TRANSITIVE 3-GROUPS OF DEGREE $3^n (n = 2, 3)$

M. S. Audu$^1$, A. Afolabi$^2$ and E. Apine$^1$

$^1$ Department of Mathematics, Faculty of Natural Sciences, University of Jos, Jos, Nigeria  
$^2$ Department of Mathematics, Faculty of Science, University of Lagos, Lagos, Nigeria

(Received April 26, 2005)

Abstract. In this paper we achieve a classification of transitive 3-groups of degrees 9 and 27. Other unique properties of these groups are discovered as a result.

INTRODUCTION

Let $G$ be a group acting on a non-empty set $\Omega$. The action of $G$ on $\Omega$ is said to be transitive if for any $\alpha, \beta$ in $\Omega$ there exists some $g$ in $G$ such that $\beta = \alpha g$. In this case $|\Omega|$ is called the degree of $G$ on $\Omega$. In [4], M. S. Audu, determined the number of transitive $p$-groups of degree $p^2$ and in [10], E. Apine, achieved a classification of transitive and faithful $p$-groups (abelian and non-abelian) of degrees at most $p^3$ whose center is elementary abelian of rank two. In this paper, we determine, up to equivalence, the actual transitive $p$-groups (abelian and non-abelian) of degrees $p^2$ and $p^3$ for $p = 3$ and achieve a classification according to small degrees.
1. RESULTS

1.1 TRANSITIVE 3-GROUPS OF DEGREE $3^2 = 9$

Let $G$ be a transitive 3-group of degree $3^2$, then $|G| = 3^n$, $n=1,2,3,4$. Clearly, $n \neq 1$ and when $n = 2$, then $|G|=9$, $G$ is essentially abelian and either $G \cong C_9$ or $G \cong C_3 \times C_3$. For transitivity, $|\alpha^G| = 9$, $|\alpha^G| = 1$, $\forall \alpha \in \Omega$ If $G \cong C_9$, then $G \cong G_{1,2} = \langle a \rangle$, with generator, say, $a = (1,2,3,4,5,6,7,8,9)$. If $G \cong C_3 \times C_3$, then $G \cong G_{2,2} = \langle a, b : a^3 = 1, b^3 = 1, ab = ba >$ with generators, say, $a = (1,4,7)(2,5,8)(3,6,9)$ and $b = (1,2,3)(4,5,6)(7,8,9)$.

Clearly $G_{1,3}$ and $G_{2,2}$ are transitive on $\Omega$ and we have:

**Lemma 1.1.1.** There are, up isomorphism, two transitive 3-groups of degree 9 and order 9, namely the abelian groups $G_{1,2}$ and $G_{2,2}$ described above.

When $n=3$, then $|G|=27$ and for transitivity we must have $|\alpha^G|=9$, $|\alpha^G|=3, \forall \alpha \in \Omega$.

Here $G$ is non-abelian and we have the following possibilities for $G$: $G \cong G_{1,3} = \langle a, b : a^9 = 1, b^3 = 1, ab = ba \rangle$ or $G \cong G_{2,3} = \langle a, b : a^9 = 1, b^3 = 1, ab = ba >$ with $c^3=1$, $G_{2,2} \triangleleft G_{2,3}$.

Consider first $G_{1,3} = \langle a, b : a^9 = 1, b^3 = 1, ab = ba \rangle$ with $a=(1,2,3,4,5,6,7,8,9)$, then, $b=(2,5,8)(3,9,6)$ (obtained by a Gap-programme (see PROGRAMME 1)).

For the case $G_{2,3}$, we obtain a presentation as follows:

$G_{2,3} = \langle a, b, c : a^3 = 1, ab = ba, c^3 = 1, ac = cab, bc = cb \rangle$, with, say, generators $a = (1,3,2)(4,6,5)(7,9,8)$, $b = (1,5,8)(3,4,7)(6,9,2)$ and $c = (2,9,6)(3,4,7)$ (obtained by a modification to PROGRAMME 1). Clearly the above groups are transitive on $\Omega$ and thus:

**Lemma 1.1.2.** There are, up to isomorphism, two transitive 3-groups of degree 9 and order 27, namely the non-abelian groups $G_{1,3}$ and $G_{2,3}$ described above.

When $n = 4$, $|G|=81$ and for transitivity, $|\alpha^G| = 9, |\alpha^G| = 9, \forall \alpha \in \Omega$.

Thus $G$ is non-abelian and the following are the possibilities for $G$: $G \cong G_{1,4} = \langle a, b, c \rangle$, where $c^3=1$, $G_{1,3} \leq G_{1,4}$ or $G \cong G_{2,4} = \langle a, b, c \rangle$, where $d^3=1$, $G_{2,3} \leq G_{2,4}$

For $G_{1,4}$, we have as a presentation:

$G_{1,4} = \langle a, b, c : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca^7b, bc = cb \rangle$, where $a$ and $b$
are the same generators as those of $G_{1,3}$ and $c = (3, 6, 9)$.

For $G_{2,4}$, we have as a presentation:

$$G_{2,4} = < a, b, c, d : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cab, bc = cb, d^3 = 1, ad = dac, bd = db, cd = dc >,$$

with the same generators $a, b, c$ as those of $G_{2,3}$ and $d = (3, 4, 7)$. Here we notice that $G_{1,4} \cong G_{2,4}$. Thus we have:

**Lemma 1.1.3.** There is, up to isomorphism, only one transitive 3-group of degree 9 and order 81, namely the non-abelian group $G_{1,4}$ described above.

We summarize our findings into the table below:

| $|G| = 3^n$ | Number of transitive abelian 3-group of degree 9 up to isomorphism | Number of transitive non-abelian 3-group of degree 9 up to isomorphism | Number of transitive 3-groups of degree 9 up to isomorphism |
|------------|-------------------------------------------------|-------------------------------------------------|---------------------------------------------------|
| $n = 1$    | 3                                               | 0                                               | 0                                                 |
| $n = 2$    | 9                                               | 2                                               | 0                                                 | 2                                                 |
| $n = 3$    | 27                                              | 0                                               | 2                                                 | 2                                                 |
| $n = 4$    | 81                                              | 0                                               | 1                                                 | 1                                                 |
| **Total**  | **2**                                           | **3**                                           | **5**                                             |

Hence we have:

**Proposition 1.1.4.** There are, up to isomorphism, 5 transitive 3-groups of degree $3^2$, 2 of these are abelian and of the remaining 3 non-abelian, 2 are of exponent 9 and 1 is of exponent 3.

