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ABOUT SOME TYPES OF BOUNDARY VALUE PROBLEMS WITH INTERFACES

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Abstract. We report results concerning different types of boundary value problems with interfaces. A broad class of such problems is defined. The corresponding abstract problems are investigated and some a priori estimates are presented. Intrinsic function spaces containing solutions of considered problems are introduced.

1. INTRODUCTION

Interface problems occur in many applications in science and engineering [16], [18]. Let us mention, for example, heat transfer in presence of concentrated capacity, oscillations with concentrated mass, Pupin's induction coils etc. Such kind of problems can be modelled by partial differential equations with discontinuous or singular input data. First partial derivatives of their solutions have discontinuities across one or several interfaces, which have lower dimension than the domain where the problem is defined. Numerical methods designed for the solution of problems with smooth solutions do not work efficiently for the interface problems.

In the present work, we report results concerning different types of boundary value problems with interfaces [5], [7]-[13], [21]. In particular, we analyze different ways to set an interface problem (strong form of equation with singular coefficients, problem with conjugation conditions), abstract models, a priori estimates, intrinsic function spaces containing the solution etc.

2. PARTIAL DIFFERENTIAL EQUATIONS WITH SINGULAR COEFFICIENTS

Let Ω be a bounded domain in $\mathbb{R}^n$ with the boundary $\Gamma = \partial\Omega$. Let $S \subset \Omega$ be a hypersurface of dimension $n-1$ (interface) dividing Ω into two disjoint parts Ω_1 and Ω_2. As a model problem we consider the following elliptic boundary value problem

$$\begin{gathered} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\Big) + [c(x) + k(x)\delta_S(x)]u = f(x), \quad x \in \Omega, \\ u(x) = 0, \quad x \in \Gamma, \end{gathered} \tag{1}$$

where

$$\begin{gathered} a_{ij} = a_{ji}, \quad \sum_{i,j=1}^{n} a_{ij} y_i y_j \geq c_0 \sum_{i=1}^{n} y_i^2, \quad c_0 > 0, \\ c(x) \geq c_1 > 0, \quad k(x) \geq k_0 > 0, \end{gathered}$$

and $\delta_S(x)$ is the Dirac distribution [19] concentrated on S.

We also consider analogous initial boundary value problems of parabolic

$$\begin{gathered} [c(x) + k(x)\delta_S(x)]\,\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\Big) = f(x,t), \quad (x,t) \in Q, \\ u(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega \end{gathered} \tag{2}$$

and hyperbolic types

$$\begin{gathered} [c(x) + k(x)\delta_S(x)]\,\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\Big) = f(x,t), \quad (x,t) \in Q, \\ u(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T), \\ u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in \Omega, \end{gathered} \tag{3}$$

where $Q = \Omega \times (0,T)$.

3. CONJUGATION CONDITIONS

The considered interface problems can be formulated in an alternative manner, without explicit use of the Dirac distribution. It is well known that for a piecewise smooth function $\varphi \in C^1[a,\xi] \cap C^1[\xi,b]$ the derivative in distributional sense can be expressed in the following way [19]:

$$\varphi'(x) = \{\varphi'(x)\} + [\varphi]_\xi \, \delta(x-\xi) \tag{4}$$

where $\{\varphi'(x)\}$ is the derivative in the classic sense and $[\varphi]_\xi = \varphi(\xi+0) - \varphi(\xi-0)$ is the jump of the function φ in the point ξ.

Because of $\delta_S(x) = 0$ for $x \notin S$ from (1) immediately follows

$$\begin{gathered} -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\Big) + c(x)u = f(x), \quad x \in \Omega_1 \cup \Omega_2, \\ u(x) = 0, \quad x \in \Gamma, \end{gathered} \tag{5}$$

Taking into account (4), we conclude that on the interface S the following conjugation conditions are satisfied

$$[u]_S = 0, \quad \left[\frac{\partial u}{\partial \nu}\right]_S = ku, \quad x \in S, \tag{6}$$

where $\frac{\partial u}{\partial \nu}$ denote the co-normal derivative:

$$\frac{\partial u}{\partial \nu} = \sum_{i,j=1}^{n} a_{ij}(x)\,\frac{\partial u}{\partial x_j}\,\cos(\nu, x_i).$$

