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ON SOME SEMINORMED SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION

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Abstract. In this paper we define the space $BV_\sigma(p, f, q, s)$ on a seminormed complex linear space, by using modulus function. We give various properties and some inclusion relations on this space. Furthermore, we construct the sequence space $BV_\sigma(p, f^k, q, s)$and we give properties and some inclusion relations on this space.

1. INTRODUCTION

Let ℓ_∞ and c denote the Banach spaces of real bounded and convergent sequences $x = (x_n)$ normed by $\|x\| = \sup_n |x_n|$, respectively.

Let σ be a one to one mapping of the set of positive integers into itself such that $\sigma^m(n) = \sigma(\sigma^{m-1}(n))$, $m = 1, 2, ...$. A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or a $\sigma-$mean if and only if

i) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

ii) $\varphi(e) = 1$ where $e = (1, 1, 1, ...)$ and

iii) $\varphi\left(\left\{x_{\sigma(n)}\right\}\right)=\varphi\left(\left\{x_n\right\}\right)$ for all $x\in\ell_\infty$.

If σ is the translation mapping $n\to n+1$, a $\sigma-$mean is often called a Banach limit [2], and V_σ, the set of $\sigma-$convergent sequences, that is, the set of bounded sequences all of whose invariant means are equal, is the set $\hat{f}$ of almost convergent sequences [6].

If $x=(x_n)$, set $Tx=(Tx_n)=\left(x_{\sigma(n)}\right)$. It can be shown (see Schaefer [13]) that

$$V_\sigma=\left\{x=(x_n):\lim_m t_{mn}(x)=Le\text{ uniformly in }n,\ L=\sigma-\lim x\right\},\tag{1.1}$$

where

$$t_{mn}(x)=\frac{1}{m+1}\sum_{j=0}^{m}T^jx_n.$$

The special case of (1.1) , in which $\sigma(n)=n+1$ was given by Lorentz [6].

Subsequently invariant means have been studied by Ahmad and Mursaleen[1], Mursaleen [9], Raimi [11] and many others.

The space

$$BV_\sigma=\left\{x\in\ell_\infty:\sum_m\left|\phi_{m,n}(x)\right|<\infty,\quad\text{uniformly in }n\right\}$$

were defined by Mursaleen [8], where

$$\phi_{m,n}(x)=t_{mn}(x)-t_{m-1,n}(x)$$

assuming that $t_{mn}(x)=0$, for $m=-1$.

A straightforward calculation shows that

$$\phi_{m,n}(x)=\begin{cases}\frac{1}{m(m+1)}\sum_{j=1}^{m}j\left(x_{\sigma^j(n)}-x_{\sigma^{j-1}(n)}\right) & (m\geq 1)\\ x_n, & (m=0)\end{cases}.$$

Note that for any sequences x, y and scalar λ, we have

$$\phi_{m,n}(x+y)=\phi_{m,n}(x)+\phi_{m,n}(y)\quad\text{and}\quad\phi_{m,n}(\lambda x)=\lambda\phi_{m,n}(x).$$

The notion of a modulus function was introduced by Nakano [10] in 1953. We recall that a modulus f is a function from $[0,\infty)$ to $[0,\infty)$ such that *(i)* $f(x)=0$

if and only if $x = 0$, *(ii)* $f(x+y) \le f(x) + f(y)$, for all $x \ge 0$, $y \ge 0$, *(iii)* f is increasing, *(iv)* f is continuous from the right at 0.

Since $|f(x) - f(y)| \le f(|x-y|)$, it follows from condition (iv) that f is continuous on $[0, \infty)$. Furthermore, we have $f(nx) \le nf(x)$ for all $n \in \mathbb{N}$, from condition (ii) and so

$$f(x) = f\left(nx\frac{1}{n}\right) \le nf\left(\frac{x}{n}\right) \text{ hence}$$

$$\frac{1}{n}f(x) \le f\left(\frac{x}{n}\right) \text{ for all } n \in \mathbb{N}.$$

A modulus may be bounded or unbounded. For example, $f(x) = x^p$, $(0 < p \le 1)$ is unbounded and $f(x) = \frac{x}{1+x}$ is bounded. Maddox [7] and Ruckle [12] used a modulus function to construct some sequence spaces.