### 1.2 Transitive 3-Groups of Degree $3^3 = 27$

Let $G$ be a transitive 3-group of degree 27, then $|G| = 3^n, n = 1, 2, \ldots, 13$. Clearly $n \neq 1, n \neq 2$. When $n = 3$, then $|G| = 27$ and for transitivity we must have $|a^G| = 27$, $|G_a| = 1, \forall a \in \Omega$.

Assuming first $G$ abelian, then either $G \cong C_{27}$ or $G \cong C_3 \times C_9$ or $G \cong C_3 \times C_3 \times C_3$. If $G \cong C_{27}$, then $G \cong G_{1,3} = < a >$, where we may take $a = (1, 2, \ldots, 27)$.
If \( G \cong C_3 \times C_9 \), then \( G \cong G_{2,3} = \langle a, b : a^9 = 1, b^3 = 1, ab = ba \rangle \), with, say,

\[
a = (1,2,3,4,5,6,7,8,9)(10,11,12,13,14,15,16,17,18)(19,20,21,22,23,24,25,26,27),
\]

and

\[
\]

If \( G \cong C_3 \times C_3 \times C_3 \), then \( G \cong G_{3,3} = \langle a, b : a^3 = 1, b^3 = 1, c^3 = 1, ab = ba, ac = ca, bc = cb \rangle \), with, say,

\[
a = (1,4,7)(2,5,8)(3,6,9)(10,13,16)(11,14,17)(12,15,18)(19,22,25)(20,23,26)(21,24,27),
\]

\[
b = (1,5,6)(2,3,7)(4,8,9)(10,14,15)(11,12,16)(13,17,18)(19,23,24)(20,21,25)(22,26,27),
\]

\[
\]

We next assume \( G \) non-abelian. Then the following are the possibilities for \( G \):

\[
G \cong G_{4,3} = \langle a, b : a^9 = 1, b^3 = 1, ab = ba \rangle 4 >
\]

or

\[
G \cong G_{5,3} = \langle K, c : c^3 = 1, K \cong C_3 \times C_3, K \triangleleft \! G_{6,3} \rangle.
\]

Taking \( a = (1,2,3,4,5,6,7,8,9)(10,11,12,13,14,15,16,17,18)(19,20,21,22,23,24,25,26,27) \) and

\[
b = (1,10,19)(2,14,26)(3,18,24)(4,13,22)(5,17,20)(6,12,27)(7,16,25)(8,11,23)(9,15,21)
\]

satisfy the requirement of \( G_{4,3} \).

For \( G_{5,3} \), we obtain a presentation as follow:

\[
G_{5,3} = \langle a, b, c : a^3 = 1, b^3 = 1, c^3 = 1, ab = ba, ac = ca, bc = cb \rangle
\]

with generators:

\[
a = (1,4,7)(2,5,8)(3,6,9)(10,13,16)(11,14,17)(12,15,18)(19,22,25)(20,23,26)(21,24,27),
\]

\[
b = (1,5,6)(2,3,7)(4,8,9)(10,14,15)(11,12,16)(13,17,18)(19,23,24)(20,21,25)(22,26,27),
\]

\[
\]

We easily check that the above-named groups are transitive on \( \Omega \) and we conclude:

\textbf{Lemma 1.2.1.} There are, up to isomorphism, five transitive 3-groups of degree \( 3^3 \) and order 27, namely the groups \( G_{1,3} \) (of exponent 27), \( G_{2,3} \) and \( G_{4,3} \) (of exponent 9) and \( G_{3,3} \) and \( G_{5,3} \) (of exponent 3) described above.

When \( n = 4 \), then \(|G| = 81 \) and for transitivity we must have

\[
|\alpha^G| = 27, |G_{\alpha}| = 3, \forall \alpha \in \Omega.
\]

Thus \( G \) must not be abelian and we have the following possibilities for \( G \):

\[
G \cong G_{1,4} = \langle a, b : a^{27} = 1, b^3 = 1, ab = ba^{10} \rangle \text{ or } G \cong G_{2,4} = \langle G_{2,3}, c \rangle,
\]

with \( c^3 = 1 \), \( G_{2,3} \triangleleft G_{2,4} \). or \( G \cong G_{3,4} = \langle G_{3,3}, d \rangle \text{ with } d^3 = 1 \), \( G_{3,3} \triangleleft G_{3,4} \) or \( G_{4,4} = \langle G_{4,3}, c \rangle \)

with \( c^3 = 1 \), \( G_{4,3} \triangleleft G_{4,4} \). or \( G \cong G_{5,4} = \langle G_{5,3}, d \rangle \text{ where } d^3 = 1 \), \( G_{5,3} \triangleleft G_{5,4} \) or \( G \cong G_{6,4} = \langle G_{6,3}, e \rangle \text{ with } e^3 = 1 \), \( G_{6,3} \triangleleft G_{6,4} \).
possibilities for $G$: (obtained by a Gap-programme (see PROGRAMME 2)). For $c$ generators $a$ and $b$ are the same for $G$

For $G$ get $b$, namely $G \cong C_3 \times C_3$, $K \leq G_{7,4}$. Of these groups only four satisfy the requirements for $G$, namely $G_{1,4}, G_{3,4}, G_{4,4}$ and $G_{5,4}$.

Now taking $a = (1,2,\ldots,27)$ and by an argument similar to the case $n = 3$, we get $b=(1,19,10)(3,12,21)(4,22,13)(6,15,24)(7,25,16)(9,18,27)$.

For $G_{4,4}$, we have a presentation as follows:

$$G_{4,4} = \langle a, b, c : a^9 = 1, b^3 = 1, ab = ba^4 \rangle$$

where the generators $a$ and $b$ are the same for $G_{4,3}$ and $c=(1,4,7)(2,5,8)(3,6,9)(19,25,22)(20,26,23)(21,27,24)$ (obtained by a Gap-programme (see PROGRAMME 2)). For $G_{3,4}$, we have a presentation as follows:

$$G_{3,4} = \langle a, b, c : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = ca, bc = cb, d^3 = 1, ad = dab, bd = db, cd = dc \rangle$$

where $a$, $b$ and $c$ are the same generators of $G_{3,3}$ and $d=(1,27,18)(2,24,14)(3,19,15)(4,21,11)(5,22,12)(6,26,16)(7,23,10)(8,20,12)(9,20,16)$.

For $G_{5,4}$, we have the presentation as follows:

$$G_{5,4} = \langle a, b, c, d : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2, d^3 = 1, ad = da, bd = db, cd = dc \rangle$$

when $a$, $b$, $c$ are the same generators of $G_{6,3}$ and $d = (1,27,11)(2,19,18)(3,23,13)(4,21,14)(5,22,12)(6,26,16)(7,24,17)(8,25,15)(9,20,10)$.