Analogously, the parabolic initial boundary value problem (2) can be represented in an equivalent form

$$\begin{gathered} c(x)\,\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\Big) = f(x,t), \quad (x,t) \in Q_1 \cup Q_2, \\ u(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T), \\ [u]_S = 0, \quad \left[\frac{\partial u}{\partial \nu}\right]_S = k\,\frac{\partial u}{\partial t}, \quad x \in S, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega, \end{gathered} \tag{7}$$

where $Q_i = \Omega_i \times (0,T)$, while the hyperbolic problem (3) reduces to

$$\begin{gathered}
c(x)\,\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\Big) = f(x,t), \quad (x,t) \in Q_1 \cup Q_2, \\
u(x,t) = 0, \quad (x,t) \in \Gamma \times (0,T), \\
[u]_S = 0, \quad \left[\frac{\partial u}{\partial \nu}\right]_S = k\,\frac{\partial^2 u}{\partial t^2}, \quad x \in S, \quad t \in (0,T), \\
u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in \Omega.
\end{gathered} \tag{8}$$

Our aim is to investigate the relation between differentiable properties of the solutions of the boundary value problems (1)-(3) and the smoothness of its coefficients and right hand sides. It will be shown that this task can be resolved using methods of functional analysis to construct a priori estimates for the solutions of linear equations and the Cauchy problems for linear differential equations in Hilbert space.

4. ABSTRACT MODELS

Let H be a real separable Hilbert space endowed with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$. Let A be an unbounded, selfadjoint, positive definite linear operator acting in H, with domain $D(A)$ dense in H. The product $(u,\, v)_A = (Au,\, v)$ $(u,\ v \in D(A))$ satisfies the inner product axioms. Reinforcing $D(A)$ in the norm $\|u\|_A = (u,\, u)_A^{1/2}$ we obtain so called energy space $H_A \subset H$. The inner product $(u,\, v)$ continuously extends to $H_A^* \times H_A$, where $H_A^* = H_{A^{-1}}$ is the adjoint space for H_A. The spaces H_A, H and $H_{A^{-1}}$ form Gel'fand triple: $H_A \subset H \subset H_{A^{-1}}$. Operator A extends to a mapping $A : H_A \to H_{A^{-1}}$. There exists unbounded selfadjoint positive definite linear operator $A^{1/2}$ such that $D(A^{1/2}) = H_A$ and $(u,\, v)_A = (Au,\, v) = (A^{1/2}u,\, A^{1/2}v)$ (see [15], [17]).

For $1 \le p \le \infty$ we define the Lebesgue space $L_p((a,b),H)$ of functions mapping real interval (a,b) into H, with the norm

$$\|u\|_{L_p((a,b),H)} = \Big(\int_a^b \|u(t)\|^p dt \Big)^{1/p}, \quad 1 \le p < \infty$$

$$\|u\|_{L_\infty((a,b),H)} = \operatorname*{supess}_{t\in(a,b)} \|u(t)\|$$

We also introduce the spaces of continuous functions $C([a,b],H)$ and $C^k([a,b],H)$ with the norms:

$$\|u\|_{C([a,b],H)} = \max_{t\in[a,b]} \|u(t)\|, \quad \|u\|_{C^k([a,b],H)} = \max_{0\le j\le k} \|u^{(j)}\|_{C([a,b],H)},$$

where $u^{(j)}(t) = d^j u/dt^j$, and the Sobolev spaces $W_p^k((a,b),H)$ with the norms [22]:

$$\|u\|_{W_p^k((a,b),H)} = \Big(\sum_{j=0}^k \|u^{(j)}\|^p_{L_p((a,b),H)} \Big)^{1/p}, \quad k = 0,1,2,\dots, \quad 1 \le p < \infty,$$

$$\|u\|_{W_p^{k+\alpha}((a,b),H)} = (\|u\|^p_{W_p^k((a,b),H)} + |u^{(k)}|^p_{W_p^\alpha((a,b),H)})^{1/p}, \quad 0<\alpha<1,$$

$$|u|_{W_p^\alpha((a,b),H)} = \Big(\int_a^b \int_a^b \frac{\|u(t)-u(t')\|^p}{|t-t'|^{1+\alpha p}}\, dt\, dt' \Big)^{1/p},$$

with standard modification for $p=\infty$.

