-level operator-difference scheme with weight

$$\frac{y^{n+1} - y^n}{\tau} + A(\sigma y^{n+1} + (1-\sigma)y^n) = \varphi^n, \quad n = 0, 1, \ldots, N_0 - 1, \quad y^0 = y_0, \qquad (43)$$

where σ is weighting parameter, y_0 is given element of space H, $\varphi^n = \varphi(t_n)$ is given function and $y^n = y(t_n)$ is unknown function on time-level $t = t_n$ with values in H.

Together with (43) we consider the following difference Cauchy problem with perturbed operator:

$$\frac{\tilde{y}^{n+1} - \tilde{y}^n}{\tau} + \sigma\tilde{A}(\tilde{y}^{n+1} + (1-\sigma)\tilde{y}^n) = \varphi^n, \quad n = 0, 1, \ldots, N_0 - 1, \quad \tilde{y}^0 = y_0. \qquad (44)$$

As in the case of differential-operator equation we set the problem to estimate the value of perturbation of solution $\delta y^n = \tilde{y}^n - y^n$ via perturbation of operator $\delta A = \tilde{A} - A$.

3.2. STABILITY IN THE CASE OF SELF-ADJOINT OPERATOR

Let in problem (43) A be self-adjoint positive-definite operator. Let perturbed operator satisfies the same assumption. By H_A denote the space with the inner product $(u,v)_A = (Au, v)$ and the norm $\|u\|_A = \sqrt{(Au,u)}$.

In [14, 16] it is proved the following estimate of asymptotic stability

$$\|y^{n+1}\|_B^2 \le e^{-\bar{m}t_{n+1}} \left(\|y_0\|_B^2 + \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|\varphi^k\|_{A^{-1}}^2 \right), \qquad (45)$$

where $B = E + \sigma\tau A$, $\bar{m} = \bar{m}(\tau, \|A\|)$.

To prove the difference analogue of inequality (8) we take the inner product of equation (43) with $2\tau e^{\bar{m}t_{n+1}}y^n$. Taking into account the identity

$$\sigma y^{n+1} + (1-\sigma)y^n = y^n + (\sigma-1)\tau\frac{y^{n+1}-y^n}{\tau},$$

$$2\tau e^{\bar{m}t_{n+1}}\left(y^n, R\frac{y^{n+1}-y^n}{\tau}\right) = e^{\bar{m}t_{n+1}}\|y^{n+1}\|_R^2 - e^{\bar{m}t_n}\|y^n\|_R^2 + \tau^2 e^{\bar{m}t_{n+1}}\left\|\frac{y^{n+1}-y^n}{\tau}\right\|_R^2 -$$
$$-e^{\bar{m}t_{n+1}}(1-e^{-\bar{m}\tau})\|y^n\|_R^2, \quad R = E + (\sigma-1)\tau A = R^*,$$

we obtain the following energy equality

$$e^{\bar{m}t_{n+1}}\|y^{n+1}\|_R^2 - e^{\bar{m}t_n}\|y^n\|_R^2 + \tau^2 e^{\bar{m}t_{n+1}}\left\|\frac{y^{n+1}-y^n}{\tau}\right\|_R^2 - e^{\bar{m}t_{n+1}}(1-e^{-\bar{m}\tau})\|y^n\|_R^2 +$$
$$+2e^{\bar{m}t_{n+1}}\tau\|y^n\|_A^2 = 2\tau e^{\bar{m}t_{n+1}}(y^n, \varphi^n).$$

Hence using Cauchy—Schwartz inequality, ε-inequality, relation $1-e^{-\bar{m}\tau} \le \bar{m}\tau$, and positive definiteness of the operator A, we get

$$e^{\bar{m}t_{n+1}}\|y^{n+1}\|_R^2 - e^{\bar{m}t_n}\|y^n\|_R^2 + \frac{\varepsilon(1-(\sigma-1)\tau)-1}{\varepsilon}e^{\bar{m}t_{n+1}}\bar{m}\tau\|y^n\|_A^2 \le$$
$$\le \varepsilon\tau e^{\bar{m}t_{n+1}}\|\varphi^n\|_{A^{-1}}^2 \qquad \forall\varepsilon > 0.$$

Under conditions $\sigma \le 1 + (1+\varepsilon)/(\varepsilon\tau)$, summing the last inequality in $k = 0, 1, \ldots, n$, we obtain

$$\sum_{k=0}^{n}\tau e^{\bar{m}t_{k+1}}\|y^k\|_A^2 \le M\left(\|y_0\|_R^2 + \sum_{k=0}^{n}\tau e^{\bar{m}t_{k+1}}\|\varphi^k\|_{A^{-1}}^2\right), \tag{46}$$

where a positive constant M depends on ε only. Similar estimate holds for the solution $\tilde{y}$ of perturbed problem (44). Acting on equation (44) by the operator $\tilde{A}^{1/2}$ and using inequality (46), we get the following estimate:

$$\sum_{k=0}^{n}\tau e^{\bar{m}t_{k+1}}\|\tilde{A}\tilde{y}^k\|^2 \le M\left(\|\tilde{A}^{1/2}y_0\|_R^2 + \sum_{k=0}^{n}\tau e^{\bar{m}t_{k+1}}\|\varphi^k\|^2\right). \tag{47}$$

The obtained estimates we shall use to prove the following statement.