We easily check that the above-named groups are transitive on $\Omega$ and we conclude:

**Lemma 1.2.2.** There are, up to isomorphism, four transitive 3-groups of degree $3^3$ and order 81, namely the non-abelian groups $G_{1,4}$ (exponent 27), $G_{4,4}$ (exponent 9), $G_{3,4}$ and $G_{5,4}$ (both of exponent 3) described above.

When $n=5$, then $|G| = 243$ and for transitivity we must have $|\alpha^G| = 27$, $|\alpha| = 9 \forall \alpha \in \Omega$. Thus $G$ must be non-abelian and we have the following possibilities for $G$:

$$G \cong G_{1,5} = \langle G_{1,4}, c \rangle$$

with $c^3 = 1$, $G_{1,4} \cong G_{1,5} \Rightarrow G \cong G_{2,5} = \langle G_{4,4}, d \rangle$ with $d^3 = 1$, $G_{4,4} \cong G_{2,5}$ or $G \cong G_{3,5} = \langle G_{3,4}, d \rangle$ with $e^3 = 1$, $G_{5,4} \cong G_{4,5}$ or $G \cong G_{5,5} = \langle K, c \rangle$ with $c^3 = 1$,

$$K \cong C_{27} \times C_3, K \cong C_{9,5} \text{ or } G \cong C_{6,5} = \langle G_{2,3}, c \rangle$$

with $c^9 = 1$, $G_{2,3} \cong G_{6,5}$ or $G \cong G_{7,5} = \langle K, c \rangle$ with $c^{27} = 1$, $G_{2,3} \cong G_{6,5}$ or $G \cong G_{7,5} = \langle K, c \rangle$ with $c^{27} = 1$,
For obvious reasons, only $G_{1,5}$, $G_{3,5}$, $G_{4,5}$ and $G_{2,5}$ satisfy the requirements for $G$.

For $G_{1,5}$, we obtain as a presentation:

$G_{1,5} = < a, b, c : a^2 = 1, b^3 = 1, ab = ba^4, c^3 = 1, bc = cb, ac = ca^5b^2 >$, where $a$, $b$ are the same generators of $G_{1,4}$ and $c = (1,19,10)(4,22,13)(7,25,16)$ (obtained by a modification to PROGRAMME 2).

For $G_{3,5}$, we have a presentation as follows:

$G_{3,5} = < a, b, c, d, e : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = ca, bc = cb, d^3 = 1, ad = dab, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = eb^2c, de = eb^2d >$, where $a$, $b$, $c$, $d$ are the same generators of $G_{3,4}$ and $e = (1,14,21)(2,12,22)(3,16,26)(4,17,24)(5,15,25)(6,10,20)(7,11,27)(8,18,19)(9,13,23)$.

For $G_{4,5}$, we have a presentation as follows:

$G_{4,5} = < a, b, c, d, e : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2, d^3 = 1, ad = da, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ec, de = eab^2d >$, where $a$, $b$, $c$, $d$ are the same generators of $G_{5,4}$ and $e = (1,3,8)(2,4,6)(5,7,9)(10,13,16)(11,14,17)(12,15,18)(19,23,24)(20,21,25)(22,26,27)$.

For $G_{2,5}$, we have:

$G_{2,5} = < a, b, c, d : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b, d^3 = 1, ad = da, bd = da^6bc^2, cd = da^6c >$, where the generators $a$, $b$, $c$ are the same for $G_{4,4}$ and $d=(1,27,11)(2,19,12)(3,20,13)(4,21,14)(5,22,15)(6,23,16)(7,24,17)(8,25,18)(9,26,10)$.

Hence we have:

**Lemma 1.2.3.** There are, up to isomorphism, four transitive 3-groups of degree $3^4$ and order $243$, namely the non-abelian groups $G_{1,5}$ (exponent 27), $G_{2,5}$ (of exponent 9), $G_{3,5}$ and $G_{4,5}$ (of exponent 3) described above.
When \( n=6 \), then \( G=729 \) and for transitivity we must have
\( |\alpha^G|=27, \ G_\alpha|=27, \ \forall \alpha \in \Omega \). Thus \( G \) must be non-abelian and we have the following possibilities for \( G \): \( G \cong G_{1,6}=<\ G_{1,5},d > \) with \( d^3=1 \), \( G_{1,5}\trianglelefteq G_{9,6} \) or \( G \cong G_{2,6}=<\ G_{2,5},e > \) with \( e^3=1 \),

\[
G_{1,5}\trianglelefteq G_{2,6} \text{ or } G \cong G_{3,6}=<\ G_{3,5},f > \text{ with } f^3=1, \\
G_{3,5}\trianglelefteq G_{3,6} \text{ or } G \cong G_{4,6}=<\ G_{4,5},f > \text{ with } f^3=1, \\
G_{4,5}\trianglelefteq G_{4,6} \text{ or } G \cong G_{13,6}=<\ K,c > \text{ with } c^{27}=1, \\
K \cong C_9 \times C_3, K \trianglelefteq G_{13,6} \text{ or } G \cong G_{14,6}=<\ G_{4,3},c > \text{ with } c^{27}=1, \\
G_{4,3}\trianglelefteq G_{14,6} \text{ or } G \cong G_{15,6}=<\ G_{3,3},d > \text{ where } d^{27}=1, \\
G_{3,3}\trianglelefteq G_{15,6} \text{ or } G \cong G_{16,6}=<\ G_{5,3},d > \text{ where } d^{27}=1, \\
G_{5,2}\trianglelefteq G_{16,6} \text{ or } G \cong G_{17,6}=<\ K,d > \text{ with } d^3=1, \\
K \cong C_9 \times C_9 \times C_3, K \trianglelefteq G_{17,6} \text{ or } G \cong G_{18,6}=<\ K,e > \text{ where } e^9=1, \\
K \cong C_3 \times C_3 \times C_3 \times C_3, K \trianglelefteq G_{18,6} \text{ or } G \cong G_{19,6}=<\ K,e > \text{ where } e^3=1, \\
K \cong C_9 \times C_9 \times C_3 \times C_3, K \trianglelefteq G_{19,6} \text{ or } G \cong G_{20,6}=<\ G_{3,4},e >, \text{ where } e^9=1, \\
G_{3,4}\trianglelefteq G_{20,6} \text{ or } G \cong G_{21,6}=<\ G_{5,4},e > \text{ where } e^9=1, \\
G_{5,4}\trianglelefteq G_{21,6} \text{ or } G \cong G_{22,6}=<\ K,f >, \text{ where } f^3=1, \\
K \cong C_3 \times C_3 \times C_3 \times C_3 \times C_3, K \trianglelefteq G_{22,6}.
\]

It is readily seen that of the above groups, \( G_{9,6}, G_{10,6}, G_{11,6} \) and \( G_{12,6} \) are acceptable.