Let B be another unbounded, selfadjoint, positive definite linear operator acting in H, such that $D(A) \subset D(B) \subset H$. In general, A and B are noncommutative. We assume that the quotient $\|u\|_A/\|u\|_B$ is unbounded on $D(A)$. Under these assumptions there exists a countable set of eigenvalues $\{\lambda_i\}$ of the spectral problem

$$Au = \lambda Bu. \tag{9}$$

All eigenvalues are positive and $\lambda_i \to \infty$ when $i \to \infty$. Further,

$$\|u\|_A \ge \sqrt{\lambda_1}\, \|u\|_B\,, \quad \|u\|_{AB^{-1}A} \ge \sqrt{\lambda_1}\, \|u\|_A\,,$$

$$\|u\|_{(AB^{-1})^n A} \ge \sqrt{\lambda_1}\, \|u\|_{(AB^{-1})^{n-1}A}\,, \quad n = 2,3,\dots$$

where λ_1 is the first (minimal) eigenvalue of the spectral problem (9).

We consider the following linear equation of the first kind in H

$$(A+B)u = f, \tag{10}$$

which can be treated as an abstract model for the boundary value problem (1). Its solution satisfies the following a priori estimates:

$$\|u\|_B \le \|f\|_{A^{-1}BA^{-1}}, \tag{11}$$

$$\|u\|_A \le \|f\|_{A^{-1}}, \tag{12}$$

$$\|u\|_{AB^{-1}A} \le \|f\|_{B^{-1}}, \tag{13}$$

etc. Note that (11) and (12) also hold even for nonnegative B.

We also consider the following abstract Cauchy problem

$$Bu' + Au = f(t), \quad 0 < t < T; \qquad u(0) = u_0 \tag{14}$$

where $u(t)$ is an unknown function from (0,T) into H. Setting $B^{1/2}u = v$ equation (14) reduces to more simple

$$v' + \Lambda v = g(t), \quad 0 < t < T; \qquad v(0) = v_0 \tag{15}$$

where $\Lambda = B^{-1/2}AB^{-1/2}$, $g = B^{-1/2}f$ and $v_0 = B^{1/2}u_0$. Using the known a priori estimates for the solution of the problem (15) (see [15], [14], [6])

$$\int_0^T \|v(t)\|^2\,dt \le M\Big(\|v_0\|^2_{\Lambda^{-1}} + \int_0^T \|\Lambda^{-1}g(t)\|^2\,dt\Big),$$

$$\int_0^T \|v(t)\|^2_\Lambda\,dt + \int_0^T\int_0^T \frac{\|v(t)-v(t')\|^2}{|t-t'|^2}\,dt\,dt' \le M\Big(\|v_0\|^2 + \int_0^T \|g(t)\|^2_{\Lambda^{-1}}\,dt\Big),$$

$$\int_0^T (\|\Lambda v(t)\|^2 + \|v'(t)\|^2)\,dt \le M\Big(\|v_0\|^2_\Lambda + \int_0^T \|g(t)\|^2\,dt\Big), \quad \text{etc.}$$

we immediately obtain the corresponding a priori estimates for the problem (14):

$$\|u\|^2_{L_2((0,T),H_B)} \le M\,(\|u_0\|^2_{BA^{-1}B} + \|f\|^2_{L_2((0,T),H_{A^{-1}BA^{-1}})}), \tag{16}$$

$$\|u\|^2_{L_2((0,T),H_A)} + \|u\|^2_{W_2^{1/2}((0,T),H_B)} \le M\,(\|u_0\|^2_B + \|f\|^2_{L_2((0,T),H_{A^{-1}})}), \tag{17}$$

$$\|u\|^2_{L_2((0,T),H_{AB^{-1}A})} + \|u\|^2_{W_2^1((0,T),H_B)} \le M\,(\|u_0\|^2_A + \|f\|^2_{L_2((0,T),H_{B^{-1}})}), \tag{18}$$

etc. Here and in the sequel M denotes positive generic constant which can take different values in different formulas. Note that obtained a priori estimates contain whole information about the relation between the differentiability of the solution of Cauchy problem (14) and the smoothness properties of input data. For example, from (18) immediately follows: if $u_0 \in H_A$ and $f \in L_2(0,T;\, H_{B^{-1}})$ then $u \in L_2((0,T), H_{AB^{-1}A}) \cap W_2^1((0,T), H_B)$. Estimates (16) and (17) are satisfied also for nonnegative B.