Theorem 5. *Let $1-(\tau\|A\|)^{-1} \le \sigma \le 1 + (1+\varepsilon)/(\varepsilon\tau)$. Then the solution of two-level operator-difference scheme (43) is stable with respect to perturbation of operator and the following estimate holds:*

$$\|\tilde{y}^{n+1} - y^{n+1}\|_B \le M_1 e^{-\bar{m}t_{n+1}}\|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{A^{-1}}}, \tag{48}$$

where $M_1 = \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|\tilde{y}^{k+1}\|_{\tilde{A}} \le M \left(\|y_0\|_{\tilde{R}}^2 + \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|\varphi^k\|_{\tilde{A}^{-1}}^2 \right)$

If $1-(1-\varepsilon)/(\tau\|A\|) \le \sigma \le 1+(1+\varepsilon)/(\varepsilon\tau)$, *then for perturbation of the solution of two-level operator-difference scheme (43) the following estimate holds:*

$$\|\tilde{y}^{n+1} - y^{n+1}\|_A \le M_2 e^{-\bar{m}t_{n+1}} \|\tilde{A} - A\|_{H_{\tilde{A}^2} \to H}, \tag{49}$$

where $M_2 = \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|\tilde{A}\tilde{y}^k\|^2 \le M \left(\|\tilde{A}^{1/2} y_0\|_R^2 + \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|\varphi^k\|^2 \right)$.

Proof. Consider the problem for perturbation of solution$\delta y^n = \tilde{y}^n - y^n$

$$\frac{\delta y^{n+1} - \delta y^n}{\tau} + A(\sigma \delta y^{n+1} + (1-\sigma)\delta y^n) = (\tilde{A} - A)(\sigma \tilde{y}^{n+1} + (1-\sigma)\tilde{y}^n). \tag{50}$$

For the solution of problem (50) it is fulfilled estimate (45), whence it follows that

$$\|\delta y^{n+1}\|_B^2 \le e^{-\bar{m}t_{n+1}} \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|(\tilde{A} - A)(\sigma \tilde{y}^{k+1} + (1-\sigma)\tilde{y}^k)\|_{A^{-1}}^2 \le$$

$$\le e^{-\bar{m}t_{n+1}} \|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{A^{-1}}} \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|\sigma \tilde{y}^{k+1} + (1-\sigma)\tilde{y}^k\|_{\tilde{A}}^2 \le$$

$$\le M e^{-\bar{m}t_{n+1}} \|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{A^{-1}}} \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|\tilde{y}^k\|_{\tilde{A}}^2.$$

Hence taking into account inequality (46) we obtain the desired estimate (48).

Moreover, acting on equation (50) by the operator $A^{1/2}$, using estimate (45) and $1-(1-\varepsilon)/(\tau\|A\|) \le \sigma \le 1+(1+\varepsilon)/(\varepsilon\tau)$, we get the inequality

$$\|\delta y^{n+1}\|_A^2 \le e^{-\bar{m}t_{n+1}} \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|(\tilde{A} - A)(\sigma \tilde{y}^{k+1} + (1-\sigma)\tilde{y}^k)\|^2 \le$$

$$\le e^{-\bar{m}t_{n+1}} \|\tilde{A} - A\|_{H_{\tilde{A}^2} \to H} \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|\tilde{A}(\sigma \tilde{y}^{k+1} + (1-\sigma)\tilde{y}^k)\|^2 \le$$

$$\le M e^{-\bar{m}t_{n+1}} \|\tilde{A} - A\|_{H_{\tilde{A}^2} \to H} \sum_{k=0}^{n} \tau e^{\bar{m}t_{k+1}} \|\tilde{A}\tilde{y}^k\|^2.$$

Hence using (47) and upper bound of weight σ, we obtain the desired estimate (49).

Using the method of energy inequality for $\sigma \ge 0.5(1+\varepsilon) - (\tau\|A\|)^{-1}$ it is easy to construct the grid analogues of *a priori* estimates (13), (14) and (15) (see [7]):

$$\sum_{k=0}^{n}{}' \tau \|Ay^k\|^2 + \sum_{k=0}^{n-1} \tau \left\| \frac{y^{k+1} - y^k}{\tau} \right\|^2 \le M_2 \left(\|y_0\|_A^2 + \tau \|Ay_0\|^2 + \sum_{k=0}^{N_0-1} \tau \|\varphi^k\|^2 \right), \tag{51}$$

$$\sum_{k=0}^{n}{}'\tau\|y^k\|_A^2+\sum_{k=0}^{n-1}{}'\tau\sum_{\substack{l=0\\l\neq k}}^{n-1}{}'\tau\frac{\left\|y^k-y^l\right\|^2}{|t_k-t_l|^2}\leq M_3\left(\|y_0\|^2+\tau\|y_0\|_A^2+\sum_{k=0}^{N_0-1}\tau\|\varphi^k\|_{A^{-1}}^2\right), \quad (52)$$

and

$$\sum_{k=0}^{n}{}'\tau\|y^k\|^2\leq M_4\left(\|y_0\|_{A^{-1}}^2+\tau\|y_0\|^2+\sum_{k=0}^{N_0-1}\tau\|A^{-1}\varphi^k\|^2\right), \quad (53)$$

where $\sum_{k=0}^{n}{}'v^k=v^0/2+\sum_{k=1}^{N_0-1}v^k+v^{N_0}/2$.

The following theorem holds.