For \( G_{9,6} \), we have a presentation:

\[
G_{9,6} =<\ a, b, c, d : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = cb, ac = ca^{10}b^2, d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc >, \text{ where } a, b, c \text{ are the same for } G_{10,5} \text{ and } d = (1,10,19)(3,21,12)(4,22,13)(5,14,23)(8,26,17)(9,18,27)
\]

For \( G_{10,6} \), we have a presentation:

\[
G_{10,6} =<\ a, b, c, d, e, f : a^{3} = 1, b^{3} = 1, ab = ba, c^{3} = 1, ac = ca, bc = cb, d^{3} = 1, ad = dab, bd = db, cd = dc, e^{3} = 1, ae = ea, be = eb, ce = eb^{2}c, de = eb^{2}d, f^{3} = 1, af = fa, bf = fb, cf = fab^{2}e, ef = fabc^{2}e^{2} >, \text{ where } a, b, c, d, e \text{ are the same generators of } G_{16,5} \text{ and } f = (1,3,8)(2,4,6)(5,7,9)(10,12,17)(11,13,15)(14,16,18)(19,20,27)(21,22,23)(24,25,26).
\]

For \( G_{11,6} \), we have a presentation as follows:
\[ G_{11,6} = \langle a, b, c, d, e, f : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2, d^3 = 1, \]
\[ ad = da, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ec, de = eab^2d, \]
\[ f^3 = 1, af = fa, bf = fb, cf = fac, df = fde^2, ef = fe, g^3 = 1, ag = ga, \]
\[ bg = gb, cg = ga^2bc, dg = ga^2b^2d, eg = ge, fg = gf >, \]
where \( a, b, c, d, e \) are the same generators of \( G_{5,4} \) and \( f = (1,5,6)(2,3,7)(4,8,9)(10,16,13)(11,17,14)(12,18,15). \)

We notice here that \( G_{10,6} \) and \( G_{11,6} \) are non-isomorphic and are of exponent 9.

For \( G_{12,6} \), we have a representation as follows:
\[ G_{12,6} = \langle a, b, c, d, e : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b, d^3 = 1, \]
\[ ad = da, bd = da^6b^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ea^3c, de = ecd >, \]
where the generators \( a, b, c \) and \( d \) are the same for \( G_{2,5} \) and \( e = (1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25)(8,17,26)(9,18,27). \)

Clearly, \( G_{12,6} \) is neither isomorphic to \( G_{10,6} \) nor to \( G_{11,6} \). Moreover, Gap-programme and computations in Sym(27) show that there are no transitive \( p \)-groups of degree \( p^3 \), exponent \( p \) and orders greater than and equal to 36. Hence we have:

**Lemma 1.2.4.** There are, up to isomorphism, four transitive 3-groups of degree 33 and order 729, namely the non-abelian groups \( G_{9,6} \) (of exponent 27), \( G_{11,6} \), \( G_{12,6} \) and \( G_{10,6} \) (of exponent 9) described above.

When \( n = 7 \), then \(|G| = 2187\) and for transitivity we must have \(|\alpha^G| = 27, |G_\alpha| = 81, \forall \alpha \in \Omega. \)

Thus, \( G \) must be non-abelian and arguing in a fashion similar to the case \( n = 6 \), we have the following five representations for \( G \) as follows:
\[ G_{1,7} = \langle a, b, c, d, e, f, g : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = cb, ac = ca^{10}b^2, \]
\[ d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ec, de = ed >, \]
where \( a, b, c, d \) are the same generators of \( G_{9,6} \) and \( e = (1,19,10)(2,20,11)(5,14,23)(7,16,25). \)
\[ G_{2,7} = \langle a, b, c, d, e, f, g : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2, \]
\[ d^3 = 1, ad = da, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ec, de = eab^2d, \]
\[ f^3 = 1, af = fa, bf = fb, cf = fac, df = fde^2, ef = fe, g^3 = 1, ag = ga, \]
\[ bg = gb, cg = ga^2bc, dg = ga^2b^2d, eg = ge, fg = gf >, \]
where \( a, b, c, d, e, f \) are the same generators of \( G_{11,6} \) and
\[ g = (1,4,7)(2,5,8)(3,6,9)(10,15,14)(11,16,12)(13,18,17). \]

\[ G_{3,7} = < a, b, c, d, e, f : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b, d^3 = 1, \\
\quad ad = da, bd = da^9bc^2, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ea^3c, de = ecd, f^3 = 1, \\
\quad af = f a^7b^2c^2e, bf = f a^3b^2ce^2, cf = fc, df = fd, ef = fbc >, \]

where \( a, b, c, d, e, \) are the same generators of \( G_{12,6} \) and

\[ f = (1,4,7)(3,6,9)(10,13,16)(11,14,17)(20,23,26)(21,24,27). \]

\[ G_{4,7} = < a, b, c, d, e, f, g : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = ca, bc = cb, d^3 = 1, \\
\quad ad = dab, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = eb^2c, de = eb^2d, f^3 = 1, \\
\quad af = fa, bf = fb, cf = fab^2e, ef = fabc^2e^2, g^3 = 1, ag = gab^2c^2d^2, bg = gb, cg = gc, \\
\quad eg = gac^2d^2, f g = gabdef >, \]

where \( a, b, c, d, e, f \) are the same generators of \( G_{10,6} \) and

\[ g = (1,27,18)(2,24,14)(3,19,15)(4,21,12)(5,22,13)(6,26,17)(7,23,10)(8,25,18)(9,26,10). \]

Now we easily see that \( G_{3,7} \cong G_{4,7}. \) Hence, we have:

**Lemma 1.2.5.** There are, up to isomorphism, four transitive 3-groups of degree \( 3^3 \) and order 2187, namely the non-abelian groups \( G_{1,7}, G_{5,7} \) (both of exponent 27), \( G_{2,7}, G_{3,7} \) (both of exponent 9) described above.

