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OPTIMAL INEQUALITIES FOR EMBEDDED SPACE-TIMES

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Abstract. In the search for ideally embedded space-times we present an optimal inequality for a m-dimensional Riemannian or Lorentzian manifold embedded as a hypersurface in a (m+1)-dimensional Ricci flat space. We give some examples and discuss its applications in higher-dimensional physics.

1. INTRODUCTION

Soon after Riemann (1854) introduced the notion of a manifold, Schläfli (1873) conjectured that every Riemannian manifold could be locally considered as a submanifold of an Euclidean space with sufficient high codimension. This was later proven in different steps by Janet (1926), Cartan (1927) and Burstin (1931) and extended

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to semi-Riemannian manifolds by Friedmann [12]. The idea was to obtain intrinsic information about the manifold using the knowledge of the extrinsic world.

Based on these embedding theorems some inequalities between intrinsic and extrinsic curvatures of the submanifold were obtained. For example, an inequality for surfaces in the four-dimensional Euclidean space, proved by Wintgen (1979), states that the squared mean curvature is always greater then or equal to the sum of the normalised scalar curvature of the surface and the normal curvature in the normal bundle. Equality holds if and only if the ellipse of curvature is a circle. This inequality was generalised by Rouxel [23] and Guadalupe and Rodriguez [14] to surfaces with general codimension in a real space-form and by De Smet, Dillen, Verstraelen and Vrancken [7] for general m-dimensional submanifolds with codimension two. Recently the inequality was proven in the semi-Riemannian case and space-times satisfying the equality were found by Dillen, Haesen, Petrović-Torgašev and Verstraelen [8]. Further, new relations between intrinsic and extrinsic curvatures using arbitrary dimensional normal sections and their projections on appropriate subspaces were considered in [15].

In 1993 Chen introduced new intrinsic curvatures which satisfy an optimal inequality with the squared mean curvature in the Riemannian case [2, 3, 4, 25]. These were applied to Lagrangian and Sasakian spaces and adapted for Einstein and conformally flat spaces. Recently a generalisation was made to semi-Riemannian spaces which are locally and isometrically embedded in a pseudo-Euclidean space [16].

In this paper we present a further generalisation to semi-Riemannian spaces embedded in a Ricci flat semi-Riemannian space and consider applications in higher-dimensional physics.

2. IDEALLY EMBEDDED HYPERSURFACES

Take $(\mathcal{M}, g)$ to be a m-dimensional semi-Riemannian manifold which is locally and isometrically embedded in a n-dimensional manifold $(\mathcal{N}, \widetilde{g})$. Denote with ∇ the

Levi-Civita connection on $\mathcal{M}$ and with $\widetilde{\nabla}$ the corresponding Levi-Civita connection on $\mathcal{N}$.

We can decompose the covariant derivative in $\mathcal{N}$ between two tangent vector fields X and Y on $\mathcal{M}$ as

$$\widetilde{\nabla}_X Y = \nabla_X Y + \Omega(X,Y) \ ,$$

with $\Omega : T\mathcal{M} \times T\mathcal{M} \to N(\mathcal{M})$ the second fundamental form.

Let $\{\vec{\xi}_A\}$ be an orthonormal basis in the normal space $N(\mathcal{M})$ of $\mathcal{M}$. We have

$$\Omega(X,Y) = \sum_{A=m+1}^{n} \varepsilon_A \, \widetilde{g}(\widetilde{\nabla}_X Y, \vec{\xi}_A) \, \vec{\xi}_A \ ,$$

with $\varepsilon_A = \widetilde{g}(\vec{\xi}_A, \vec{\xi}_A) = \pm 1$.

The mean curvature vector is defined as

$$\vec{H} = \frac{1}{m} \sum_{A=m+1}^{n} \varepsilon_A \, g^{\alpha\beta} \Omega^A_{\alpha\beta} \, \vec{\xi}_A \ ,$$

with summation convention on the indices $\alpha, \beta = 1, ..., m$.

Let $\{\vec{e}_\alpha\}$ be an orthonormal basis of $\mathcal{M}$. The sectional curvature of a two-plane in $T_p\mathcal{M}$ spanned by $\{\vec{e}_\alpha, \vec{e}_\beta\}$ is given by

$$K(\vec{e}_\alpha \wedge \vec{e}_\beta) = \varepsilon_{\alpha\beta} \, g(\mathcal{R}(\vec{e}_\alpha, \vec{e}_\beta)\vec{e}_\alpha, \vec{e}_\beta) \ ,$$

with $\varepsilon_{\alpha\beta} = \varepsilon_\alpha \varepsilon_\beta$ and $\mathcal{R}$ is the curvature operator.