Theorem 6. *Let* $\sigma\geq 0.5(1+\varepsilon)-(\tau\|A\|)^{-1}$. *Then the solution of problem (43) is stable with respect to perturbation of operator and the following estimates holds:*

$$\sum_{k=0}^{n}{}'\tau\|A\delta y^k\|^2+\sum_{k=0}^{n-1}\tau\left\|\frac{\delta y^{k+1}-\delta y^k}{\tau}\right\|^2\leq M_5\|\tilde{A}-A\|_{H_{\tilde{A}^2}\to H}, \quad (54)$$

$$\sum_{k=0}^{n}{}'\tau\|\delta y^k\|_A^2+\sum_{k=0}^{n-1}{}'\tau\sum_{\substack{l=0\\l\neq k}}^{n-1}{}'\tau\frac{\left\|\delta y^k-\delta y^l\right\|^2}{|t_k-t_l|^2}\leq M_6\|\tilde{A}-A\|_{H_{\tilde{A}}\to H_{A^{-1}}}, \quad (55)$$

and

$$\sum_{k=0}^{n}{}'\tau\|\delta y^k\|^2\leq M_7\|\tilde{A}-A\|_{H\to H_{A^{-2}}}, \quad (56)$$

where

$$M_5=\sum_{k=0}^{N_0-1}\tau\|\tilde{A}\tilde{y}^k\|^2\leq M_2\left(\|y_0\|_{\tilde{A}}^2+\tau\|\tilde{A}y_0\|^2+\sum_{k=0}^{N_0-1}\tau\|\varphi^k\|^2\right),$$

$$M_6=\sum_{k=0}^{N_0-1}\tau\|\tilde{y}^k\|_{\tilde{A}}^2\leq M_3\left(\|y_0\|^2+\tau\|y_0\|_{\tilde{A}}^2+\sum_{k=0}^{N_0-1}\tau\|\varphi^k\|_{\tilde{A}^{-1}}^2\right),$$

$$M_7=\sum_{k=0}^{N_0-1}\tau\|\tilde{y}^k\|^2\leq M_4\left(\|y_0\|_{\tilde{A}^{-1}}^2+\tau\|y_0\|^2+\sum_{k=0}^{N_0-1}\tau\|\tilde{A}^{-1}\varphi^k\|^2\right),$$

Proof. Estimates (54)–(56) follows directly from estimates (51)–(53), which is applied to the solution of problem (50).

3.3. EXAMPLE

Consider the difference approximation of the initial-boundary value problem (32)–(34). On interval $[0,1]$ we introduce the uniform grid

$$\bar{\omega}_h = \omega_h \cup \{0,1\} = \omega_h^+ \cup \{0\} = \omega_h^- \cup \{1\} = \{x \mid x_i = ih, \quad i = 0,1,\dots,N, \quad Nh = 1\}.$$

By y_i^n denote the value of the grid function y at the node (x_i, t_n) of the grid $\bar{\omega}_{h\tau} = \bar{\omega}_h \times \bar{\omega}_\tau$. For difference derivatives we use the following notation:

$$y_{x,i}^n = \frac{y_{i+1}^n - y_i^n}{h}, \quad y_{\bar{x},i}^n = \frac{y_i^n - y_{i-1}^n}{h}.$$

On the grid $\bar{\omega}_{h\tau}$ we approximate problem (32)–(34) by the following difference scheme with weight

$$\frac{y_i^{n+1} - y_i^n}{\tau} = (a(\sigma y_{\bar{x}}^{n+1} + (1-\sigma) y_{\bar{x}}^n))_{x,i} + \varphi_i^n, \quad i = 1,2,\dots,N, \quad n = 0,1,\dots,N_0, \tag{57}$$

$$y_0^n = y_N^n = 0, \quad n = 0,1,\dots,N_0, \tag{58}$$

$$y_i^0 = u_0(x_i), \quad i = 1,2,\dots,N-1, \tag{59}$$

where a_i and φ_i^n are certain stencil functionals approximating coefficient $k(x)$ and right-hand side $f(x,t)$ of equation (32), respectively. For example, $a_i = k(x_{i-1/2})$, $\varphi_i^n = f(x_i, t_{n+1/2})$, $x_{i-0.5} = x_i - h/2$, $t_{n+1/2} = t_n + \tau/2$. Problem (57)–(59) can be reduced to the form of (43), where H is the space of grid functions that is given on $\bar{\omega}_h$ and vanish at $x = 0$ and $x = 1$; $(y,v) = \sum_{i=1}^{N-1} h y_i v_i$ and $(Ay)_i = -(a y_{\bar{x}})_{x,i}$. The inverse operator A^{-1} is defined by

$$(A^{-1}v)_i = -\sum_{j=1}^{i} \frac{h}{a_j} \sum_{p=1}^{j-1} h v_p + \left(\sum_{j=1}^{N} \frac{h}{a_j}\right)^{-1} \left(\sum_{j=1}^{i} \frac{h}{a_j}\right) \left(\sum_{j=1}^{N} \frac{h}{a_j} \sum_{p=1}^{j-1} h v_p\right).$$

The following inequalities holds (see [3])

$$m_6 \|v\|_{H^1(\omega_h)}^2 \le (Av, v) = \sum_{i=1}^{N} h a_i |v_{\bar{x},i}|^2 \le m_7 \|v\|_{H^1(\omega_h)}^2, \quad v \in H, \tag{60}$$

$$m_8 \|v\|_{H^2(\omega_h)}^2 \le \|Av\| \le m_9 \|v\|_{H^2(\omega_h)}^2, \quad v \in H. \tag{61}$$

where the discrete Sobolev norms are defined by the following way:

$$\|v\|^2_{H^1(\omega_h)} = \|v\|^2 + (\|v_{\bar{x}}\|^+)^2, \quad \|v\|^2_{H^2(\omega_h)} = \|v\|^2 + (\|v_{\bar{x}}\|^+)^2 + \|v_{\bar{x}x}\|^2,$$