When \( n = 8, \) then \(|G| = 6561 \) and for transitivity we must have

\[ |\alpha G| = 27, \ |G_\alpha| = 243, \ \forall \alpha \in \Omega. \]

Thus \( G \) must be non-abelian and arguing in a fashion similar to case \( n=6, \) we have the following presentations for \( G: \)

\[ G_{1,8} = < a, b, c, d, e, f : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10}b^2, \\
\quad d^3 = 1, ad = da^{10}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, \\
\quad de = ed, f^3 = 1, af = fa^{10}b^2c, bf = fb, cf = fc, df = fd, ef = fe >, \]

where \( a, b, c, d, e, \) are the same for \( G_{1,7} \) and

\[ f=(1,10,19)(3,12,21)(4,22,13)(6,24,15). \]
\[ G_{2,8} = < a, b, c, d, e, f, g, h : a^3 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = cb, bc = ca^2b^2, \\
d^3 = 1, ad = da, bd = db, cd = dc, e^3 = 1, ae = ea, be = eb, ce = ec, de = eab^2d, \\
af = fa, bf = fb, cf = face^2, df = fa^2de, ef = fe, g^3 = 1, ag = ga, bg = gb, \\
cg = gce^2, dg = gabde^2, eg = ge, fg = gf, h^3 = 1, ah = ha, bh = hb, ch = hce^2g, \\
dh = habdfg, eh = he, fh = hf, gh = hg >, \] where a, b, c, d, e, f, g are the same generators for \( G_{11,7} \) and \( h = (1,3,8)(2,4,8)(5,7,9). \)

\[ G_{3,8} = < a, b, c, d, e, f, g : a^9 = 1, b^9 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^5b, \\
d^3 = 1, ad = da, bd = da^6bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ea^5c, \\
dc = ec, f^3 = 1, af = fa^7b^2c, bf = fa^3b^2ce^2, cf = fc, df = fd, ef = fbc, \\
g^3 = 1, ag = gac, bg = gac^3e, cg = gc, dg = ga^4bc^2de, eg = gb^2ce^2, fg = gf >, \] where the generators a, b, c, d are the same for \( G_{3,7} \) and \( g = (1,4,7)(2,8,5)(19,22,25)(21,27,24). \)

We easily see that \( G_{3,8} \cong G_{2,8}. \) \( G_{4,8} = < a, b, c, d, e : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1, \\
ac = ca, bc = cb, d^3 = 1, ad = da, bd = dab^6c^4, cd = db^2, e^3 = 1, ae = ca^4, be = eb^4, \\
ce = ec^4, de = ea^5d >, \] where the generators a, b, c and d are the same for \( G_{1,7} \) and \( e=(2,5,8)(3,9,6)(11,14,17)(12,18,15)(20,23,26)(21,27,24). \)

Hence we have:

**Lemma 1.2.6.** There are, up to isomorphism, three transitive 3-groups of degree \( 3^3 \) and order 6561, namely the non-abelian groups \( G_{1,8}, G_{4,8} \) (both of exponent 27) and \( G_{3,8} \) (of exponent 9) described above.

When \( n=9 \), then \(|G|=19683\) and for transitivity we must have \(|\alpha^G| = 27, G_{\alpha} = 729, \forall \alpha \in \Omega.\)

Thus, \( G \) must be non-abelian and arguing in a fashion similar to case \( n=6 \), we have as presentations for \( G: \)

\[ G_{1,9} = < a, b, c, d, e, f, g : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10}b^2, \\
d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, \\
dc = ed, f^3 = 1, af = fa^{10}bcde^2, bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1, \\
ag = ga^{19}cd^2e^2f^2, bg = gb, cg = gc, dg = gd, eg = ge, fg = gf >, \] where a, b, c, d, e, f are the same for \( G_{1,8} \) and \( g=(1,10,19)(211,20,9,18,27). \)

\[ G_{2,9} = < a, b, c, d, e, f, g, h : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^5b, \\
d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, \\
dc = ed, f^3 = 1, af = fa^{10}bcde^2, bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1, \\
ag = ga^{19}cd^2e^2f^2, bg = gb, cg = gc, dg = gd, eg = ge, fg = gf >, \] where a, b, c, d, e, f are the same for \( G_{2,8} \) and \( g=(1,10,19)(211,20,9,18,27). \)
\[d^3 = 1, \ \text{ad} = da, bd = da^6b^2, cd = da^6c, e^3 = 1, \text{ae} = ea, be = eb, ce = ea^2c, de = ecdf^3 = 1, af = f a^9b^2c^2e, bf = fa^3b^2c^2e, cf = fc, df = fd, ef = fbc, g^3 = 1,\]

\[ag = gac, bg = ga^3c, cg = gc, dg = gac^3cde^2, eg = gb^2c^2e, fg = gf, h^3 = 1, ah = ha^3b^2c^2d^2, bh = ha^4b^2c^2dg, ch = hc, dh = hbdce^2g, eh = ha^4bcdeg, hg = gh, fh = hf >,\]

where the generators \(a, b, c, d, e, f, g\) are the same for \(G_{3,8}\) and \(h = (1,4,7)(3,6,9)(12,18,15)(21,27,24).\)

\[G_{3,9} = \langle a, b, c, d, e, f : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1, ac = ca, bc = cb, d^3 = 1,\]

\[ad = da, bd = da^6bc^2, cd = db^2, e^3 = 1, ae = ea^4, be = eb^4, ce = ec^4, de = ea^6d, f^3 = 1,\]

\[af = fa^9b^2c^3, bf = fb^7, cf = fc, df = fad^6c, e^3 = 1, ag = gac, bg = gc, cg = gc, dg = gac^3cde^2, eg = gb^2c^2e, fg = gf, h^3 = 1,\]

\[ah = ha^3b^2c^2d^2, bh = ha^4b^2c^2dg, ch = hc, dh = hbdce^2g, eh = ha^4bcdeg, hg = gh, fh = hf >,\]

where the generators \(a, b, c, d, e, f\) are the same for \(G_{1,8}\) and \(f = (1,7,4)(2,5,8)(19,22,25)(20,26,23).\)

Hence, we have:

**Lemma 1.2.7.** There are, up to isomorphism, three transitive 3-groups of degree 3\(^3\) and order 19683, namely the non-abelian groups \(G_{1,9}, G_{3,9}\) (both of exponent 27) and \(G_{2,9}\) (of exponent 9) described above.

