Kragujevac J. Math. 29 (2006) 113–120.

FIXED POINTS OF MAPPING ON THE NORMED AND REFLEXIVE SPACES

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(Received October 25, 2004)

Abstract. In this work we shall prove the theorems of the fixed points of some classes of mapping on the sets from the normed and reflexed space, which have some of the properties NST, NSTN or KŠ.

1. INTRODUCTION

For some of the mappings on Banach's spaces having the normal structure there are many results about the existence of the fixed point. For the example, in paper [2] it has been proved that if K is a subset of reflexive Banach's space, having the normal structure, the nonexpansive mapping $T : K \to K$ has a fixed point. In paper [5] it has been proved the existence of the fixed point for mappings $T : K \to K$ which are diametral contractions, under the condition that K has the normal structure. In the paper [3] there is the proof of the existence of the fixed point for general nonexpansive mapping on the sets of Banach's space having the normal structure. The existence of the fixed point for one class of mapping on the sets which have the normal structure is given in the paper [1].

Now we introduce the conditions for subsets on the normed and reflexive spaces which extend the normal structure.

Let M be nonempty, bounded, closed and convex subset of the normed and reflexive space X and S any closed and convex subset of M having more than one element. Let use $\delta(S)$ to denote diameter of the set S. For the set $M \subset X$ we say that it has the property NST if there is a mapping $T : M \to M$, so that for some point $x_0 \in S$ there is

$$\sup_{z \in S} \| x_0 - T^k z \| < \delta(S), \tag{1.1}$$

and there is for some $k \in N$, $T^k(S) \subset S$.

Let's assume that sets M, S satisfy conditions of the above definition. For the set $M \subset X$ we say that it has the property NSTN if there is a mapping $T : M \to M$, having the property

$$\sup_{n \in N} \| x_0 - T^n x_0 \| < \delta(S). \tag{1.2}$$

for some point $x_0 \in S$ for which $Tx_0 \in S$.

2. THE NEW RESULTS

Let us prove the existence of the fixed point for T defined on the sets from normed and reflexive space X, satisfying the condition NST or NSTN.

Theorem 1. *Let K be nonempty, bounded, closed and convex subset of the normed and reflexive space X, and let K have the property NST, where T is one of the mappings defining the property NST. If for every closed and convex subset $E \subset K$ it is valid that $T(E) \subset E$ and for the number $k \in N$ for which set K has the property NST is valid that*

$$\| T_x - T_y \| \leq \sup_{z \in E} \| x - T^k z \|, \tag{2.1}$$

then the mapping T has a fixed point.

Proof. Let G be the set of all nonempty and closed and convex subsets E of set K for which $T(E) \subset E$. The set G is nonempty, because $K \subset G$. Let us introduce into the set G the relation of the order being the set relation of inclusion. From the norming and mapping of the space X, it follows that the space X is complete, so on the basis caracterisation of Šmulijan's [4] every chain in G, which consists of nonempty, bounded, closed and convex set of G has nonempty intersection. By Zorn's lemma there is the minimal element S of the set G.

If S consists only of one element, on the basis of supposition that $T(S) \subset S$, this element is also the fixed point of the mapping T.

If S has more than one point on the property NST it is valid that

$$\sup_{z \in S} \| x_0 - T^k z \| < \delta(S),$$

for certain $k \in N$, and $x_0 \in S$.

If in the inequality (2.1) we put that $x = x_0$ we have that

$$\| Tx_0 - T_y \| \leq \sup_{z \in S} \| x_0 - T^k z \|,$$

so that all $Ty, y \in S$ are in the ball with the center in Tx_0 and radius $\sup\limits_{z \in S} \| x_0 - T^k z \| = r$, i.e. $T(S) \subset B(Tx_0, r)$, and that is also $T^k(S) \subset B(Tx_0, r)$.

Since $T(S) \subset S$ it implies that $T^k(S) \subset S$ so that $T^k(S) \subset B(Tx_0, r) \cap S$, and on the basis of minimality of the set S, it is valid $B(Tx_0, r) \cap S = S$, so that $S \subset B(Tx_0, r)$. From the relation $S \subset B(Tx_0, r)$ it implies that

$$\| Tx_0 - y \| \leq \sup_{z \in S} \| x_0 - T^k z \|, \tag{2.2}$$

for all $y \in S$.