After then some sequence spaces, defined by a modulus function, were introduced and studied by Bhardwaj [3], Connor [5], Waszak [14], and many others.

Proposition 1.1. *Let f be a modulus and $0 < \delta < 1$. Then for $n \in N$ and $t \in [0, \infty)$*

$$f^{n-1}(t) > \delta \Rightarrow f^{n-1}(t) \le \frac{2f(1)}{\delta}\left\{f^{n-1}(t)\right\} \quad [4].$$

Definition 1.2. *Let q_1, q_2 be seminorms on a vector space X. Then q_1 is said to be stronger than q_2 if whenever (x_m) is a sequence such that $q_1(x_m) \to 0$, than also $q_2(x_m) \to 0$. If each is stronger than the others q_1 and q_2 are said to be equivalent (one may refer to Wilansky [15]).*

Lemma 1.3. *Let q_1 and q_2 be seminorms on a linear space X. Then q_1 is stronger than q_2 if and only if there exists a constant M such that $q_2(x) \le Mq_1(x)$ for all $x \in X$ (see for instance Wilansky [15]).*

A sequence space E is said to be solid (or normal) if $(\alpha_m x_m) \in E$ whenever $(x_m) \in E$ for all sequences (α_m) of scalars with $|\alpha_m| \le 1$.

It is well known that a sequence space E is normal implies that E is monotone.

Lemma 1.4. *If f is a modulus then f^k is also modulus for each $k = 1, 2, \ldots$, where $f^k = f \circ f \circ \ldots \circ f$ (k times).*

Let $p = (p_m)$ be a sequence of strictly positive real numbers and X be a seminormed space over the field $\mathbb{C}$ of complex numbers with the seminorm q. We define the sequence space as follows:

$$BV_\sigma(p, f, q, s) = \left\{ x = (x_m) \in X : \sum_{m=1}^{\infty} m^{-s} \left[f\left(q\left(\phi_{m,n}(x) \right) \right) \right]^{p_m} < \infty, \quad s \geq 0, \quad \text{uniformly in } n \right\},$$

where f is a modulus function.

We get the following sequence spaces from $BV_\sigma(p, f, q, s)$ by choosing some of the special p, f, and s :
For $f(x) = x$ we get

$$BV_\sigma(p, q, s) = \left\{ x = (x_m) \in X : \sum_{m=1}^{\infty} m^{-s} \left[\left(q\left(\phi_{m,n}(x) \right) \right) \right]^{p_m} < \infty, \quad s \geq 0, \quad \text{uniformly in } n \right\},$$

for $p_m = 1$, for all m, we get

$$BV_\sigma(f, q, s) = \left\{ x = (x_m) \in X : \sum_{m=1}^{\infty} m^{-s} \left[f\left(q\left(\phi_{m,n}(x) \right) \right) \right] < \infty, \quad s \geq 0, \quad \text{uniformly in } n \right\},$$

for $s = 0$ we get

$$BV_\sigma(p, f, q) = \left\{ x = (x_m) \in X : \sum_{m=1}^{\infty} \left[f\left(q\left(\phi_{m,n}(x) \right) \right) \right]^{p_m} < \infty, \quad \text{uniformly in } n \right\},$$

for $f(x) = x$ and $s = 0$ we get

$$BV_\sigma(p, q) = \left\{ x = (x_m) \in X : \sum_{m=1}^{\infty} \left[\left(q\left(\phi_{m,n}(x) \right) \right) \right]^{p_m} < \infty, \quad \text{uniformly in } n \right\},$$

for $p_m = 1$, for all m, and $s = 0$ we get

$$BV_\sigma(f, q) = \left\{ x = (x_m) \in X : \sum_{m=1}^{\infty} \left[f\left(q\left(\phi_{m,n}(x) \right) \right) \right] < \infty, \quad \text{uniformly in } n \right\},$$

for $f(x) = x$, $p_m = 1$, for all m, and $s = 0$ we have

$$BV_\sigma(q) = \left\{ x = (x_m) \in X : \sum_{m=1}^{\infty} q\left(\phi_{m,n}(x) \right) < \infty, \quad \text{uniformly in } n \right\},$$

The following inequalities will be used throughout the paper. Let $p = (p_m)$ be a bounded sequence of strictly positive real numbers with $0 < p_m \leq \sup p_m = H$, $C = \max\left(1, 2^{H-1}\right)$, then

$$|a_m + b_m|^{p_m} \leq C\left\{|a_m|^{p_m} + |b_m|^{p_m}\right\}, \tag{1.2}$$

where a_m, $b_m \in \mathbb{C}$ and

$$\sum_{m=1}^{n} (a_m + b_m)^i \leq \sum_{m=1}^{n} a_m^i + \sum_{m=1}^{n} b_m^i \tag{1.3}$$

where $a_1, a_2, ..., a_n \geq 0$, $b_1, b_2, ..., b_n \geq 0$ and $0 < i \leq 1$.

