Kragujevac J. Math. 28 (2005) 87–96.

CONFORMALLY OSSERMAN LORENTZIAN MANIFOLDS[1]

Novica Blažić

Faculty of Mathematics, University of Belgrade,
P. O. Box 550, 11001 Belgrade, Serbia and Montenegro

Abstract. Let (M^n, g) be a pseudo-Riemannian manifold of which the Jacobi operator associated to the Weyl conformal curvature tensor has constant eigenvalues on the bundle of unit timelike (spacelike) tangent vectors (known as *conformally Osserman manifolds*). In this work we study the conformally Osserman Lorentzian manifolds. The established characterizations indicate the rigidity of conformally Osserman Lorentzian manifolds. We additionally illustrate that rigidity by reviewing analog recent characterizations in the case of metrics of other signatures.

1. INTRODUCTION

Let R be the curvature tensor of pseudo-Riemannian n-dimensional manifold (M^n, g) of signature (p, q), $p + q = n$. *The Jacobi operator* $\mathcal{J}_R(x) : T_pM \longrightarrow T_pM$ is defined by $\mathcal{J}_R(x)y = R(y, x)x$, for tangent vectors x and y. It is a symmetric operator which eigenvalues could be used to characterize the geometry of M. We say (M, g) is a *timelike (spacelike) Osserman manifold* if the eigenvalues of the Jacobi operators $\mathcal{J}_R(x)$ are independent on the choice of the unit timelike (spacelike) directions

[1]Research partially supported by MNTS, Project 1854 (Serbia).

x at a given point. If both conditions are satisfied then (M, g) is *Osserman*. Notice that in [10] was shown that for $p, q > 1$ the algebraic curvature tensor R is timelike Osserman if and only if it is spacelike Osserman. Moreover, if the eigenvalues of $\mathcal{J}_R(x)$ are also independent on the point p, M is *globally Osserman*. The basic examples are 2-point homogeneous spaces. Osserman conjectured that the inverse is true in Riemannian setting: globally Osserman manifolds have to be locally isometric to: $\mathbf{R^n}$, complex projective space $\mathbf{C}P^n$, quaternionic-projective space $\mathbf{HP^n}$, Cayley projective plane CaP^2 or to their noncompact duals (see [15]). Chi ([9]) was first to confirm the conjecture when the dimension of the manifold is not divisible by 4. Then in [12],the two-step approach to classify Osserman manifolds was suggested. The first step is to show the existence of a Clifford algebra structure on a tangent space which is compatible with a given Osserman algebraic curvature tensor. Later on Nikolayevsky proved that is the case when $n \neq 16$ ([13], [14]).

The Weyl curvature tensor W as an conformal invariant is important in the understanding of conformal pseudo-Riemannian geometry. The corresponding *Weyl Jacobi* operator is given by $\mathcal{J}_W(x)y = \mathcal{J}(x)y = W(y, x)x$, for tangent vectors x, y. We say that (M, g) is *conformally Osserman* if the eigenvalues of the symmetric Weyl Jacobi operator $\mathcal{J}_W(x) = \mathcal{J}(x) : \{x\}^\perp \longrightarrow \{x\}^\perp$ are independent on the choice of the unit timelike (spacelike) direction x, as introduced in [7]. This is a conformal notion. Conformally Osserman manifolds of some particular dimensions and metric signatures were studied in [4, 7, 5, 6].

The main goal is to extend the study of conformally Osserman manifolds from the Riemannian to the Lorentzian setting. The only 2-point homogeneous Lorentzian manifolds are spaces of constant sectional curvature (see [2]). Thus, manifolds conformally equivalent to spaces of constant curvature are examples of conformally Osserman manifolds. We will show in Section 4 that the inverse is also true: conformally Osserman manifolds are conformally flat (Theorem 3.1).