Finally, let us consider the Cauchy problem for the second order abstract differential equation

$$Bu'' + Au = f(t), \quad 0 < t < T; \quad u(0) = u_0; \quad u'(0) = u_1. \tag{19}$$

Similarly as in the previous case one obtains the following a priori estimates for its solution

$$\|u\|_{C([0,T],H_B)} \le M\,(\|u_0\|_B + \|u_1\|_{BA^{-1}B} + \|f\|_{L_1((0,T),H_{A^{-1}})}), \tag{20}$$

$$\|u\|_{C([0,T],H_A)} + \|u'\|_{C([0,T],H_B)} \le M\,(\|u_0\|_A + \|u_1\|_B + \|f\|_{L_1((0,T),H_{B^{-1}})}), \tag{21}$$

etc. Estimate (16) also holds for nonnegative B.

5. IDENTIFICATION OF FUNCTION SPACES AND NORMS

Let us choose $H = L_2(\Omega)$. Then the boundary value problem (1) reduces to the abstract form (10), where

$$Au = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\,\frac{\partial u}{\partial x_j}\Big), \qquad Bu = [c(x) + k(x)\delta_S(x)]u.$$

Analogously, the initial boundary value problem (2) reduces to the abstract Cauchy problem (14), while the problem (3) reduces to (19).

For $u \in D(A) = W_2^2(\Omega) \cap \overset{\circ}{W}{}_2^1(\Omega)$ using partial integration we get

$$\|u\|_A^2 = \int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)\,\frac{\partial u}{\partial x_i}\,\frac{\partial u}{\partial x_j}\,dx.$$

For $a_{ij} \in L_\infty(\Omega)$ under the assumption of strong ellipticity one immediately obtains

$$\|u\|_A \asymp \|u\|_{W_2^1(\Omega)} \quad \text{i.e.} \quad M_1 \|u\|_{W_2^1(\Omega)} \leq \|u\|_A \leq M_2 \|u\|_{W_2^1(\Omega)} \cdots$$

In such a way, we conclude that $H_A = \overset{\circ}{W}{}_2^1(\Omega)$. Consequently, $H_{A^{-1}} = H_A^* = (\overset{\circ}{W}{}_2^1(\Omega))^* = W_2^{-1}(\Omega)$ and

$$\|u\|_{A^{-1}} \asymp \|u\|_{W_2^{-1}(\Omega)}.$$

If $c \in L_\infty(\Omega)$, $k \in L_\infty(S)$, $c(x) \geq c_1 > 0$ and $k(x) \geq k_0 > 0$ then

$$\|u\|_B^2 = \int_\Omega c(x)\, u^2(x)\, dx + \int_S k(x)\, u^2(x)\, dS \asymp \|u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(S)}.$$

In such a way $H_B = L_2(\Omega) \cap L_2(S)$, $H_{B^{-1}} = (L_2(\Omega) \cap L_2(S))^*$ and

$$\|u\|_{B^{-1}} \asymp \sup_{v \in L_2(\Omega) \cap L_2(S)} \frac{|{}_{H_{B^{-1}}}\langle u, v\rangle_{H_B}|}{(\|v\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(S)})^{1/2}}.$$

Here ${}_{H_{B^{-1}}}\langle u, v\rangle_{H_B}$ denotes the duality pairing in $H_{B^{-1}} \times H_B$.