When \(n=10\), then \(|G| = 59049\) and for transitivity we must have

\[|\alpha_G| = 27, |G_\alpha| = 2187.\]

Thus \(G\) must be non-abelian and arguing in a fashion similar to case \(n=6\), we have the following presentations for \(G\):

\[G_{1,10} = \langle a, b, c, d, e, f, g, h : a^{27} = 1, b^9 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10}b^2,\]

\[d^3 = 1, ad = da^9bc^2, bd = db^2, cd = dc, e^3 = 1, ae = eab^2d, be = eb, ce = ec, de = ed,\]

\[f^3 = 1, af = f a^{10}bcd^2e, bf = fb^7, cf = fc, df = fad^6c, e^3 = 1, ag = gac,\]

\[bg = gb, cg = gc, dg = gd, eg = ge, f^3 = 1, ah = ha^3b^2c^2d^2, bh = hb^c, ch = hc, dh = hbdce^2g, eh = ha^4bcdeg, hg = gh, fh = hf >,\]

where \(a, b, c, d, e, f, g, h\) are the same for \(G_{1,9}\) and \(h=(1,10,19)(3,21,12)(4,22,13)(7,16,25).\)

\[G_{3,10} = \langle a, b, c, d, e, f, g, h, k : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6b,\]

\[d^3 = 1, ad = da^9bc^2, cd = da^6bc, e^3 = 1, ae = ea, be = eb, ce = ea^3c, de = ecd,\]

\[f^3 = 1, af = f a^{10}b^2c^2e, bf = f a^3b^2c^2e, cf = fc, df = fd, e^3 = 1, ag = gac,\]

\[bg = gb^3c, cg = gc, dg = gb^3cde^2, eg = gb^2c^2e, fg = gf, h^3 = 1, ah = ha^3b^2c^2d^2,\]

\[bh = ha^4b^2c^2dg, ch = hc, dh = hbdce^2g, eh = ha^4bcdeg, hg = gh, fh = hf, k^3 = 1,\]

\[ak = kd^2c^2g, bk = ka^4bcdef, ck = kc, dk = kcd^2g^2, ek = ka^7d^2g^2, fk = kf, gk = kg,\]
The generators $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$ are the same for $G_{3,9}$ and $k=(1,4,7)(10,13,16)(12,18,15)(21,27,24)$.

\[ G_{2,10} = \langle a, b, c, d, e, f, g : a^9 = 1, b^3 = 1, ab = ba, c^3 = 1, ac = ca, bc = cb, d^3 = 1, ad = da, bd = dabc^2, cd = dc, e^3 = 1, ace = ca^4, be = eb^4, ce = ec^4, de = eabc^2d, f^3 = 1, af = fabc^3, bf = fbc^2, cf = fc, df = fabc^3e de^2, ef = fe, ag = gabc^5, bg = gb^4, cg = gc, dg = gb^5cd^2, eg = ge, fg = gf \rangle >, \]

where the generators $a$, $b$, $c$, $d$, $e$, $f$ are the same for $G_{1,9}$ and $g=(1,7,4)(3,6,9)$.

Hence we have:

**Lemma 1.2.8.** There are, up to isomorphism, three transitive 3-groups of degree $3^3$ and order 59049, namely the non-abelian groups $G_{1,10}$, $G_{2,10}$ (of exponent 27) and $G_{3,10}$ (of exponent 9) described above.

When $n=11$, then $|G|=177147$ and for transitivity we must have $|\alpha|^G=27$, $|G\alpha|=6561 \forall \alpha \in \Omega$.

Thus $G$ must be non-abelian and arguing in a fashion similar to case $n=6$, we have the following presentations for $G$ as follows:

\[ G_{1,11} = \langle a, b, c, d, e, f, g, h, k : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = cb, ac = ca^{10}b^2, d^3 = 1, ad = da^{19}c^2, bd = db, cd = dc, e^3 = 1, ace = aeb^2d, be = eb, ce = ec, de = ed, f^3 = 1, af = fabc^{10}bcde^2, bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1, ag = gabc^{19}cd^2e^2f^2, bg = gb, cg = gc, dg = gd, eg = ge, fg = gf, h^3 = 1, ah = hac^2dg^2, bh = hb, ch = hc, dh = hd, eh = he, fh = hf, gh = hg, k^3 = 1, ak = kab^2cd^2e^2fgh, bk = kb, ck = kc, dk = kd, ek = ke, f k = kf, gk = kg, hk = kh \rangle >, \]

where the generators $a$, $b$, $c$, $d$, $e$, $f$, $g$, $h$ are the same for $G_{1,10}$ and $k=(1,10,19)(5,14,23)(6,24,15)(8,17,26)$.

\[ G_{3,11} = \langle a, b, c, d, e, f, g, h, k, m : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^5b, d^3 = 1, ad = da, bd = dabc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb, ce = ca^3c, de = ecd, f^3 = 1, af = fabc^2c^2e, bf = fabc^3cc^2, cf = fc, df = fd, ef = fbc, g^3 = 1, ag = gac, bg = gac^3e, cg = gc, dg = gabc^2de^2, eg = gb^2ce^2, fg = gf, h^3 = 1, ah = ha^3b^2c^2d^2, bh = ha^3b^2c^2dg, ch = hc, dh = hbde^2g^2, eh = ha^4bcde^2g, gh = gh, fh = hf, k^3 = 1, ak = ka^2e^2g, bk = ka^4bcdef, ck = kc, dk = kcdfe^2g, ek = ka^7de^2f, fk = kf, gk = kg, hk = kh, m^3 = 1, am = ma^8cdefgh^2, bm = ma^4bdef^2g^2h^2k^2, cm = mc, dm = mbe^2de^2f^2g^2h^2k, em = mae^2de^2f^2g^2h^2k, \]
have the following presentations for $G_{3,10}$ and $m=(a,b,c,d,e,f,g,h,k)$ are the same for $G_{3,10}$ and $m=(1,4,7)(10,13,16)(11,14,17)(12,15,18)$.

$G_{2,11}=< a,b,c,d,e,f,g,h : a^9 = 1, b^9 = 1, ab = ba, c^9 = 1, ac = ca, bc = cb, d^3 = 1, ad = da, bd = da^b c^4, cd = db^2, e^3 = 1, ae = ea^4, be = eb^4, ce = ec^4, de = eg, af = f a^3 b^2 c^3, bf = fb^7, cf = fc, df = f a^3 b^6 c^6 de^2, ef = fe, ag = g a b^3, bg = gb^4, cg = gc, dg = gb^3 c^6 df^2, eg = ge, fg = gf, h^3 = 1, ah = hab^6 c^3 e, bh = hb, ch = ha^6 b^3 c^7 e f g^2, dh = ha^6 c^3 de f^2, he = eh, hf = fh, hg = gh >$, where the generators $a, b, c, d, e, f, g$ are the same for $G_{1,10}$ and $h = (1,7,4)(2,8,5)(12,15,18)(20,23,26)$.

Hence we have:

**Lemma 1.2.9.** There is, up to isomorphism, three transitive 3-groups of degree $3^3$ and order $177147$, namely the non-abelian groups $G_{1,11}, G_{2,11}$ (of exponent 27) and $G_{3,11}$ (of exponent 9) described above.