The scalar curvature of an r-plane section L in $T_p\mathcal{M}$ spanned by $\{\vec{e}_1, \ldots, \vec{e}_r\}$ is then given by

$$\tau(L) = \sum_{1 \le \alpha < \beta \le r} K(\vec{e}_\alpha \wedge \vec{e}_\beta) \ .$$

Analogously we define the scalar curvature of the same plane considered as a subspace of $T_p\mathcal{N}$,

$$\widetilde{\tau}(L) = \sum_{1 \le \alpha < \beta \le r} \widetilde{K}(\vec{e}_\alpha \wedge \vec{e}_\beta) \ .$$

We denote the difference of these two scalar curvatures of the same plane in a point p by

$$\sigma(L) = \tau(L) - \widetilde{\tau}(L) \ .$$

Definition 1 *For any given set of mutually orthogonal plane sections $\{L_j\}$ with dimensions $(n_1, \ldots, n_k)$ such that $n_1 + \ldots + n_k \leq m$, the Λ-curvatures of Chen in the semi-Riemannian case are given by*

$$\Lambda(n_1, \ldots, n_k) = \tau - \mathit{inf}\{\sigma(L_1) + \ldots + \sigma(L_k) \mid L_j \text{ a non-null plane section}, L_i \perp L_j\} \ ,$$

and

$$\hat{\Lambda}(n_1, \ldots, n_k) = \tau - \mathit{sup}\{\sigma(L_1) + \ldots + \sigma(L_k) \mid L_j \text{ a non-null plane section}, L_i \perp L_j\} \ ,$$

Remark that these newly defined Λ-curvatures reduce to the δ-curvatures of Chen in the case $\mathcal{N}$ is Euclidean.

Let $\{\vec{e}_1, \ldots, \vec{e}_m, \vec{\xi}_{m+1}, \ldots, \vec{\xi}_n\}$ be an orthonormal basis of $\mathcal{N}$. If $\mathcal{M}$ is Lorentzian, and because we have space-time applications in mind, we take $\mathcal{M}$ to be time-orientable, i.e., there exists a global nowhere-zero timelike vector field which we denote with $\vec{e}_m$.

Definition 2 *An embedding $f : \big(\mathcal{M}_{(m-1,1)}, g\big) \to \big(\mathcal{N}_{(n-1,1)}, \widetilde{g}\big)$ is called causal-type preserving if $\widetilde{\nabla}_{\vec{e}_\alpha}\vec{\xi}_A$ is spacelike, $\forall \alpha = 1, \ldots, m$ and $\forall A = m+1, \ldots, n$.*

Definition 3 *An embedding $f : \big(\mathcal{M}_{(m-1,1)}, g\big) \to \big(\mathcal{N}_{(m-1,n-m+1)}, \widetilde{g}\big)$ is called causal-type preserving if $\widetilde{\nabla}_{\vec{e}_m}\vec{\xi}_A$ is timelike, $\forall A = m+1, \ldots, n$.*

Because $\Omega^A_{m\alpha} = -\widetilde{g}(\vec{e}_m, \widetilde{\nabla}_{\vec{e}_\alpha}\vec{\xi}_A) = -\widetilde{g}(\vec{e}_\alpha, \widetilde{\nabla}_{\vec{e}_m}\vec{\xi}_A)$, we have that causal-type preserving embeddings satisfy $\Omega^A_{m\alpha} = 0$, $\alpha = 1, \ldots, m-1$.

In the following theorem we consider the embedding of a Riemannian or Lorentzian manifold as a hypersurface into a semi-Riemannian space. If $\mathcal{M}$ is Lorentzian, the embedding is understood to be causal-type preserving.

Theorem 1 *Let a m-dimensional Riemannian or Lorentzian manifold $(\mathcal{M}, g)$ be locally and isometrically embedded in a $(m+1)$-dimensional semi-Riemannian manifold $(\mathcal{N}, \widetilde{g})$ with diagonalisable Ricci tensor $\widetilde{S}$ (i.e., there exists an orthonormal basis $\{\vec{e}_a\}$ of $\mathcal{N}$ such that $\widetilde{S} = \sum_{a=1}^{m+1} \lambda_a \vec{e}_a \otimes \vec{e}_a$).*