Even this classification is in the spirit of known classification in the Riemannian setting, it shows some rigidity of the Lorentzian signature. In Section 5, we review known recent results concerning conformally Osserman manifolds in other signatures to illustrate the rigidity of Lorentzian signature. Results in Riemannian and $(2,2)$ setting (also known as Kleinian geometry) were discussed and compared to the established characterization in the Lorenzian signature. In Section 2, main notions and notation are introduced and some basic results are commented. In Section 3 the proof of the Theorem 3.1 is given.

2. PRELIMINARIES

Let (M^n, g) be a pseudo-Riemannian manifold, $R(x,y)z = [\nabla_x, \nabla_y]z - \nabla_{[x,y]}z$ its *curvature operator* and $R(x,y,z,w) = g(R(x,y)z, w)$ its *curvature tensor*. Let $\{e_i\}$ be a local orthonormal frame for the tangent bundle, $|e_i|^2 = \epsilon_i$, $\epsilon_i = \pm 1$. A vector $x \in T_pM$ is *timelike (spacelike) unit* if $|x|^2 = -1$, ($|x|^2 = 1$). The *Ricci curvature* ρ and the *scalar curvature* τ are defined by $\rho(x,y) = \sum_i \epsilon_i R(x, e_i, e_i, y)$ and $\tau = \sum_k \epsilon_k \rho(e_k, e_k)$.

The *Weyl conformal curvature operator* is defined by

$$W(x,y) = R(x,y) + c_1(n)\tau R_0(x,y) + c_2(n)L(x,y), \tag{1}$$

where

$$L(x,y)z = g(\rho y, z)x - g(\rho x, z)y + g(y,z)\rho x - g(x,z)\rho y, \tag{2}$$

$$R_0(x,y)z = g(y,z)x - g(x,z)y, \tag{3}$$

$c_1(n)^{-1} = (n-1)(n-2)$, and $c_2(n)^{-1} = n-2$, $n > 2$. Notice that R_0 is the curvature tensor of the constant curvature space.

We say that pseudo-Riemannian metrics g_1 and g_2 are *conformally equivalent* if $g_1 = \alpha \cdot g_2$ where α is a smooth positive function. The conformal class of a metric g is denoted by $[g]$. The Weyl curvature operator is invariant on a conformal class $[g]$. It is well known that the pseudo-Riemannian n-dimensional manifold $n \geq 4$ is conformally flat if and only if its Weyl operator W vanishes. It is important to notice that the notion of conformally Osserman manifolds is a conformal invariant, i.e. it is a property of conformal class of metric $[g]$.

Here recall some basic definitions related to the algebraic curvature tensors. Let $(V, \langle\cdot,\cdot\rangle)$ be an n-dimensional vector space and $\langle\cdot,\cdot\rangle$ a scalar product on $V$of signature (p, q). A tensor $R \in \otimes^4 V^*$ is said to be an *algebraic curvature tensor* if it has the following standard symmetries:

$$R(x, y, z, w) = -R(y, x, z, w) = -R(x, y, w, z), \tag{4}$$

$$R(x, y, z, w) = R(z, w, x, y), \tag{5}$$

$$R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \tag{6}$$

The curvature operator $R(x, y)z$ is determined by the algebraic curvature tensor.

In the space of algebraic curvature tensors, an important role is played by the following curvature tensor introduced by Gilkey, (see [11]) determined by the skew-symmetric endomorphism Φ, $\Phi^2 = -1$,

$$R_\Phi(x, y)z = g(\Phi y, z)\Phi x - g(\Phi x, z)\Phi y - 2g(\Phi x, y)\Phi z. \tag{7}$$

A *quaternionic structure* (Φ_1, Φ_2, Φ_3) on a vector space $(V, \langle\cdot,\cdot\rangle)$ is determined by the skew-symmetric endomorphisms Φ_i on $(V, \langle\cdot,\cdot\rangle)$ such that

$$\Phi_1^2 = \Phi_2^2 = -\mathrm{Id}\,, \quad \Phi_1\Phi_2 = -\Phi_2\Phi_1 = \Phi_3.$$