2. MAIN RESULTS

In this section we will prove the general results of this paper on the sequence space $BV_\sigma(p, f, q, s)$, those characterize the structure of this space.

Theorem 2.1. *The sequence space $BV_\sigma(p, f, q, s)$ is a linear space over the field C of complex numbers.*

Proof. Let $x, y \in BV_\sigma(p, f, q, s)$ and $\lambda, \mu \in \mathbb{C}$. Then there exist integers M_λ and N_λ such that $|\lambda| \leq M_\lambda$ and $|\mu| \leq N_\mu$. Since f is subadditive, q is a seminorm

$$\begin{aligned}
&\sum_{m=1}^{\infty} m^{-s}\left[f\left(q\left(\lambda\phi_{m,n}(x) + \mu\phi_{m,n}(y)\right)\right)\right]^{p_m} \\
&\quad \leq \sum_{m=1}^{\infty} m^{-s}\left[f\left(|\lambda|\, q\left(\phi_{m,n}(x)\right)\right) + f\left(|\mu|\, q\left(\phi_{m,n}(y)\right)\right)\right]^{p_m} \\
&\quad \leq C\left(M_\lambda\right)^H \sum_{m=1}^{\infty} m^{-s}\left[f\left(q\left(\phi_{m,n}(x)\right)\right)\right]^{p_m} + C\left(N_\mu\right)^H \sum_{m=1}^{\infty} m^{-s}\left[f\left(q\left(\phi_{m,n}(y)\right)\right)\right]^{p_m} \\
&\quad < \infty.
\end{aligned}$$

This proves that $BV_\sigma(p, f, q, s)$ is a linear space. □

Theorem 2.2. *$BV_\sigma(p, f, q, s)$ is a paranormed (need not be total paranormed) space with*

$$g(x) = \left(\sum_{m=1}^{\infty} m^{-s}\left[f\left(q\left(\phi_{m,n}(x)\right)\right)\right]^{p_m}\right)^{\frac{1}{M}},$$

where $M = \max(1, \sup p_m)$, $H = \sup_m p_m < \infty$.

Proof. It is clear that $g(\bar{\theta}) = 0$ and $g(x) = g(-x)$, for all $x \in BV_\sigma(p, f, q, s)$, where $\bar{\theta} = (\theta, \theta, ...)$. It also follows from (1.2), Minkowski's inequality and definition of f that g is subadditive and

$$g(\lambda x) \leq K_\lambda^{H/M} g(x),$$

where K_λ is an integer such that $|\lambda| < K_\lambda$. Therefore the function $(\lambda, x) \rightarrow \lambda x$ is continuous at $\lambda = 0$, $x = \bar{\theta}$ and that when λ is fixed, the function $x \rightarrow \lambda x$ is continuous at $x = \bar{\theta}$. If x is fixed and $\varepsilon > 0$, we can choose m_0 such that

$$\sum_{m=m_0}^{\infty} m^{-s} [f(q(\phi_{m,n}(x)))]^{p_m} < \frac{\varepsilon}{2}$$

and $\delta > 0$, so that $|\lambda| < \delta$ and definition of f gives

$$\sum_{m=1}^{m_0} m^{-s} [f(q(\lambda\phi_{m,n}(x)))]^{p_m} = \sum_{m=1}^{m_0} m^{-s} [f(|\lambda| q(\phi_{m,n}(x)))]^{p_m} < \frac{\varepsilon}{2}.$$

Therefore $|\lambda| < \min(1, \delta)$ implies that $g(\lambda x) < \varepsilon$. Thus the function $(\lambda, x) \rightarrow \lambda x$ is continuous at $\lambda = 0$ and $BV_\sigma(p, f, q, s)$ is a paranormed space. □