Then, for every $k \geq 0$ and every set $(n_1, \ldots, n_k)$ such that $n_1 < m$ and $n_1 + \ldots + n_k \leq m$, we have

$$\|H\|^2 \geq c(n_1, \ldots, n_k)\Lambda(n_1, \ldots, n_k) - \frac{1}{2}c(n_1, \ldots, n_k)\left\{\sum_{\alpha=1}^{m} \varepsilon_\alpha \lambda_\alpha - \lambda_{m+1}\right\} \ , \qquad (1)$$

if sign($\mathcal{N}$) $= (s_{\mathcal{M}} + 1, t_{\mathcal{M}})$, and

$$\|H\|^2 \leq c(n_1, \ldots, n_k)\hat{\Lambda}(n_1, \ldots, n_k) - \frac{1}{2}c(n_1, \ldots, n_k)\left\{\sum_{\alpha=1}^{m} \varepsilon_\alpha \lambda_\alpha + \lambda_{m+1}\right\} \ , \qquad (2)$$

if sign($\mathcal{N}$) $= (s_{\mathcal{M}}, t_{\mathcal{M}} + 1)$.

The constant $c(n_1, \ldots, n_k)$ is defined as

$$c(n_1, \ldots, n_k) = \frac{2(m + k - \sum_{j=1}^{k} n_j)}{m^2(m + k - 1 - \sum_{j=1}^{k} n_j)} \ .$$

Proof. The Gauss equation in this case reads,

$$\widetilde{R}_{\alpha\beta\gamma\mu} = R_{\alpha\beta\gamma\mu} - \widetilde{\varepsilon}(\Omega_{\alpha\gamma}\Omega_{\beta\mu} - \Omega_{\alpha\mu}\Omega_{\beta\gamma}) \ ,$$

with $\widetilde{\varepsilon} = \widetilde{g}(\vec{e}_{m+1}, \vec{e}_{m+1})$ and $\Omega_{\alpha\beta}$ the second fundamental form of the embedding. Since we only have one normal direction, we have

$$\vec{H} = \frac{1}{m}\widetilde{\varepsilon}\, g^{\alpha\beta}\Omega_{\alpha\beta}\, \vec{e}_{m+1} \ .$$

Remark that $\Omega_{\alpha\beta}$ is not necessarily diagonal with respect to the orthonormal basis $\{\vec{e}_a\}$ which diagonalises $\widetilde{R}_{ab}$. If we contract the Gauss equations two times using the metric $g_{\alpha\beta}$ we find,

$$2\tau = \sum_{\alpha=1}^{m} \varepsilon_\alpha \, \lambda_\alpha - \widetilde{\varepsilon}\, \lambda_{m+1} + \widetilde{\varepsilon}\{(\Omega_\mu{}^\mu)^2 - \Omega_{\alpha\beta}\Omega^{\alpha\beta}\} \ . \qquad (3)$$

Introducing the notation $a_\alpha = \varepsilon_\alpha \Omega_{\alpha\alpha}$ (no sum) and

$$\begin{aligned} \phi &= 2\tau - \left(\sum_{\alpha=1}^{m} \varepsilon_\alpha \lambda_\alpha - \widetilde{\varepsilon}\lambda_{m+1}\right) - \frac{m^2(m + k - 1 - \sum_{j=1}^{k} n_j)}{m + k - \sum_{j=1}^{k} n_j}\|H\|_\perp^2 \ , \\ \gamma &= m + k - \sum_{j=1}^{k} n_j \ , \end{aligned}$$

equation (3) becomes

$$\widetilde{\varepsilon}\left(\sum_{\alpha=1}^{m} a_\alpha\right)^2 = \gamma\left\{\phi + \widetilde{\varepsilon}\sum_{\alpha=1}^{m}(a_\alpha)^2 + \widetilde{\varepsilon}\sum_{\alpha\neq\beta=1}^{m}\varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2\right\} \ . \tag{4}$$

If we use the notation

$$\begin{aligned}
\bar{a}_1 &= a_1 \ , \\
\bar{a}_2 &= a_2 + \ldots + a_{n_1} \ , \\
\bar{a}_3 &= a_{n_1+1} + \ldots + a_{n_1+n_2} \ , \\
&\vdots \\
\bar{a}_{k+1} &= a_{n_1+\ldots+n_{k-1}+1} + \ldots + a_{n_1+\ldots+n_k} \ , \\
\bar{a}_{k+2} &= a_{n_1+\ldots+n_k+1} \ , \\
&\vdots \\
\bar{a}_\gamma &= a_{m-1} \ , \\
\bar{a}_{\gamma+1} &= a_m \ ,
\end{aligned}$$

we have

$$\left(\sum_{\alpha=1}^{\gamma+1}\bar{a}_\alpha\right)^2 = \left(\sum_{\alpha=1}^{m} a_\alpha\right)^2 \ ,$$