When $n=12$, then $|G| = 531441$ and for transitivity we must have $|\alpha^G| = 27, |G_\alpha| = 19683, \forall \alpha \in \Omega$.

Thus $G$ must be non-abelian and, arguing in a fashion similar to case $n=6$, we have the following presentations for $G$:

$G_{1,12}=< a,b,c,d,e,f,g,h,k,m : a^{27} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc, ac = ca^{10} b^2, d^3 = 1, ad = da^{19} b^2 c, bd = db, cd = dc, e^3 = 1, ae = ea b^2 d, be = eb, ce = ec, de = ed, f^3 = 1, af = f a^{10} b c d e f g h, bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1, ag = g a^{19} c d^2 e f^2, bg = gb, cg = gc, dg = gd, eg = ge, fg = gf, h^3 = 1, ah = hac^2 d^2 g h, bh = hb, ch = hc, dh = hd, eh = he, fh = hf, gh = hg, k^3 = 1, ak = kab^6 c^2 e f g h, bk = kb, ck = kc, dk = kd, ek = ke, fk = kf, gk = kg, hk = kh, m^3 = 1, am = ma^{16} b c d e f g h, bm = mb, cm = mc, dm = ma^9 b d, em = ma^{18} b c, fm = mb^2 c f, gm = m c d^2 e g, hm = ma^{18} b^2 c^2 f^2 h, km = m b c d e k >$, where $a, b, c, d, e, f, g, h, k$ are the same for $G_{1,11}$ and $m = (1,10,19)(2,17,5)(3,6,27)(7,16,25)(8,23,20)(9,12,15)(11,26,14)(18,21,24)$.

$G_{3,12}=< a,b,c,d,e,f,g,h,k,m,n : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca, bc = ca^6 b, d^3 = 1, ad = da, bd = da^6 b c^2, cd = da^6 c, e^3 = 1, ae = ea, be = eb, ce = ea^3 c, de = e c d, f^3 = 1, af = f a^3 b^2 c^2 e, bf = f a^3 b^2 c e^2, cf = f c, df = f d,$
ef = fbc, g³ = 1, ag = gac, bg = ga³ce, cg = gc, dg = ga³bc²de², eg = gb²ce²;
fg = gf, h³ = 1, ah = ha³b²c²d², bh = ha³b²c²dg, ch = hc, dh = hbd²e²g²,

\[eh = ha^4bcde, h_g = gh, f_h = h_f, k^3 = 1, ak = kda^2e²g, bk = ka^4bcdef, ck = kc,\]
dk = kcd²g²e², ck = ka⁷c²d²f, fk = kf, gk = kg, hk = kh, {m^3 = 1, am = ma^8cdefgh²,}

\[bm = ma^4bcdefg²h²k², cm = me, dm = mbc²d²f⁴g²h²k, em = mae^2d²f²g²h²k²,\]

\[fm = mf, gm = mg, hm = mh, km = mk, n^3 = 1, an = nab²d²e, bn = nbcdf²g,\]

cn = na^6bc²e², dn = nd, en = na^3bc²df²g, fn = nf, gn = nbce²f, hn = na^6b²cef²gk,\]

\[kn = na^6b²cef²gh²k², mn = nbe²f²h²km,\]

where the generators a, b, c, d, e, f, g, h, k, m are the same for G_{3,11} and

\[n=(1,11,27)(2,25,15)(4,14,21)(5,19,18)(7,17,24)(8,22,12).\]

\[G_{2,12} =< a, b, c, d, e, f, g, h, k : a^9 = 1, b^3 = 1, ab = ba, c^9 = 1, ac = ca, bc = cb,\]

d³ = 1, ad = da, bd = db, cd = dc, e³ = 1, ae = ea^4, be = eb^4, ce = ec^4,\]

dc = ec^6d, f^3 = 1, af = fa^4b^6c^3, bf = fb^7, cf = fc, df = fa^3b^6c^6d^2, ef = fe,\]

ag = gab³, bg = gb^4, cg = gc, dg = gb^5c^6d^2, eg = ge, fg = gf, h^3 = 1,\]

ah = hab^⁵c^e, hh = bb, ch = ha^6b^³c^⁷ef²g², dh = ha^6c³d²ef², he = eh, hf = fh,\]

hg = gh, ak = ka⁴c³f, bk = kb⁷g², ck = kc, dk = kc⁵d²fg²h², ek = ke, fk = kf,\]

gk = kg, hk = kh, where the generators a, b, c, d, e, f, g, h are the same for G_{1,11} and k = (2,8,5)(21,24,27).

Hence we have:

**Lemma 1.2.10.** There are, up to isomorphism, three transitive 3-groups of degree 3³ and order 531441, namely the non-abelian groups G_{1,12}, G_{2,12} (of exponent 27) and G_{3,12} (of exponent 9) described above.

When n=13, then |G|=1594323 and for transitivity we must have

\[|\alpha|^G=27, |G\alpha|=19683, \forall \alpha \in \Omega.\]

Thus G must be non-abelian and arguing in a fashion similar to case n=6, we have the following presentations for G:

\[G_{1,13} =< a, b, c, d, e, f, g, h, k, m, n : a^{2^7} = 1, b^3 = 1, ab = ba^{10}, c^3 = 1, bc = bc,\]

ac = ca^{10}b^2, d^3 = 1, ad = da^{10}c^2, bd = db, cd = dc, e³ = 1, ae = eab²d, be = eb,\]

ce = ec, de = ed, f^3 = 1, af = fa^{10}bcde², bf = fb, cf = fc, df = fd, ef = fe, g^3 = 1,

ag = ga^9cd²e²f², bg = gb, cg = gc, dg = gd, eg = ge, fg = gf, h^3 = 1, ah = hac²dg²,
$bh = hb, ch = hc, dh = hd, eh = he, fh = hf, gh = hg, k^3 = 1, ak = kab^2d^2e^2fgh,$

$bk = kb, ck = kc, dk = kd, ek = ke, fk = kf, gk = kg, hk = kh, m^3 = 1,$

$am = ma^{16}bcdefg^2h, bm = mb, cm = mc, dm = ma^9bd, em = ma^{18}bc,$

$f m = mb^2cf, gm = mcd^2eg, hm = ma^{18}b^2c^2f^2h, km = mbcde^2k, n^3 = 1,$

$an = na^{16}cd^2fh^2km^2, bn = nb, cn = nc, dn = nb^2cd, en = ne, fn = nb^2ef,$

$gn = na^9c^2de^2f^2g^2h, kn = na^{18}bd^2e^2f^2k, mn = na^{18}b^2d^2f m >,$

where $a, b, c, d, e, f, g, h, k, m$ and $n$ are the same for $G_{1,12}$ and


$G_{3,13} = < a, b, c, d, e, f, g, h, k, m, n, p : a^9 = 1, b^3 = 1, ab = ba^4, c^3 = 1, ac = ca,$