$$(\|v_{\bar{x}}\|^+)^2 = \sum_{i=1}^{N} h|y_{\bar{x},i}|^2.$$

Together with (57)–(59) we consider the perturbed problem

$$\frac{\tilde{y}_i^{n+1} - \tilde{y}_i^n}{\tau} = (\tilde{a}(\sigma \tilde{y}_{\bar{x}}^{n+1} + (1-\sigma)\tilde{y}_{\bar{x}}^n))_{x,i} + \varphi_i^n, \quad i = 1, 2, \ldots, N, \quad n = 0, 1, \ldots, N_0, \tag{62}$$

$$\tilde{y}_0^n = \tilde{y}_N^n = 0, \quad n = 0, 1, \ldots, N_0, \tag{63}$$

$$\tilde{y}_i^0 = u_0(x_i), \quad i = 1, 2, \ldots, N-1, \tag{64}$$

which can be written in the form of (44) with the operator $(\tilde{A}\tilde{y})_i = -(\tilde{a}\tilde{y}_{\bar{x}})_{x,i}$.

The following inequalities holds:

$$\|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{A^{-1}}} \le M\|\tilde{a} - a\|_{C(\bar{\omega}_h)} = M \max_{0 \le i \le N} |\tilde{a}_i - a_i|, \tag{65}$$

$$\|\tilde{A} - A\|_{H_{\tilde{A}^2} \to H} \le M\|\tilde{a} - a\|_{C^1(\bar{\omega}_h)} = M \left(\max_{0 \le i \le N} |\tilde{a}_i - a_i| + \max_{1 \le i \le N} |\tilde{a}_{\bar{x},i} - a_{\bar{x},i}| \right), \tag{66}$$

$$\|\tilde{A} - A\|_{H \to H_{A^{-2}}} \le M\|\tilde{a} - a\|_{C^1(\bar{\omega}_h)}. \tag{67}$$

In fact, inequality (65) follows from relation

$$\|\tilde{A} - A\|_{H_{\tilde{A}} \to H_{A^{-1}}} = \sup_{v \in H_{\tilde{A}}} \frac{\|(\tilde{A} - A)v\|_{A^{-1}}}{\|v\|_{\tilde{A}}} = \sup_{v \in H_{\tilde{A}}} \frac{|((\tilde{A} - A)v, v)|}{\|v\|_{\tilde{A}}\|v\|_A} =$$

$$= \sup_{v \in H_{\tilde{A}}} \frac{\left|\sum_{i=1}^N h(\tilde{a}_i - a_i)|v_{\bar{x},i}|^2\right|}{\left(\sum_{i=1}^N h|\tilde{a}_i v_{\bar{x},i}|\right)^{1/2} \left(\sum_{i=1}^N h|a_i v_{\bar{x},i}|\right)^{1/2}}$$

and the properties of coefficients $\tilde{a}$ a.

Inequality (66) follows form (61) and equality

$$\|\tilde{A} - A\|_{H_{\tilde{A}^2} \to H} = \sup_{v \in H_{\tilde{A}^2}} \frac{\|(\tilde{A} - A)v\|}{\|\tilde{A}v\|} = \frac{\sum_{i=1}^{N-1} h|((\tilde{a}_i - a_i)v_{\bar{x}})_{x,i}|^2}{\sum_{i=1}^{N-1} h|(\tilde{a}_i v_{\bar{x}})_{x,i}|^2}.$$

Inequality (67) follows from definition of the norm of operator

$$\|\tilde{A} - A\|_{H \to H_{A^{-2}}} = \sup_{v \in H} \frac{\|A^{-1}(\tilde{A} - A)v\|}{\|v\|},$$

presentation

$$(A^{-1}(\tilde{A}-A)v)_i = \frac{\tilde{a}_i - a_i}{a_i} v_i - \sum_{j=1}^{i-1} h \left[\frac{\tilde{a}_{x,j} - a_{x,j}}{a_{j+1}} - \frac{a_{x,j}(\tilde{a}_j - a_j)}{a_j a_{j+1}} \right] v_j +$$

$$+ \left(\sum_{j=1}^{N} \frac{h}{a_j} \right)^{-1} \left(\sum_{j=1}^{i} \frac{h}{a_j} \right) \left(\sum_{j=1}^{N-1} h \left[\frac{\tilde{a}_{x,j} - a_{x,j}}{a_{j+1}} - \frac{a_{x,j}(\tilde{a}_j - a_j)}{a_j a_{j+1}} \right] v_j \right)$$

and the properties of coefficients $\tilde{a}_i$, a_i and their difference derivatives.

Thus for the perturbation of the solution of problem (57)–(59) using Theorems 5 and 6 under corresponding restrictions on the weighting parameter σ we obtain the following estimates of coefficient stability:

$$\|\delta y^{n+1}\|^2 \le M_1^{(0)} e^{-\bar{m} t_{n+1}} \|\tilde{a} - a\|_{C(\bar{\omega}_h)},$$

$$\|\delta y^{n+1}\|^2_{H^1(\omega_h)} \le M_2^{(0)} e^{-\bar{m} t_{n+1}} \|\tilde{a} - a\|_{C^1(\bar{\omega}_h)},$$

$$\sum_{k=0}^{n}{}' \tau \|\delta y^k\|^2_{H^2(\omega_h)} + \sum_{k=0}^{n-1} \tau \left\| \frac{\delta y^{k+1} - \delta y^k}{\tau} \right\|^2 \le M_5^{(0)} \|\tilde{a} - a\|_{C^1(\bar{\omega}_h)},$$

$$\sum_{k=0}^{n}{}' \tau \|\delta y^k\|^2_{H^1(\omega_h)} + \sum_{k=0}^{n-1}{}' \tau \sum_{\substack{l=0 \\ l \ne k}}^{n-1}{}' \tau \frac{\left\|\delta y^k - \delta y^l\right\|^2}{|t_k - t_l|^2} \le M_6^{(0)} \|\tilde{a} - a\|_{C(\bar{\omega}_h)},$$

$$\sum_{k=0}^{n}{}' \tau \|\delta y^k\|^2 \le M_7^{(0)} \|\tilde{a} - a\|_{C^1(\bar{\omega}_h)}$$

with corresponding constants $M_1^{(0)}$, $M_2^{(0)}$, $M_5^{(0)}$, $M_6^{(0)}$ $M_7^{(0)}$, depending on initial data u_0 and right-hand side φ.