Let us form the set

$$S' = \{v \in S : \sup_{z \in S} \| v - z \| \leq \sup_{z \in S} \| x_0 - T^k z \|\}.$$

On the basis of definition of the set S' and the relation 2.2 we conclude that the set S' is bounded and closed, regarding that $Tx_0 \in S'$, than S' also nonempty set.

Let us prove that for all $v \in S'$ it is valid that $Tv \in S'$. Since S is a nonempty, limited, closed and convex set having more than one element and it is a minimal element of the family G and is valid that $TS \subset S$, then is $S = \overline{CoTS}$.

If $z \in S$, than z can be calculated as convex combination of the elements from TS, i.e.

$$z = \sum_{i=1}^{n} \alpha_i T z_i \ , \ \sum_{i=1}^{n} \alpha_i = 1 \ , \ \alpha_i \geq 0 \ and \ z_i \in S.$$

Now

$$\begin{aligned} \| \ Tv - z \ \| &= \left\| Tv - \sum_{i=1}^{n} \alpha_i T z_i \right\| \leq \sum_{i=1}^{n} \alpha_i \left\| Tv - Tz_i \right\| \\ &\leq \sum_{i=1}^{n} \alpha_i \sup_{z \in S} \| \ v - T^k z \ \| \leq \sum_{i=1}^{n} \alpha_i \sup_{z \in S} \| \ v - z \ \| \\ &\leq \sup_{z \in S} \| \ x_0 - T^k z \ \| \sum_{i=1}^{n} \alpha_i = \sup_{z \in S} \| \ x_0 - T^k z \ \|, \end{aligned}$$

so that

$$TS' \subset S'.$$

Let us give the sequence $\{\alpha_n\} \subset S'$, for all $n \in N$ and let $\alpha_n \to \alpha \in S$ when $n \to \infty$.

Now

$$\begin{aligned} \sup_{z \in S} \| \ \alpha - z \ \| &\leq \sup_{z \in S} (\| \ \alpha - \alpha_n \ \| + \| \ \alpha_n - z \ \|) \\ &= \| \ \alpha - \alpha_n \ \| + \sup_{z \in S} \| \ \alpha_n - z \ \| \\ &\leq \| \ \alpha - \alpha_n \ \| + r \end{aligned}$$

When $n \to \infty$ we get that

$$\sup_{z \in S} \| \ \alpha - z \ \| \leq r,$$

then the set S' is closed.

Let u and v be two points from S'.

For $\lambda \in [0, 1]$ we have that

$$\begin{aligned} \| \ \lambda v + (1 - \lambda) u - z \ \| &= \| \ \lambda v + (1 - \lambda) u - \lambda z + \lambda z - z \ \| \\ &\leq \lambda \sup_{z \in S} \| \ x_0 - T^k z \ \| + (1 - \lambda) \sup_{z \in S} \| \ x_0 - T^k z \ \| \\ &= r, \end{aligned}$$

so the set S' is convex.

For $v, w \in S'$ we have that

$$\delta(S') = \sup_{\substack{v \in S' \\ w \in S'}} \| v - w \| \leq \sup_{\substack{v \in S' \\ z \in S}} \| v - z \| \leq \sup_{z \in S} \| x_0 - T^k z \| < \delta(S)$$

Now S' is a nonempty, closed, bounded and convex subset of K for which $T(S') \subset S'$, S' belongs to the family G, and it is valid that $S' \subset S$ and $S' \neq S$, which is impossible for the reason of minimality of the set S, so the set S has only one point, and it is fixed point of mapping T. By this the proof of Theorem 1. is completed. □

Theorem 2. *Let us introduce the mapping $T : K \to K$ where K is a nonempty, bounded, closed and convex subset of the normed and reflexive space X, and let K have the property NSTN, where T is one of the mappings defining the property NSTN. If for any closed and convex subset $E \subset K$ having more than one element it is valid that $T(E) \subset E$ and the condition is valid that*

$$\| Tx - Ty \| \leq \sup_{k \in N} \| x - T^k x \|, \tag{2.3}$$

for all $x, y \in E$, then the mapping T has the fixed point.

Proof. In the same way as in Theorem 1. we come to the set S. If set S has one element, regarding that $TS \subset S$ it is also the fixed point of mapping T.

Let us presume that the set S has more than one element. On the basis of the property NSTN and the condition 2.3 for $x = x_0$ we get that

$$\| Tx_0 - Ty \| \leq \sup_{k \in N} \| x_0 - T^k x_0 \|,$$

for all $y \in S$.