If the structures are defined locally in the neighborhood of an arbitrary point of a manifold, we say that (Φ_1, Φ_2, Φ_3) defines a *quaternionic structure* on the manifold. A curvature tensor R is a *quaternionic* (or Cliff(2)) curvature tensor if there exist a quaternionic structure (Φ_1, Φ_2, Φ_3) such that R is a linear combination of the curvature tensors R_0, R_{Φ_i} i.e. if

$$R = \lambda_0 R_0 + \lambda_1 R_{\Phi_1} + \lambda_2 R_{\Phi_2} + \lambda_3 R_{\Phi_3}. \tag{8}$$

The curvatures of the complex projective space $\mathbf{C}P^n$ and the quaternionic space $\mathbf{HP^n}$ are of the form (8) if $\lambda_2 = \lambda_3 = 0$ and $\lambda_1 = \lambda_2 = \lambda_3$ respectively. Thus, the spaces which curvature tensors R are as in (8) are *generalized complex space forms*, if $\lambda_2 = \lambda_3 = 0$ and *generalized quaternionic space forms* in a general case. The conformal notions, *generalized conformally complex space forms* and *generalized* *conformally quaternionic space forms* are defined by requiring that the Weyl curvature tensor W is of the corresponding form.

3. LORENTZIAN GEOMETRY

In this section the following characterization will be established.

Theorem 3.1. *Let (M, g) be an n-dimensional Lorentzian manifold, $n \geq 4$. Then the following conditions are equivalent:*

(a) *M is a conformally Osserman manifold,*

(b) *M is a conformally flat manifold.*

Remark 3.2. *For $n = 3$ the theorem holds if we replace (b) with: the Weyl tensor W vanishes.*

The proof of the theorem will follow from the following purely algebraic lemma.

Theorem 3.3. *Let V be the Lorentzian vector space of the signature $(1, n-1)$, let R be an algebraic curvature tensor and W the corresponding Weyl operator. The following conditions are equivalent:*

(a) *R is a conformally Osserman curvature tensor,*

(b) $W = \lambda R_0$,

(c) $W = 0$.

Proof. Let $e_1, ..., e_m$ be a local orthonormal frame for a Lorentzian vector space V. We choose the notation so that e_1 is timelike and e_2,..., e_n are spacelike. Let R_{ijkl} be the components of the curvature tensor. The proof that (a) implies (b) in Theorem 3.3 will follow from the following Lemma proved in [2] (see also [11], Theorem 3.1.8 and Lemma 1.7.5).

Lemma 3.4. **[2]** *Let $1 < i, j, k \leq m$ and let k be distinct from i, j. If R is an algebraic Osserman curvature tensor on V, then*

(a) $R_{kimj} + R_{mikj} = 0$ *and* $R_{kikj} + R_{mimj} = 0$,

(b) $R_{mkmi} = 0$ *and* $R_{kmkj} = 0$.

In the proof of this Lemma the special algebraic properties of the Lorentzian signature were used. Lemma 3.4 implies that an Osserman algebraic curvature tensor in the Lorentzian signature is of constant sectional curvature, i.e. proportional to the tensor R_0 (see also [2]). Thus, under the assumptions of Theorem 3.3, (a), $W = \lambda R_0$.

The Weyl curvature tensor is a trace-free component in the decomposition of the space of curvature tensors. Hence from $W = \lambda R_0$ follows $\lambda = 0$ and (b) implies (c). The checking of the last implication being directly, we complete the proof of the theorem. Q.E.D.

From the proof of the theorem also follows

Corollary 3.5. *Assume that at every point $p \in M$, $\mathrm{Tr}\,\{\mathcal{J}^2_{\mathrm{W}}(\mathrm{x})\}$ is independent on the timelike (spacelike) unit direction x. Then (M, g) is conformally flat for $n \geq 4$ and the Weyl tensor W wanishes for $n = 3$.*

From the proof of Theorem 3.1 we show the following equivalence.