$bc = ca^6b, d^3 = 1, ad = da, bd = da^8bc^2, cd = da^6c, e^3 = 1, ae = ea, be = eb,$

$ce = ea^3c, de = ecd, f^3 = 1, af = f a^7b^2c^2e, bf = f a^3b^2c^2e, cf = fc, df = fd,$

$ef = fbc, g^3 = 1, ag = gac, bg = ga^3ce, cg = gc, dg = ga^3bc^2de^2, eg = gb^2c^2e^2,$

$fg = gf, h^3 = 1, ah = ha^9b^2c^2d^2, bh = ha^8b^2c^2dg, ch = hc, dh = hbde^2g^2,$

$eh = ha^4bcdeg, hg = gh, fh = hf, k^3 = 1, ak = kd^2e^2g, bk = ka^4bcdef, ek = kc,$

$dk = kcd^2g^2, ek = ka^7de^2f, fk = kf, gk = kg, hk = kh, m^3 = 1, am = ma^8cdefgh^2,$

$bn = ma^4bdef^2g^2h^2k^2, cm = mc, dm = mbcde^2f^2g^2h^2k, em = mae^2de^2f^2g^2k^2,$

$fm = mf, gm = mg, hm = mh, km = mk, n^3 = 1, an = nab^2d^2e, bn = nbcd^2f^2g,$

$cn = na^4bc^2d^2, dn = nd, en = na^3bc^2df^2g, fn = nf, gn = nbce^2f^2g, hn = na^6b^2c^2e^2gk,$

$kn = na^6b^2c^2ef^2gh^2k^2, mn = nbc^2f^2h^2km^3, n^3 = 1, ap = pa^7bcde^2f^2n, bp = pecg^2h^2kn^2,$

$cp = pa^3c^2f^2, dp = pd, ep = pc^2ef^2g^2h^2kn^2, fp = pf, gp = po^6fg, hp = pb^2ef^2g^2h^2k^2,$

$k p = pb^2ef^2g^2h, np = pn >,$ where the generators $a, b, c, d, e, f, g, h, k, m, n$ are the same for $G_{3,12}$ and

$p=(1,4,7)(2,12,19)(5,15,22)(8,18,25)(11,14,17)(21,24,27).$
Let \( fl = lf, gl = lg, hl = lh, kl = lk \), where the generators \( a, b, c, d, e, f, g, k \) are the same for \( G_{1,12} \) and \( l=(21,27,24) \).

Hence we have:

**Lemma 1.2.11.** There are, up to isomorphism, three transitive 3-groups of degree \( 3^3 \) and order 1594323, namely the non-abelian groups \( G_{1,13}, G_{2,13} \) (of exponent 27) and \( G_{3,13} \) (of exponent 9) described above.

We summarize our findings:

| \( n \) | \( |G| = 3^n \) | Number of transitive abelian 3-group of degree 27 up to isomorphism | Number of transitive non-abelian 3-group of degree 27 up to isomorphism | Number of transitive 3-groups of degree 27 up to isomorphism |
|--------|-----------------|-------------------------------------------------|-------------------------------------------------|-------------------------------------------------|
| n=1   | 3               | 0                                               | 0                                               | 0                                               |
| n=2   | 9               | 0                                               | 0                                               | 0                                               |
| n=3   | 27              | 3                                               | 2                                               | 5                                               |
| n=4   | 81              | 0                                               | 4                                               | 4                                               |
| n=5   | 243             | 0                                               | 4                                               | 4                                               |
| n=6   | 729             | 0                                               | 4                                               | 4                                               |
| n=7   | 2187            | 0                                               | 4                                               | 4                                               |
| n=8   | 6561            | 0                                               | 3                                               | 3                                               |
| n=9   | 19683           | 0                                               | 3                                               | 3                                               |
| n=10  | 59049           | 0                                               | 3                                               | 3                                               |
| n=11  | 177147          | 0                                               | 3                                               | 3                                               |
| n=12  | 531441          | 0                                               | 3                                               | 3                                               |
| n=13  | 1594323         | 0                                               | 3                                               | 3                                               |
| Total |                 | 3                                               | 36                                              | 39                                              |

We may state:

**Proposition 1.2.12.** There are, up to isomorphism, 39 transitive 3 - groups of degree \( 3^3 \), three of these are abelian. Of the remaining 36 non - abelian, 17 are of exponent 27, 13 are of exponent 9 and 6 are of exponent 3.
PROGRAMME 1:

\begin{verbatim}
gap s8:=Group((1,2),(1,2,3,4,5,6,7,8));
gap a:=(1,2,3,4,5,6,7,8);
b:=(1,7)(3,5)(4,8);
gap h:=Subgroup(s8,[a,b]);
gap req:=[ ];
gap for c in diff do
  if c^2=() then
    if b^c=b then
      if a^c=a^7 then
        Add(req,c);
    fi;
  fi;
  od;
gap req;[(1,3)(4,8)(5,7),(1,7)(2,6)(3,5)]
\end{verbatim}

PROGRAMME 2:

\begin{verbatim}
gap s8:=SymmetricGroup(8);
gap a:=(1,2,3,4,5,6,7,8);
b:=(1,7)(3,5)(4,8);
c:=(1,3)(4,8)(5,7);
gap H:=Subgroup(s8,[a,b,c]);
gap req:=[ ];
gap for r in diff do
  if r^2=() then
    if Order(s8,r)<=4 then
      if a^r in H then
        if b^r in H then
          if c^r in H then
            if Size(Subgroup(s8,[a,b,c,r]))=64 then
              Add(req,r);
            fi;
          fi;
        fi;
      fi;
    fi;
  fi;
  od;
gap req;
[(3,7)(4,8),(2,6)(3,7),(1,2)(3,4)(5,6)(7,8),
 (1,3)(2,4)(5,7)(6,8),(1,3)(2,8)(4,6)(5,7),
 (1,4)(2,7)(3,6)(5,8),(1,5)(4,8)(1,5)(2,6),
 ((1,6)(2,5)(3,8)(4,7),(1,7)(2,4)(3,5)(6,8),
 (1,7)(2,8)(3,5)(4,6),(1,8)(2,3)(4,5)(6,7)]
\end{verbatim}
References