4. STABILITY OF THE SOLUTION OF THE SECOND-ORDER DIFFERENTIAL-OPERATOR EQUATIONS WITH RESPECT TO PERTURBATION OF OPERATOR

In this section we investigate the stability of the solution of the second-order differential-operator equations with respect to perturbation of operator.

4.1. PROBLEM DEFINITION

Consider the Cauchy problem for the second-order differential-operator equation

$$\frac{d^2u}{dt^2} + \mathcal{A}u = f, \quad 0 < t \leq T, \quad u(0) = u_0, \quad \frac{du(0)}{dt} = u_1, \tag{68}$$

where $\mathcal{A}$ is unbounded self-adjoint positive-definite operator mapping the Hilbert space $\mathscr{H}$ to Hilbert space $\mathscr{G}$, $u(t)$ is unknown function taking interval $(0, \infty)$ (or $(0, T)$) to $\mathscr{H}$, $f(t)$ is given function taking interval $(0, \infty)$ (or $(0, T)$) to $\mathscr{G}$, u_0, u_1 are given elements of the space $\mathscr{H}$.

Together with (68) we consider the problem with perturb operator $\mathcal{A}$

$$\frac{d^2\tilde{u}}{dt^2} + \tilde{\mathcal{A}}\tilde{u} = f, \quad 0 < t \leq T, \quad \tilde{u}(0) = u_0, \quad \frac{du(0)}{dt} = u_1, \tag{69}$$

where $\tilde{\mathcal{A}} : \mathscr{H} \to \mathscr{G}$ is unbounded self-adjoint positive-definite linear operator.

4.2. STABILITY ESTIMATE UNDER PERTURBATION OF OPERATOR $\mathcal{A}$

Let us obtain the stability estimates under perturbation of operator $\mathcal{A} : \mathscr{H} \to \mathscr{H}$ with domain of definition $\mathscr{D}(\mathcal{A})$, which is everywhere dense in $\mathscr{H}$. For that we need the corresponding estimates of stability with respect to initial data and right-hand side. Let the operator $\tilde{\mathcal{A}}$ satisfies the same conditions, which is fulfilled for operator $\mathcal{A}$ of the non-perturbed problem, and $\mathscr{D}(\mathcal{A}) = \mathscr{D}(\mathcal{A})$.

Taking the inner product of (68) with $2\frac{du}{dt}$, we get the energy identity

$$\frac{d}{dt}\left(\left\|\frac{du(t)}{dt}\right\|^2 + \|u(t)\|^2_{\mathcal{A}}\right) = 2\left(f(t), \frac{du(t)}{dt}\right), \tag{70}$$

whence

$$\frac{d}{dt}\left(\left\|\frac{du(t)}{dt}\right\|^2 + \|u(t)\|^2_{\mathcal{A}}\right) \leq 2\left\|\frac{du(t)}{dt}\right\| \|f(t)\| \leq \left(\left\|\frac{du(t)}{dt}\right\|^2 + \|u(t)\|^2_{\mathcal{A}}\right)^{1/2} \|f(t)\|.$$

Integrating the last inequality in t, we obtain the estimate

$$\left(\left\|\frac{du(t)}{dt}\right\|^2 + \|u(t)\|^2_{\mathcal{A}}\right)^{1/2} \leq \left(\|u_1\|^2 + \|u_0\|^2_{\mathcal{A}}\right)^{1/2} + \int_0^t \|f(s)\| ds. \tag{71}$$

Moreover from (70) we have

$$\frac{d}{dt}\left(\left\|\frac{du(t)}{dt}\right\|^2 + \|u(t)\|^2_{\mathcal{A}}\right) = 2\frac{d(f(t),u(t))}{dt} - 2\left(\frac{df(t)}{dt}, u(t)\right),$$

and

$$\frac{d}{dt}\left(\left\|\frac{du(t)}{dt}\right\|^2 + \|u(t)\|^2_{\mathcal{A}} - 2(u(t),f(t)) + \|f(t)\|^2_{\mathcal{A}^{-1}}\right)$$
$$= 2\left(\frac{df(t)}{dt}, \mathcal{A}^{-1}f(t)\right) - 2\left(\frac{df(t)}{dt}, u(t)\right) = 2\left(\frac{df(t)}{dt}, \mathcal{A}^{-1}f(t) - u\right).$$

Whence we obtain

$$\frac{d}{dt}\left(\left\|\frac{du(t)}{dt}\right\|^2 + \|u(t) - \mathcal{A}^{-1}f(t)\|^2_{\mathcal{A}}\right) \le 2\left\|\frac{df(t)}{dt}\right\|_{\mathcal{A}^{-1}} \|u(t) - \mathcal{A}^{-1}f(t)\|_{\mathcal{A}}$$
$$\le 2\left\|\frac{df(t)}{dt}\right\|_{\mathcal{A}^{-1}} \left(\left\|\frac{du(t)}{dt}\right\|^2 + \|u(t) - \mathcal{A}^{-1}f(t)\|^2_{\mathcal{A}}\right)^{1/2}$$

Integrating the last relation in t, we get

$$\left(\left\|\frac{du(t)}{dt}\right\|^2 + \|u(t)\|^2_{\mathcal{A}}\right)^{1/2} \le \left(\|u_1\|^2 + \|u_0\|^2_{\mathcal{A}}\right)^{1/2} + \|f(t)\|_{\mathcal{A}^{-1}} + \|f(0)\|_{\mathcal{A}^{-1}} + \int_0^t \|f'(s)\|_{\mathcal{A}^{-1}} ds. \quad (72)$$

Similar estimates holds for the solution $\tilde{u}$ of the problem with the perturbed operator (69).