Corollary 3.6. *A Lorentzian manifold (M, g) is timelike conformally Osserman if and only if it is spacelike conformally Osserman.*

Remark 3.7. *This justifies the notion of conformally Osserman manifold which we use for either of the following two conditions: timelike conformally Osserman or spacelike conformally Osserman manifolds. Generally for a pseudo-Riemannian manifold of dimension (p, q), $p, q \geq 1$ in [10] was shown: the eigenvalues of the Jacobi operator of an algebraic curvature tensor are independent on the spacelike unit direction x if and only if they are independent on the timelike unit direction x.*

Let's recall that the variational vector field y along a geodesic γ_0 (tidal force) satisfies the Jacobi equation

$$y'' = -R(y, \gamma')\gamma' = -J_{\gamma'}(y).$$

Thus, we may say that all directions in a given point of an Osserman manifold define the same local dynamics (*dynamically isotropic* manifolds). Similarly, conformally Osserman manifolds could be understood as *conformally dynamically isotropic* manifolds. Then we can reformulate the know results as follows.

Corollary 3.8. *A Lorentzian manifold of dimension n, $n \geq 4$, is (conformally) dynamically isotropic if and only if is (conformal to) a flat space or a rank 1 symmetric space.*

4. COMPARISON TO THE CONFORMALLY OSSERMAN NON-LORENTZIAN GEOMETRY

Let (M, g) be an n-dimensional Riemannian manifold. If dimension n is not divisible by 4, it was shown that a Riemannian conformally Osserman manifold (M, g) has essentially to be locally conformal to a flat space, $\mathbf{C}P^n$ or its noncompact dual ${}^*\mathbf{C}P^n$ ([7],[4]). In the proof, along two-step approach, was shown that an algebraic conformally Osserman manifold is a *conformal space form*, i. e. it is of the form

$$W = \lambda_0 R_0 + \lambda_1 R_\Phi,$$

where W is its Weyl curvature and Φ is an almost complex structure.

In dimension 4, a positive definite conformally Osserman manifold has Weyl tensor of the following form

$$W = \lambda_0 R_0 + \lambda_1 R_{\Phi_1} + \lambda_2 R_{\Phi_2} + \lambda_3 R_{\Phi_3},$$

where Φ_i, $i = 1, 2, 3$, are almost complex structures. In the case of $(2, 2)$ signature the Weyl curvature tensor of a conformally Osserman manifold can also be expressed in terms of an appropriate quaternionic-like, Cliff(1,1)-structure (Φ_1, Φ_2, Φ_3)

$$\Phi_1^2 = -\mathrm{Id}\,, \quad \Phi_2^2 = \mathrm{Id}\,, \quad \Phi_1\Phi_2 = -\Phi_2\Phi_1 = \Phi_3.$$

The expression is more complicated than in the positive definite 4 dimensional case ([6]).

The known results show that the conformally Osserman manifolds Lorentzian and Riemannian (known examples) are conformally equivalent to some Osserman metrics. But in the Kleinnian geometry of split $(2,2)$ signature that is not true. Examples of conformally Osserman metrics (M,g) which do not contain Osserman metric in the conformal class $[g]$ were constucted in [8].

Similarly, in the relation to the (nonconformal) Osserman conditions, the Lorentzian signature is more rigid than the Riemannian. For example, Lorentzian Osserman manifolds of arbitrary dimension are of constant sectional curvature [2] but for Riemannian manifolds that doesn't hold. There exist a big family of 4-dimensional Einstein self-dual metrics which are of course Osserman (see [9, 16]). In $(2,2)$-signature there exist even more complicated examples ([3]). Generally, in dimension 4, the notion of conformally Osserman manifolds of $(++++)$ and $(--++)$ signature is closely related to the the notion of self-duality [1, 5, 6, 8, 16].

Acknowledgements: The author is grateful to the referee for helpful suggestions.

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