In addition to the previous assumptions, let $a_{ij} \in W_\infty^1(\Omega)$. Then

$$\|u\|^2_{AB^{-1}A} \asymp |u|^2_{W_2^2(\Omega_1)} + |u|^2_{W_2^2(\Omega_2)} + \|u\|^2_{W_2^1(\Omega)}.$$

In such a way $H_{AB^{-1}A} = W_2^2(\Omega_1) \cap W_2^2(\Omega_2) \cap \overset{\circ}{W}{}_2^1(\Omega)$, $H_{A^{-1}BA^{-1}} = (W_2^2(\Omega_1) \cap W_2^2(\Omega_2) \cap \overset{\circ}{W}{}_2^1(\Omega))^*$ and

$$\|u\|_{A^{-1}BA^{-1}} \asymp \sup_{v \in W_2^2(\Omega_1) \cap W_2^2(\Omega_2) \cap \overset{\circ}{W}{}_2^1(\Omega)} \frac{|{}_{H_{A^{-1}BA^{-1}}}\langle u, v\rangle_{H_{AB^{-1}A}}|}{(|u|^2_{W_2^2(\Omega_1)} + |u|^2_{W_2^2(\Omega_2)} + \|u\|^2_{W_2^1(\Omega)})^{1/2}}.$$

Finally, the norm $\|u\|_{BA^{-1}B}$ can be estimated as

$$\|u\|_{BA^{-1}B} = \|Bu\|_{A^{-1}} \asymp \|u + u\,\delta_S\|_{W_2^{-1}(\Omega)}.$$

6. DEFINITION OF INTRINSIC FUNCTION SPACES

A priori estimates (11)-(13), (16)-(18) and (20)-(21) implicitly define intrinsic function spaces containing the solutions of considered interface problems. In the elliptic case, such spaces are Sobolev-like spaces $\widetilde{W}_2^k(\Omega)$ defined in the following manner:

$$\begin{aligned}
&\widetilde{W}_2^0(\Omega) = \widetilde{L}_2(\Omega) = L_2(\Omega) \cap L_2(S),\\
&\widetilde{W}_2^1(\Omega) = \overset{\circ}{W}{}_2^1(\Omega),\\
&\widetilde{W}_2^k(\Omega) = W_2^k(\Omega_1) \cap W_2^k(\Omega_2) \cap \overset{\circ}{W}{}_2^1(\Omega), \quad k = 2, 3, \ldots
\end{aligned}$$

In the parabolic case we define anisotropic Sobolev-like spaces

$$\widetilde{W}_2^{k,k/2}(Q) = L_2((0,T), \widetilde{W}_2^k(\Omega)) \cap W_2^{k/2}((0,T), \widetilde{L}_2(\Omega)), \quad k = 0, 1, 2, \ldots$$

The a priori estimates (11)-(13) can now be rewritten as

$$\|u\|_{\widetilde{L}_2(\Omega)} \le M \, \|f\|_{\left(\widetilde{W}_2^2(\Omega)\right)^*}, \tag{22}$$

$$\|u\|_{\widetilde{W}_2^1(\Omega)} \le M \, \|f\|_{\left(\widetilde{W}_2^1(\Omega)\right)^*} \qquad \Big(\text{i.e.} \quad \|u\|_{\overset{\circ}{W}{}_2^1(\Omega)} \le M \, \|f\|_{W_2^{-1}(\Omega)}\Big), \tag{23}$$

$$\|u\|_{\widetilde{W}_2^2(\Omega)} \le M \, \|f\|_{\left(\widetilde{L}_2(\Omega)\right)^*}. \tag{24}$$

Analogously, the estimates (16)-(18) reduce to

$$\|u\|^2_{\widetilde{W}_2^{0,0}(Q)} \le M \, (\|u_0 + u_0\delta_S\|^2_{W_2^{-1}(\Omega)} + \|f\|^2_{L_2((0,T),(\widetilde{W}_2^2(\Omega))^*)}), \tag{25}$$

$$\|u\|^2_{\widetilde{W}_2^{1,1/2}(Q)} \le M \, (\|u_0\|^2_{\widetilde{L}_2(\Omega)} + \|f\|^2_{L_2((0,T),W_2^{-1}(\Omega))}), \tag{26}$$

$$\|u\|^2_{\widetilde{W}_2^{2,1}(Q)} \le M \, (\|u_0\|^2_{\overset{\circ}{W}{}_2^1(\Omega)} + \|f\|^2_{L_2((0,T),(\widetilde{L}_2(\Omega))^*)}), \tag{27}$$