Theorem 7. *Let $u_0 \in \mathscr{H}_{\mathcal{A}}$, $u_1 \in \mathscr{H}$, $f \in L_1((0,t);\mathscr{H})$. Then the solution of problem (68) is stable with respect to perturbation of operator and the following estimate holds:*

$$\|\tilde{u}(t) - u(t)\|_{\mathcal{A}} \le M_1\|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}}, \quad (73)$$

where $M_1 \le \|u_0\|_{\tilde{\mathcal{A}}} + (t+1)\left(\left(\|u_1\|^2 + \|u_0\|^2_{\mathcal{A}}\right)^{1/2} + \int_0^t \|f(s)\|ds\right)$.

Proof. For perturbation of solution $\delta u = \tilde{u} - u$ we have Cauchy problem

$$\frac{d^2\,\delta u}{dt^2} + \tilde{\mathcal{A}}\delta u = (\tilde{\mathcal{A}} - \mathcal{A})\tilde{u}, \quad 0 < t \le T, \quad \delta u(0) = 0, \quad \frac{d\,\delta u(0)}{dt} = 0. \quad (74)$$

For the solution of problem (74) it is fulfilled estimate (72), whence

$$\left(\left\|\frac{d\,\delta u(t)}{dt}\right\|^2 + \|\delta u(t)\|^2_{\mathcal{A}}\right)^{1/2}$$
$$\leq \|(\tilde{\mathcal{A}} - \mathcal{A})\tilde{u}(t)\|_{\mathcal{A}^{-1}} + \|(\tilde{\mathcal{A}} - \mathcal{A})\tilde{u}_0\|_{\mathcal{A}^{-1}} + \int_0^t \|(\tilde{\mathcal{A}} - \mathcal{A})\tilde{u}(s)\|_{\mathcal{A}^{-1}} ds$$
$$\leq \|\tilde{\mathcal{A}} - \mathcal{A}\|_{\mathscr{H}_{\mathcal{A}} \to \mathscr{H}_{\mathcal{A}^-}} \left(\|\tilde{u}(t)\|_{\tilde{\mathcal{A}}} + \|\tilde{u}_0\|_{\tilde{\mathcal{A}}} + \int_0^t \|\tilde{u}(s)\|_{\tilde{\mathcal{A}}} ds\right).$$

Taking into account (71) from the last inequality we obtain desired estimate (73).

4.3. EXAMPLE

Consider initial-boundary value problem for the second-order hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + f(x,t), \quad 0 < x < 1, \quad t > 0, \tag{75}$$

$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{76}$$

$$u(x,0) = u_0(x), \qquad \frac{\partial u(x,0)}{\partial t}, \quad 0 < x < 1. \tag{77}$$

Defining $\mathscr{H} = \mathscr{L}(,)$, $\mathcal{A}u = -\frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)$, we reduce problem (75)–(77) to the form of (68). If $k(x)$ is continuously differentiable function ($k(x) \in C^1[0,1]$) satisfying conditions $0 < m_1 \leq k(x) \leq m_2$, $|k'(x)| \leq m_3$, then the operator $\mathcal{A}$ map the set $\mathscr{D}(\mathcal{A}) = \overset{\mathscr{H}}{\circ}(,) \cap \mathscr{H}(,)$ to $L_2(0,1)$.

Together with (75)-(77) we consider the perturbed initial-boundary value problem

$$\frac{\partial^2 \tilde{u}}{\partial t^2} = \frac{\partial}{\partial x}\left(\tilde{k}(x)\frac{\partial \tilde{u}}{\partial x}\right) + f(x,t), \quad 0 < x < 1, \quad t > 0, \tag{78}$$

$$\tilde{u}(0,t) = \tilde{u}(1,t) = 0, \quad t > 0, \tag{79}$$

$$\tilde{u}(x,0) = u_0(x), \qquad \frac{\partial \tilde{u}(x,0)}{\partial t}, \quad 0 < x < 1. \tag{80}$$

Suppose that the coefficient of the perturbed problem satisfies conditions $\tilde{k}(x) \in C^1[0,1]$, $0 < \tilde{m}_1 \leq \tilde{k}(x) \leq \tilde{m}_2$, $|\tilde{k}'(x)| \leq \tilde{m}_3$. Similarly to non-perturbed problem we set $\tilde{\mathcal{A}}\tilde{u} = -\frac{\partial}{\partial x}\left(\tilde{k}(x)\frac{\partial \tilde{u}}{\partial x}\right)$.