Theorem 2.3. *Let* f, f_1, f_2 *be modulus functions* q, q_1, q_2 *seminorms and* $s, s_1, s_2 \geq 0$. *Then*

i) If $s > 1$ *then* $BV_\sigma(p, f_1, q, s) \subseteq BV_\sigma(p, f \circ f_1, q, s)$,

ii) $BV_\sigma(p, f_1, q, s) \cap BV_\sigma(p, f_2, q, s) \subseteq BV_\sigma(p, f_1 + f_2, q, s)$,

iii) $BV_\sigma(p, f, q_1, s) \cap BV_\sigma(p, f, q_2, s) \subseteq BV_\sigma(p, f, q_1 + q_2, s)$,

iv) If q_1 *is stronger than* q_2 *then* $BV_\sigma(p, f, q_1, s) \subseteq BV_\sigma(p, f, q_2, s)$,

v) If $s_1 \leq s_2$ *then* $BV_\sigma(p, f, q, s_1) \subseteq BV_\sigma(p, f, q, s_2)$.

Proof. (i) Since f is continuous at 0 from right, for $\varepsilon > 0$ there exists $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $f(c) < \varepsilon$. If we define

$$I_1 = \{m \in \mathbb{N} : f_1(q(\phi_{m,n}(x))) \leq \delta\}$$

$$I_2 = \{m \in \mathbb{N} : f_1(q(\phi_{m,n}(x))) > \delta\},$$

then, when $f_1(q(\phi_{m,n}(x))) > \delta$ we get

$$f(f_1(q(\phi_{m,n}(x)))) \leq \{2f(1)/\delta\} f_1(q(\phi_{m,n}(x))).$$

Hence for $x \in BV_\sigma(p, f_1, q, s)$ and $s > 1$

$$\begin{aligned}
&\sum_{m=1}^{\infty} m^{-s}[f \circ f_1(q(\phi_{m,n}(x)))]^{p_m} \\
&\qquad = \sum_{m \in I_1} m^{-s}[f \circ f_1(q(\phi_{m,n}(x)))]^{p_m} + \sum_{m \in I_2} m^{-s}[f \circ f_1(q(\phi_{m,n}(x)))]^{p_m} \\
&\qquad \leq \sum_{m \in I_1} m^{-s}[\varepsilon]^{p_m} + \sum_{m \in I_2} m^{-s}[\{2f(1)/\delta\} f_1(q(\phi_{m,n}(x)))]^{p_m} \\
&\qquad \leq \max\left(\varepsilon^h, \varepsilon^H\right) \sum_{m=1}^{\infty} m^{-s} \\
&\qquad\quad + \max\left(\{2f(1)/\delta\}^h, \{2f(1)/\delta\}^H\right) \sum_{m=1}^{\infty} m^{-s}[f_1(q(\phi_{m,n}(x)))]^{p_m} \\
&\qquad < \infty.
\end{aligned}$$

(Where $0 < h = \inf p_m \leq p_m \leq H = \sup_m p_m < \infty$).

(ii) The proof follows from the following inequality

$$m^{-s}[(f_1 + f_2)(q(\phi_{m,n}(x)))]^{p_m} \leq Cm^{-s}[f_1(q(\phi_{m,n}(x)))]^{p_m} + Cm^{-s}[f_2(q(\phi_{m,n}(x)))]^{p_m}.$$

(iii), (iv) and (v) follow easily. □

Corollary 2.4. *Let f be a modulus function, then we have*

i) If $s > 1$, $BV_\sigma(p, q, s) \subseteq BV_\sigma(p, f, q, s)$,

ii) If $q_1 \cong$ (equivalent to) q_2, then $BV_\sigma(p, f, q_1, s) = BV_\sigma(p, f, q_2, s)$,

iii) $BV_\sigma(p, f, q) \subseteq BV_\sigma(p, f, q, s)$,

iv) $BV_\sigma(p, q) \subseteq BV_\sigma(p, q, s)$,

v) $BV_\sigma(f, q) \subseteq BV_\sigma(f, q, s)$.

The proof is straightforward.