By the similar reasoning as in the Theorem 1. we come to the relation

$$\| Tx_0 - y \| \leq \sup_{k \in N} \| x_0 - T^k x_0 \|, \tag{2.4}$$

for all $y \in S$.

Let us form the set

$$S'' = \{v \in S : \sup_{z \in S} \| v - z \| \leq \sup_{k \in N} \| x_0 - T^k z \|\}.$$

The set S'' is nonempty. If $v \in S''$ let us prove that also $Tv \in S''$. Since the set S is a convex combination of elements from $T(S)$ for every $z_i \in S$ and $k \in N$, the inequalities are valid.

$$\begin{aligned} \| Tv - z \| &= \left\| Tv - \sum_{i=1}^{n} \alpha_i T z_i \right\| \leq \sum_{i=1}^{n} \alpha_i \| Tv - Tz_i \| \\ &\leq \sum_{i=1}^{n} \alpha_i \sup_{k \in N} \| v - T^k v \| \leq \sup_{z \in S} \| v - z \| \\ &\leq \sup_{k \in N} \| x_0 - T^k x_0 \|, \end{aligned}$$

then $T(S'') \subset S''$.

It is simple to prove that the set S'' is closed and convex. On the basis of the definition of the set S'' for all $v, w \in S''$ we have that

$$\delta(S'') = \sup_{\substack{v \in S'' \\ w \in S''}} \| v - w \| \leq \sup_{\substack{v \in S'' \\ z \in S}} \| v - z \| \leq \sup_{k \in N} \| x_0 - T^k z \| < \delta(S).$$

Now S'' is a nonempty, closed, bounded and convex subset of S and it is valid that $S'' \neq S$, which is impossible because of the minimality of the set S. This completes the proof of Theorem 2. □

For normed and reflexive space X we say that it satisfies the property KŠ, if every decreasing sequence of nonempty, bounded, closed and convex subsets from X has the compact intersection.

In paper [5] it has been proved that if M is a nonempty and compact subset of Banach's space X, and K is a closed and convex shell of M and if $\delta(M) > 0$, then there is an element $u \in K$ so that

$$\sup_{x \in M} \| x - u \| < \delta(M).$$

Theorem 3. *Let K be a nonempty, bounded, closed and convex subset of the normed and reflexive space X having the property KŠ and let there be mapping $T : K \to K$. If there is $k \in N$ so that for any subset $E \subset K$ for which $T(E) \subset E$ the condition* (2.1) *is valid, then the mapping T has a fixed point.*

Proof. By the same reasoning as in the Theorem 1. we come to the set S. If S has one element than the proof is completed.

Let us suppose that S has more than one element, and on the basis of the supposition of the Theorem 3., it is also compact so regarding the statement from paper [5] there is a point $x_0 \in S$ so that

$$\sup_{z \in S} \| x_0 - z \| < \delta(S), \tag{2.5}$$

and it means that the set K has the property NST. The further proof of the Theorem 3. resembles the proof of the Theorem 1. □

Theorem 4. *Let K be a nonempty, bounded, closed and convex subset of the normed and reflexive space X which has the property KŠ and let there be mapping $T : K \to K$. If for every subset $E \subset K$ for which $T(E) \subset E$ the inequality* (2.3) *is valid, then the mapping T has the fixed point.*

Proof. In the same manner as in Theorem 1. we get the nonempty, closed, convex and compact subset S, of the set K.If S has one element the proof is completed.

If S has more than one element then regarding the statement of the paper [5] there is an element $x_0 \in S$ so that the inequality (2.5) is valid and the set K satisfies the condition NSTN.

The further proof of the Theorem 4. resembles the proof of the Theorem 1. □

References

[1] F. Vajzović, M. Stefanović, *Two fixed point theorems*, Radovi Akad. Nauka Bosne i Herceg., **24** (1985), 63–71.

[2] W. A. Kirk, *A fixed point theorem for mappings which do not increase distance*, Amer. Math. Monthly, **72** (1965), 1004–1006.

[3] Shih-sen Chang, *Uniform convexity and the fixed point property* , Univ. u Novom sadu, rad Prirod. Mat. Fak. Ser. Mat. **21, 1** (1991), 105–115.

[4] V. Šmulian, *On the principle of inclusion in the space of the type (B)*, Math. Sb. (N. S.), **5** (1939), 237–238.

[5] M. R. Tasković, *Some results in the fixed point theory - II*, Publ. Inst. Math., (Beograd), **41** (1980), 249–258.