For the perturbation of operator $\delta\mathcal{A} = \tilde{\mathcal{A}} - \mathcal{A}$ estimate (40) holds (see Subsection 1.4), whence using Theorem 7 we obtain the following estimate of coefficient stability:

$$\|\tilde{u}(t) - u(t)\|^2_{\overset{\circ}{H}{}^1(0,1)} \leq M_1^{(0)} \|\tilde{k} - k\|_{C[0,1]}$$

with constant

$$M_1^{(0)} = M\left(\|u_0\|_{\overset{\circ}{H}{}^1(0,1)} + (t+1)\left(\left(\|u_1\|^2_{L_2(0,1)} + \|u_0\|^2_{\overset{\circ}{H}{}^1(0,1)}\right)^{1/2} + \int_0^t \|f(s)\|_{L_2(0,1)} ds\right)\right).$$

5. THREE-LEVEL OPERATOR-DIFFERENCE SCHEMES

Now we pass to construction of estimates for three-level operator-difference schemes which is similar to the estimates obtained in Section 3.

5.1. PROBLEM FORMULATION

Let H be finite-dimensional Hilbert space supplied the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$, A be linear operator in H. On the uniform time-grid $\bar{\omega}_\tau$ we consider three-level operator-difference scheme with weights

$$\frac{y^{n+1} - 2y^n + y^{n-1}}{\tau^2} + A(\sigma_1 y^{n+1} + (1-\sigma_1-\sigma_2)y^n + \sigma_2 y^{n-1}) = \varphi^n, \ n = 1, 2, \ldots, N_0 - 1,$$
$$y^0 = y_0, \quad \frac{y^1 - y^0}{\tau} = y_1, \tag{81}$$

where σ_1, σ_2 are weighting parameters, y_0, y_1 are given elements of the space H, $\varphi^n = \varphi(t_n)$ is given function and $y^n = y(t_n)$ is unknown function on time-level $t = t_n$ with values in H.

Together with (81) we consider similar Cauchy problem with perturbed operator

$$\frac{\tilde{y}^{n+1} - 2\tilde{y}^n + \tilde{y}^{n-1}}{\tau^2} + \tilde{A}(\sigma_1 \tilde{y}^{n+1} + (1-\sigma_1-\sigma_2)\tilde{y}^n + \sigma_2 \tilde{y}^{n-1}) = \varphi^n, \ n = 1, 2, \ldots, N_0 - 1, \tag{82}$$

$$\tilde{y}^0 = y_0, \qquad \frac{\tilde{y}^1 - \tilde{y}^0}{\tau} = y_1.$$

As in the case of differential-operator equation we set the problem to estimate the value of perturbation of solution $\delta y^n = \tilde{y}^n - y^n$ via perturbation of operator $\delta A = \tilde{A} - A$.

5.2. ESTIMATES OF STABILITY WITH RESPECT TO PERTURBATION OF OPERATOR

Let in problem (81) A be self-adjoint positive-definite operator. Let perturbed operator satisfies the same assumptions. By H_A denote the space with inner product $(u,v)_A = (Au, v)$ and the norm $\|u\|_A = \sqrt{(Au,u)}$.

Take the inner product of (81) with $(y^{n+1} - y^{n-1})$, we obtain the energy identity

$$\left(R\frac{y^{n+1}-y^n}{\tau}, \frac{y^{n+1}-y^n}{\tau}\right) + \left\|\frac{y^{n+1}+y^n}{2}\right\|_A^2 - \left(R\frac{y^n-y^{n-1}}{\tau}, \frac{y^{n+1}-y^n}{\tau}\right) +$$

$$+\left\|\frac{y^n+y^{n-1}}{2}\right\|_A^2 + (\sigma_1-\sigma_2)\tau\left\|\frac{y^{n+1}-y^{n-1}}{2\tau}\right\|_A^2 = (\varphi, y^{n+1}-y^n) + (\varphi, y^n - y^{n-1}),$$

where $R = E + ((\sigma_1+\sigma_2)/2 - 1/4)\,\tau^2 A$. Hence, estimating the right-hand side as in [4], under the conditions $\sigma_1 \geq \sigma_2$, $\sigma_1+\sigma_2 > (1+\varepsilon)/2 - (\tau^2\|A\|)^{-1}$, $\varepsilon > 0$ we get the discrete analogues of estimates (71) and (72)

$$\left(\left\|\frac{y^{n+1}-y^n}{\tau}\right\|_R^2 + \left\|\frac{y^{n+1}+y^n}{2}\right\|_A^2\right)^{1/2} \leq \left(\|y_1\|^2_{R+\tau^2 A/4} + \|y_0\|_A^2\right)^{1/2} + \sum_{k=1}^{n}\tau\|\varphi^k\|_{R^{-1}}. \quad (83)$$

$$\left(\left\|\frac{y^{n+1}-y^n}{\tau}\right\|_R^2 + \left\|\frac{y^{n+1}+y^n}{2}\right\|_A^2\right)^{1/2} \leq \left(\|y_1\|^2_{R+\tau^2 A/4} + \|y_0\|_A^2\right)^{1/2} + \|\varphi^n\|_{A^{-1}} + \|\varphi^0\|_{A^{-1}} + \sum_{k=1}^{n}\tau\left\|\frac{\varphi^k-\varphi^{k-1}}{\tau}\right\|_{A^{-1}}. \quad (84)$$

Similar estimates holds for the solution $\tilde{y}$ of problem (82) with perturbed operator.