Theorem 2.5. *Suppose that $0 < p_m \leq t_m < \infty$ for each $m \in N$. Then* *i)* $BV_{\sigma}(f,p,q) \subseteq BV_{\sigma}(f,t,q)$,

ii) $BV_{\sigma}(f,q) \subseteq BV_{\sigma}(f,q,s)$.

Proof. (i) Let $x \in BV_{\sigma}(f,p,q)$. This implies that

$$[f(q(\phi_{i,n}(x)))]^{p_m} \leq 1$$

for sufficiently large values of i, say $i \geq m_0$ for some fixed $m_0 \in \mathbb{N}$. Since f is non decreasing, we have

$$\sum_{m=m_0}^{\infty}[f(q(\phi_{m,n}(x)))]^{t_m} \leq \sum_{m=m_0}^{\infty}[f(q(\phi_{m,n}(x)))]^{p_m} < \infty.$$

Hence $x \in BV_{\sigma}(f,t,q)$.

The proof of (ii) is trivial. □

The following result is a consequence of the above result.

Corollary 2.6. *(i) If* $0 < p_m \leq 1$ *for each* m, *then* $BV_{\sigma}(p,f,q) \subseteq BV_{\sigma}(f,q)$.
(ii) If $p_m \geq 1$ *for all* m, *then* $BV_{\sigma}(f,q) \subseteq BV_{\sigma}(p,f,q)$.

Theorem 2.7. *The sequence space* $BV_{\sigma}(p,f,q,s)$ *is solid.*
Let $x \in BV_{\sigma}(p,f,q,s)$ *i.e*

$$\sum_{m=1}^{\infty} m^{-s}[f(q(\phi_{k,n}(x)))]^{p_m} < \infty.$$

Let (α_m) *be sequence of scalars such that* $|\alpha_m| \leq 1$ *for all* $m \in \mathbb{N}$. *Then the result follows from the following inequality*

.

$$\sum_{m=1}^{\infty} m^{-s}[f(q(\alpha_m\phi_{k,n}(x)))]^{p_k} \leq \sum_{m=1}^{\infty} m^{-s}[f(q(\phi_{k,n}(x)))]^{p_m}$$

Corollary 2.8. *The sequence space* $BV_{\sigma}(p,f,q,s)$ *is monotone.*

Proposition 2.9. *For any two sequences* $p=(p_k)$ *and* $t=(t_k)$ *of positive real numbers and any two seminorms* q_1 *and* q_2 *we have* $BV_{\sigma}(p,f,q_1,r) \cap BV_{\sigma}(t,f,q_1,s) \neq \phi$ *for all* $r>0$, $s>0$.

3. SOME RELATIONS ON $BV_\sigma\left(p, f^k, q, s\right)$

If we replace f^k by f in the definition of $BV_\sigma(p, f, q, s)$ from Lemma (1.4), then we have

$$BV_\sigma\left(p, f^k, q, s\right) = \left\{x = (x_k) \in \ell_\infty : \sum_{m=1}^{\infty} m^{-s}\left[f^k\left(q\left(\phi_{m,n}(x)\right)\right)\right]^{p_m} < \infty, \quad s \geq 0 \text{ uniformly in } n\right\}.$$

The all results obtained for $BV_\sigma(p, f, q, s)$ also hold for $BV_\sigma\left(p, f^k, q, s\right)$.

Theorem 3.1. *If* $s > 1$ *and* $k_1 < k_2$ *then*

$$BV_\sigma\left(p, f^{k_1}, q, s\right) \subseteq BV_\sigma\left(p, f^{k_2}, q, s\right).$$

Proof. The proof can be proved by using mathematical induction. Let $k_2 - k_1 = r$. So $r \geq 1$. Now we show that the assertion is true for $r = 1$. That is,

$$BV_\sigma\left(p, f^{k_1}, q, s\right) \subseteq BV_\sigma\left(p, f^{k_1+1}, q, s\right).$$

By the continuity of f, for $\varepsilon > 0$, there exists $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $f(c) < \varepsilon$. Let

$$\begin{aligned} I_1 &= \left\{m \in \mathbb{N} : f^{k_1}\left(q\left(\phi_{m,n}(x)\right)\right) \leq \delta\right\}, \\ I_2 &= \left\{m \in \mathbb{N} : f^{k_1}\left(q\left(\phi_{m,n}(x)\right)\right) > \delta\right\}. \end{aligned}$$