while the estimates (20)-(21) reduce to

$$\begin{aligned}
&\|u\|_{C([0,T],\widetilde{L}_2(\Omega))} \le M \, (\|u_0\|_{\widetilde{L}_2(\Omega)}\\
&\qquad + \|u_1 + u_1\delta_S\|_{W_2^{-1}(\Omega)} + \|f\|_{L_1((0,T),W_2^{-1}(\Omega))}),
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\|u\|_{C([0,T],\overset{\circ}{W}{}_2^1(\Omega))} + \left\|\frac{\partial u}{\partial t}\right\|_{C([0,T],\widetilde{L}_2(\Omega))} \le\\
&\qquad \le M \, (\|u_0\|_{\overset{\circ}{W}{}_2^1(\Omega)} + \|u_1\|_{\widetilde{L}_2(\Omega)} + \|f\|_{L_1((0,T),(\widetilde{L}_2(\Omega))^*)}).
\end{aligned} \tag{29}$$

7. SOME OTHER INTERFACE PROBLEMS

Initial boundary value problems with dynamical boundary conditions of the following type

$$d(x)\,\frac{\partial^k u}{\partial t^k} = \frac{\partial u}{\partial \nu}\,, \qquad x \in \Gamma_1 \subset \Gamma, \quad t \in (0,T), \quad k = 1,2$$

have similar properties as the previously investigated interface problems. Let us consider, for the sake of simplicity, the simplest one-dimensional parabolic problem with dynamical boundary condition

$$\begin{gathered} c(x)\,\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) = f(x,t), \quad (x,t) \in Q = (0,1)\times(0,T), \\ K\,\frac{\partial u}{\partial t}(0,t) = a(0)\,\frac{\partial u}{\partial x}(0,t)\,, \qquad u(1,t) = 0, \\ u(x,0) = u_0(x), \quad x \in (0,1) \end{gathered} \tag{30}$$

The problem (30) can be reduced to a problem of the type (2) using even extension of the input data: $c(x) = c(-x)$, $a(x) = a(-x)$, $u_0(x) = u_0(-x)$, and $f(x,t) = f(-x,t)$ for $x \in (-1,0)$. It easily follows that the solution $u(x,t)$ can also be evenly extended on $(-1,0)\times(0,T)$ and it satisfies the conditions

$$\begin{gathered} [c(x) + 2K\delta(x)]\,\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\Big(a(x)\,\frac{\partial u}{\partial x}\Big) = f(x,t), \quad (x,t) \in (-1,1)\times(0,T), \\ u(-1,t) = 0\,, \quad u(1,t) = 0, \\ u(x,0) = u_0(x), \quad x \in (-1,1). \end{gathered} \tag{31}$$

Analogous results hold for the corresponding hyperbolic problem with dynamical boundary condition.

Finally, let us mention so called "weakly" evolution problems, i.e. the initial boundary value problems of the form (2) and (3) where $c(x) = 0$. In this case the operator B reduces to

$$Bu = k(x)\delta_S(x)u(x)$$

and it is only nonnegative. Consequently, the inverse operator B^{-1} is not defined and the a priori estimates (18) and (21) are meaningless. We have

$$\|u\|_B \asymp \|u\|_{L_2(S)}$$

and $H_B = L_2(S)$, therefore the a priori estimate (26) reduces to

$$\begin{aligned} \|u\|^2_{\widetilde{W}_2^{1,1/2}(Q)} &\equiv \|u\|^2_{L_2((0,T),\overset{\circ}{W}{}_2^1(\Omega))} + |u|^2_{W_2^{1/2}((0,T),L_2(S))} \\ &\le M\,(\|u_0\|^2_{L_2(S)} + \|f\|^2_{L_2((0,T),W_2^{-1}(\Omega))}), \end{aligned} \tag{32}$$

Note that in this case the initial conditions (2) and (3) are determined only on interface S.

8. FINITE DIFFERENCE APPROXIMATION

For the finite difference analogues of (10), (14) and (19) similar results hold. For example, the discrete analogues of the a priori estimates (16)-(18) are obtained in [7], [8], while the discrete analogues of the a priori estimates (20)-(21) are obtained in [9]. In particular, convergence of finite difference schemes approximating interface problems in a natural way can be proved in the discrete version of norms appearing in (22)-(29) (see [7]-[13], [21]). Numerical methods for the solution of different interface problems are also investigated in [1], [2], [3], [20], [4], [23] etc.

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