BOUNDED AND L^2-SOLUTIONS OF CERTAIN THIRD ORDER NON-LINEAR DIFFERENTIAL EQUATION WITH A SQUARE INTEGRABLE FORCING TERM

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Abstract. This paper is concerned with the existence of bounded and L^2-solutions to equations of the form

$$(*) \qquad \dddot{x} + a(t)f(\dot{x})\ddot{x} + b(t)g(x)\dot{x} + c(t,x) = e(t),$$

where $e(t)$ is a continuous square integrable function. We obtain sufficient conditions which guarantee that all solutions of the equation $(*)$ are bounded are in $L^2[0, \infty)$.

1. INTRODUCTION

In this paper we are interested in the existence of bounded and L^2-solutions on the non-negative real line $[0, \infty)$ to the equation

$$\dddot{x} + a(t)f(\dot{x})\ddot{x} + b(t)g(x)\dot{x} + c(t,x) = e(t), \qquad (1.1)$$

where f and g are continuous on $\Re$, $a(t)$ and $b(t)$ are continuous on $[0, \infty)$, with $e(t)$ square integrable. By an L^2-solution, we mean a solution of (1.1) such that $\int_0^\infty x(t)^2 dt < \infty$. The equation (1.1) and its second order analogous play an important role in the phase locked loop model realised by T.V. system. See e.g. [1, 2] for more expository results.

The problem of existence of solutions which are bounded, and in $L^2[0, \infty)$ for higher order nonlinear differential equations has been of great interest to many mathematicians for decades. Such a problem has been studied mostly for second order nonlinear differential equations by many authors (see, e.g., [1, 4-7] and the references included; for the case of L^p-solutions, see also [5, 6, 8]), but only a few results (see [2]) are related to the third order nonlinear differential equations. In [1], the authors discussed the square integrable solutions of Duffing's and Lienard's types, where the physical motivation as well as application were explained in detail. The case of a more generalized Lienard equation was treated in [5]. A third order analogous of the problem discussed in [1] can be found in [2]. However, the forcing term in the equation treated in the last cited paper was not a continuous square integrable function.

Here, we would like to investigate the third order analogous of the problem discussed in [7] and obtain sufficient conditions on the equation (1.1) that will ensure the existence of bounded and L^2-solutions on the non-negative real line $[0, \infty)$. The results obtained in this work extend and generalize to third order nonlinear differential equations, the results in [7]. We also obtained sufficient conditions that made all the solutions of the equation (1.1) as well as as their first and second derivatives bounded.

2. MAIN RESULTS

The main results of this work are the following:

Theorem 2.1. *Consider the differential equation (1.1) where $e(t)$ is continu-*

ous on $[0, \infty)$ and $\int_0^\infty e(t)^2 dt < \infty$. Let $a(.)$ and $b(.)$ be continuous on $[0, \infty)$, with $a(t) > a_0 > 0, b(t) > b_0 > 0, f(.)$, and $g(.)$ also continuous on $\Re$ with $f(\dot{x}) > f_0 > 0$, $g(x) > g_0 > 0$ where a_0, b_0, f_0 and g_0 are positive constants. In addition, let $c(t, x)$ be a continuous function on $[0, \infty) \times \Re$ with $\int_0^\infty c(t, x) dt = \infty$ uniformly on t, and $x \frac{\partial c(t,x)}{\partial t} \leq 0$; then any solution x of (1.1), as well as its derivatives are bounded on $[0, \infty)$.

Theorem 2.2. *Let the hypotheses of the Theorem 2.1 hold. In addition, suppose that $c(t, x)x > c_0 x^2$ for some positive constant c_0 and $a(.), b(.)$ are decreasing i.e. $\dot{a}(t) \leq 0, \dot{b}(t) \leq 0$, then all the solutions of the equation (1.1) are L^2-solutions.*

3. PROOF OF THE MAIN RESULTS

Proof of Theorem 2.1.

By the standard existence theorem, there is a solution to the equation (1.1) which exists on $[0, T)$ for some $T > 0$. On multiplying the equation (1.1) by $\dot{x}$ and integrating the last part of LHS by parts we have

$$\dot{x}(t)\ddot{x}(t) - \int_0^t \ddot{x}(s)^2 ds + \int_0^t a(s)f(\dot{x}(s))\ddot{x}(s)\dot{x}(s) ds + \int_0^t b(s)g(x(s))\dot{x}(s)^2 ds$$
$$+ \int_{x(0)}^{x(t)} c(s, u) ds - \int_0^t \int_{x(0)}^{x(t)} \frac{\partial c(s, u)}{\partial s} du ds = \dot{x}(0)\ddot{x}(0) + \int_0^t e(s)\dot{x}(s) ds, \tag{3.1}$$

from which we obtain

$$\dot{x}(t)\ddot{x}(t) - \int_0^t \ddot{x}(s)^2 ds + \int_0^t a(s)f(\dot{x}(s))\ddot{x}(s)\dot{x}(s) ds + \int_0^t b(s)g(x(s))\dot{x}(s)^2 ds$$
$$+ \int_{x(0)}^{x(t)} c(s, u) ds - \int_0^t \int_{x(0)}^{x(t)} \frac{\partial c(s, u)}{\partial s} du ds \leq \dot{x}(0)\ddot{x}(0) + \int_0^t |e(s)\dot{x}(s)| ds. \tag{3.2}$$

Now if $x(t)$ is unbounded, then for large values of $|x|$, we have that the LHS of the inequality (3.2) is positive from our hypothesis. By using the mean value theorem to

the last part of the RHS in the inequality (3.2), we have that

$$\dot{x}(t)\ddot{x}(t) - \int_0^t \ddot{x}(s)^2 ds + \int_0^t a(s)f(\dot{x}(s))\ddot{x}(s)\dot{x}(s)ds + \int_0^t b(s)g(x(s))\dot{x}(s)^2 ds$$
$$+ \int_{x(0)}^{x(t)} c(s,u)ds - \int_0^t \int_{x(0)}^{x(t)} \frac{\partial c(s,u)}{\partial s} duds \leq \dot{x}(0)\ddot{x}(0) + \left|\dot{x}(\tilde{t})\right| D, \tag{3.3}$$

where

$$D = \int_0^\infty |e(t)|dt; 0 < \tilde{t} < t.$$

From the inequality (3.3), it is clear that if $|x| \to \infty$, then must $|\dot{x}|$ and $|\ddot{x}|$. Otherwise, the LHS becomes unbounded while the RHS stays bounded which is impossible.