Theorem 8. *Let weighting parameters satisfy the following conditions: $\sigma_2 = 0$, $\sigma_1 = \sigma \geq 0$, $\sigma > (1+\varepsilon)/2 - (\tau^2\|A\|)^{-1}$. Then the solution of problem (81) is stable* *with respect to perturbation of operator and the following estimate holds:*

$$\left\| \frac{\delta y^{n+1} - \delta y^n}{2} \right\|_A \le M_2 \|\tilde{A} - A\|_{H_A \to H_{A^{-1}}}, \tag{85}$$

where

$$M_2 \le M \left(\|y_0\|_{\tilde{A}} + \sigma\tau \|y_1\|_{\tilde{A}} + (t_n + 1) \left(\left(\|y_1\|^2_{R+\tau^2 A/4} + \|y_0\|^2_A \right)^{1/2} + \sum_{k=1}^{n} \tau \|\varphi^k\|_{R^{-1}} \right) \right).$$

Proof. For the perturbation of solution $\tilde{y}^n = \tilde{y} - y$ we have the Cauchy problem

$$\begin{gathered} \frac{\delta y^{n+1} - 2\delta y^n + \delta y^{n-1}}{\tau^2} + A(\sigma \delta y^{n+1} + (1-\sigma)\delta y^n) \\ = (\tilde{A} - A)(\sigma \tilde{y}^{n+1} + (1-\sigma)\tilde{y}^n), \ n = 1, 2, \ldots, N_0 - 1, \\ \delta y^0 = 0, \quad \frac{\delta y^1 - \delta y^0}{\tau} = 0, \end{gathered} \tag{86}$$

For the solution of problem (86) it is fulfilled estimate (84), whence we get

$$\begin{aligned} \left(\left\| \frac{\delta y^{n+1} - \delta y^n}{\tau} \right\|^2_R + \left\| \frac{\delta y^{n+1} + \delta y^n}{2} \right\|^2_A \right)^{1/2} &\le \|(\tilde{A} - A)(\sigma \tilde{y}^{n+1} + (1-\sigma)\tilde{y}^n)\|_{A^{-1}} \\ &+ \|(\tilde{A} - A)(\sigma \tilde{y}^1 + (1-\sigma)\tilde{y}^0\|_{A^{-1}} + \sum_{k=1}^{n} \tau \|(\tilde{A} - A)(\sigma \tilde{y}^{k+1} + (1-\sigma)\tilde{y}^k\|_{A^{-1}}) \\ &\le \|\tilde{A} - A\|_{H_A \to H_{A^{-1}}} \Big(\|\sigma \tilde{y}^{n+1} + (1-\sigma)\tilde{y}^n\|_{\tilde{A}} \\ &+ \|y_0\|_{\tilde{A}} + \sigma\tau \|y_1\|_{\tilde{A}} + \sum_{k=1}^{n} \tau \|\sigma \tilde{y}^{k+1} + (1-\sigma)\tilde{y}^k\|_{\tilde{A}} \Big). \end{aligned}$$

Using (83) from the last inequality we obtain to desired estimate (85).

5.3. EXAMPLE

On the grid $\bar{\omega}_{h\tau}$ (see Subsection 2.3) we approximate problem (75)–(77) by the following three-level difference scheme with weight σ:

$$\frac{y_i^{n+1} - 2y_i^n + y_i^{n-1}}{\tau^2} = (a(\sigma y_{\bar{x}}^{n+1} + (1-\sigma) y_{\bar{x}}^n))_{x,i} + \varphi_i^n, \quad i = 1, 2, \ldots, N, \ n = 0, 1, \ldots, N_0, \tag{87}$$

$$y_0^n = y_N^n = 0, \quad n = 0, 1, \ldots, N_0, \tag{88}$$

$$y_i^0 = u_0(x_i), \quad \frac{y_i^1 - y_i^0}{\tau} = \bar{u}_1(x_i), \quad i = 1, 2, \ldots, N-1, \tag{89}$$

where a_i and φ_i^n are certain stencil functionals approximating coefficient $k(x)$ and right-hand side $f(x,t)$ of equation (32), respectively. For example, $a_i = k(x_{i-1/2})$, $\varphi_i^n = f(x_i, t_n)$, $x_{i-0.5} = x_i - h/2$, $t_{n+1/2} = t_n + \tau/2$. In addition, $\bar{u}_1(x_i)$ is certain approximation of the second initial condition. Problem (87)–(89) reduced to the form of (81), where H is the space of grid functions which are given on $\bar{\omega}_h$ and vanish at $x = 0$ and $x = 1$, $(y, v) = \sum_{i=1}^{N-1} h y_i v_i$ and $(Ay)_i = -(a y_{\bar{x}})_{x,i}$.

Together with problem (87)–(89) we consider the perturbed difference scheme

$$\frac{\tilde{y}_i^{n+1} - 2\tilde{y}_i^n + \tilde{y}_i^{n-1}}{\tau^2} = (\tilde{a}(\sigma \tilde{y}_{\bar{x}}^{n+1} + (1-\sigma)\tilde{y}_{\bar{x}}^n))_{x,i} + \varphi_i^n, \quad i = 1, 2, \ldots, N, \ n = 0, 1, \ldots, N_0, \tag{90}$$

$$\tilde{y}_0^n = \tilde{y}_N^n = 0, \quad n = 0, 1, \ldots, N_0, \tag{91}$$

$$\tilde{y}_i^0 = u_0(x_i), \quad \frac{\tilde{y}_i^1 - \tilde{y}_i^0}{\tau} = \bar{u}_1(x_i), \quad i = 1, 2, \ldots, N-1. \tag{92}$$

For perturbation of operator $\delta A = \tilde{A} - A$ estimate (65) holds. Therefore from Theorem 8 we get the estimate of coefficient stability

$$\|\tilde{y}^{n+1} - y^{n+1}\|_{H^1(\omega_h)} \le M_2^{(0)} \|\tilde{a} - a\|_{C(\bar{\omega}_h)}$$

with a constant $M_2^{(0)}$ depending on initial data u_0, $\bar{u}_1$ and right-hand side φ.

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