Hence for $x \in BV_\sigma\left(p, f^{k_1}, q, s\right)$ and $s > 1$,

$$\begin{aligned} &\sum_{m=1}^{\infty} m^{-s}\left[f^{k_1+1}\left(q\left(\phi_{m,n}(x)\right)\right)\right]^{p_m} \\ &\quad = \sum_{m \in I_1} m^{-s}\left[f\left(f^{k_1}\left(q\left(\phi_{m,n}(x)\right)\right)\right)\right]^{p_m} + \sum_{m \in I_2} m^{-s}\left[f\left(f^{k_1}\left(q\left(\phi_{m,n}(x)\right)\right)\right)\right]^{p_m} \\ &\quad \leq \sum_{m \in I_1} m^{-s}[\varepsilon]^{p_m} + \sum_{m \in I_2} m^{-s}\left[\{2f(1)/\delta\} f^{k_1}\left(q\left(\phi_{m,n}(x)\right)\right)\right]^{p_m} \\ &\quad \leq \max\left(\varepsilon^h, \varepsilon^H\right) \sum_{m=1}^{\infty} m^{-s} + \max\left(a^1, a^2\right). \sum_{m=1}^{\infty} m^{-s}\left[f^{k_1}\left(q\left(\phi_{m,n}(x)\right)\right)\right]^{p_m} \end{aligned}$$

where $a^1 = \{2f(1)/\delta\}^h$, $a^2 = \{2f(1)/\delta\}^H$. Thus $x \in BV_\sigma\left(p, f^{k_1+1}, q, s\right)$. Now assume that the assertion is true for any r, that is

$$BV_\sigma\left(p, f^{k_1}, q, s\right) \subseteq BV_\sigma\left(p, f^{k_1+r}, q, s\right). \tag{3.1}$$

We show that it is also true for $r+1$, that is,

$$BV_\sigma\left(p, f^{k_1}, q, s\right) \subseteq BV_\sigma\left(p, f^{k_1+r+1}, q, s\right).$$

But from (3.1) it suffices to show that

$$BV_\sigma\left(p, f^{k_1+r}, q, s\right) \subseteq BV_\sigma\left(p, f^{k_1+r+1}, q, s\right).$$

This can be easily done as in the proof for $r=1$.

(Where $0 < h = \inf p_m \leq p_m \leq H = \sup_m p_m < \infty$).

Corollary 3.2. *Let $s > 1$ and $k \in N$, then*

i) $BV_\sigma(p, f, q, s) \subseteq BV_\sigma\left(p, f^k, q, s\right)$,

ii) $BV_\sigma(p, q, s) \subseteq BV_\sigma\left(p, f^k, q, s\right)$.

Theorem 3.3. *Let $k_1, k_2 \in N$ and $k_1 < k_2$, then*

i) If $f(c) < c$ for all $c \in [0, \infty)$, then $BV_\sigma(p, q, s) \subseteq BV_\sigma\left(p, f^{k_1}, q, s\right) \subseteq BV_\sigma\left(p, f^{k_2}, q, s\right)$

ii) If $f(c) \geq c$ for all $c \in [0, \infty)$, then $BV_\sigma\left(p, f^{k_2}, q, s\right) \subseteq BV_\sigma\left(p, f^{k_1}, q, s\right) \subseteq BV_\sigma(p, q, s)$.

Proof. Since $f(c) < c$ and f is increasing we have

$$f^{k_2}(c) \leq f^{k_2-1}(c) \leq \ldots \leq f^{k_1}(c) \leq \ldots \leq f(c) < c.$$

Thus for each m and $p_m > 0$, the proof follows from

$$\begin{aligned} m^{-s}\left[f^{k_2} q\left(\phi_{m,n}(x)\right)\right]^{p_m} &\leq m^{-s}\left[f^{k_2-1} q\left(\phi_{m,n}(x)\right)\right]^{p_m} \\ &\leq \ldots \leq m^{-s}\left[f^{k_1} q\left(\phi_{m,n}(x)\right)\right]^{p_m} \leq \ldots \\ &\leq m^{-s}\left[f\left(q\left(\phi_{m,n}(x)\right)\right)\right]^{p_m} < m^{-s} q\left(\phi_{m,n}(x)\right)^{p_m}. \end{aligned}$$

(ii) Omitted. $\square$

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