Also as $|\dot{x}(t)|$ and $|\ddot{x}(t)|$ approach ∞, so must $\left|\dot{x}(\tilde{t})\right|$. On any compact subinterval, choose t where $\dot{x}(t)$ is a maximum and divide the inequality (3.3) by $\dot{x}(t)$.

$$\ddot{x} + \frac{1}{\dot{x}(t)}\left\{-\int_0^t \ddot{x}(s)^2 ds + \int_0^t a(s)f(\dot{x}(s))\ddot{x}(s)\dot{x}(s)ds + \int_0^t b(s)g(x(s))\dot{x}(s)^2 ds\right\}$$
$$+ \frac{1}{\dot{x}(t)}\left\{\int_{x(0)}^{x(t)} c(s,u)ds - \int_0^t \int_{x(0)}^{x(t)} \frac{\partial c(s,u)}{\partial s} duds\right\} \leq \frac{\dot{x}(0)\ddot{x}(0)}{\dot{x}(t)} + \frac{\left|\dot{x}(\tilde{t})\right| D}{\dot{x}(t)}. \tag{3.4}$$

Now if $\dot{x}(t) \to \infty$, the LHS of the inequality (3.4) becomes unbounded while the RHS remains bounded, i.e.,

$$\ddot{x}(t) \leq \frac{\dot{x}(0)\ddot{x}(0)}{\dot{x}(t)} + \frac{\left|\dot{x}(\tilde{t})\right| D}{\dot{x}(t)}. \tag{3.5}$$

is a contradiction. Thus $|x(t)|, |\dot{x}(t)|$ and $|\ddot{x}(t)|$ must stay bounded on $[0, T)$. By standard arguement we are permitted to extend the solution on all $[0, \infty)$ (see [8]). As argued in [2] and [3] by imposing more stringent conditions on $c(t, x)$, all the solutions become L^2-solutions. Thus the conclusion to the proof of Theorem 2.1. $\square$

Next we prove Theorem 2.2.

Proof of Theorem 2.2.

To show that x is in $L^2[0, \infty)$, we first multiply the equation (1.1) by x and

integrate from 0 to t to get

$$x\ddot{x} - \int_0^t x(s)\dddot{x}(s)ds + \int_0^t a(s)f(\dot{x}(s))x(s)\ddot{x}(s)ds + \int_0^t b(s)g(x)x(s)\dot{x}(s)ds$$
$$+ \int_0^t x(s)c(s,x(s))ds = x(0)\ddot{x}(0) + \int_0^\infty e(s)x(s)ds. \tag{3.6}$$

From the inequality (3.3), we observe that $\dot{x}$ and $\ddot{x}$ must be square integrable since

$$\int_0^\infty \ddot{x}(s)^2 ds < \infty$$

and

$$\int_0^\infty b(s)g(s)\dot{x}(s)^2 ds > \int_0^\infty b_0 g_0 \dot{x}(s)^2 ds.$$

By Schwartz inequality, the third term of the LHS of the inequality (3.3) becomes

$$\int_0^\infty a(s)f(\dot{x}(s))\ddot{x}(s)\dot{x}(s)ds > \int_0^\infty a_0 f_0 \ddot{x}(s)\dot{x}(s)ds \le a_0 f_0 \left(\int_0^\infty \dot{x}(t)^2 dt\right)^{\frac{1}{2}} \left(\int_0^\infty \ddot{x}(t)^2 dt\right)^{\frac{1}{2}},$$

where f_0, g_0 are the lower bounds of $f(x)$ and $g(x)$ respectively, on the interval $[-k, k]$ and k is a bound of x on $[0, \infty)$. Let

$$F(\dot{x}) = \int_0^t \dot{u}f(\dot{u})du \quad \text{and} \quad G(x) = \int_0^t ug(u)du.$$

On integrating by parts the third and fourth terms of the equation (3.6), we have

$$x\ddot{x} - \int_0^t x(s)\ddot{x}(s)ds + a(t)F(\dot{x}(t)) - \int_0^t F(\dot{x}(s))\dot{a}(s)ds$$
$$+b(t)G(x(t)) - \int_0^t G(x(s))\dot{b}(s)ds + \int_0^t c(s,x(s))ds \le D_1, \tag{3.7}$$

where $D_1 = |x(0)\ddot{x}(0)| + |a(0)F(\dot{x}(0)| + |b(0)G(x(0))| + \int_0^t |e(s)x(s)|ds$. Since the RHS of the inequality (3.7) is bounded and all terms on the LHS of the equation (3.6) are either bounded or positive, the result thus follows. $\qquad \square$

Remark. $\int_0^t \dot{x}\ddot{x}ds \le \left(\int_0^t \dot{x}^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \ddot{x}^2 ds\right)^{\frac{1}{2}}$ since each is positive, and by the hypotheses, all terms on the LHS is either positive or bounded.

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