and

$$\sum_{\alpha=1}^{\gamma+1}(\bar{a}_\alpha)^2 = \sum_{\alpha=1}^{m}(a_\alpha)^2 + \sum_{2\leq\alpha_1\neq\beta_1\leq n_1} a_{\alpha_1}a_{\beta_1} + \sum_{\alpha_2\neq\beta_2\in Q_2} a_{\alpha_2}a_{\beta_2} + \ldots + \sum_{\alpha_k\neq\beta_k\in Q_k} a_{\alpha_k}a_{\beta_k} \ ,$$

with $Q_1 = \{1,\ldots,n_1\}$, $Q_2 = \{n_1+1,\ldots,n_1+n_2\}$,..., $Q_k = \{n_1+\ldots+n_{k-1}+1,\ldots,n_1+\ldots+n_k\}$. Equation (4) becomes

$$\begin{aligned}
\widetilde{\varepsilon}\left(\sum_{\alpha=1}^{\gamma+1}\bar{a}_\alpha\right)^2 = \gamma\Bigg\{\phi + \widetilde{\varepsilon}\sum_{\alpha=1}^{\gamma+1}(\bar{a}_\alpha)^2 + \widetilde{\varepsilon}\sum_{\alpha\neq\beta=1}^{m}\varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2 \\
-\widetilde{\varepsilon}\sum_{2\leq\alpha_1\neq\beta_1\leq n_1} a_{\alpha_1}a_{\beta_1} - \ldots - \widetilde{\varepsilon}\sum_{\alpha_k\neq\beta_k\in Q_k} a_{\alpha_k}a_{\beta_k}\Bigg\}
\end{aligned} \tag{5}$$

We need the following algebraic lemma.

Lemma 1 ([4]) *If $\bar{a}_1, \ldots, \bar{a}_n, c$ are $n+1$ $(n \geq 2)$ real numbers such that*

$$\left(\sum_{i=1}^{n} \bar{a}_i\right)^2 = (n-1)\left(\sum_{i=1}^{n} (\bar{a}_i)^2 + c\right) ,$$

we have that $2\bar{a}_1\bar{a}_2 \geq c$ and equality holds if and only if $\bar{a}_1 + \bar{a}_2 = \bar{a}_3 = \ldots = \bar{a}_n$.

Two separate cases appear. We first look at the case when $\vec{H}$ is spacelike, i.e. $\widetilde{\varepsilon} = 1$. Using the above lemma, eqn.(5) becomes

$$\bar{a}_1\bar{a}_2 \geq \frac{1}{2}\phi + \frac{1}{2}\sum_{\alpha\neq\beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2 - \frac{1}{2}\sum_{2\leq\alpha_1\neq\beta_1\leq n_1} a_{\alpha_1}a_{\beta_1} - \ldots - \frac{1}{2}\sum_{\alpha_k\neq\beta_k\in Q_k} a_{\alpha_k}a_{\beta_k} .$$

Because

$$\sum_{\alpha_j\neq\beta_j} a_{\alpha_j}a_{\beta_j} = 2\sum_{\alpha_j<\beta_j} a_{\alpha_j}a_{\beta_j} ,$$

we have

$$\sum_{j=1}^{k}\sum_{\alpha_j<\beta_j\in Q_j} a_{\alpha_j}a_{\beta_j} \geq \frac{1}{2}\phi + \sum_{\alpha<\beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2 . \tag{6}$$

Let L_j be a n_j-dimensional subspace of $T_p\mathcal{M}$ such that

$$L_j = \operatorname{span}\{\vec{e}_{n_1+\ldots+n_{j-1}+1}, \ldots, \vec{e}_{n_1+\ldots+n_j}\} .$$

The scalar curvature of the plane section is given by

$$\tau(L_j) = \sum_{\alpha_j<\beta_j\in Q_j} \varepsilon_{\alpha_j\beta_j}\,\widetilde{\varepsilon}\left[\Omega_{\alpha_j\alpha_j}\Omega_{\beta_j\beta_j} - \left(\Omega_{\alpha_j\beta_j}\right)^2\right] .$$

Then, using the above notation, we find

$$\tau(L_1) + \ldots + \tau(L_k) = \sum_{j=1}^{k}\sum_{\alpha_j<\beta_j\in Q_j} a_{\alpha_j}a_{\beta_j} - \sum_{j=1}^{k}\sum_{\alpha_j<\beta_j\in Q_j} \varepsilon_{\alpha_j\beta_j}(\Omega_{\alpha_j\beta_j})^2 .$$

If we use the inequality (6) and the notation

$$\begin{aligned}
Q_{k+1} &= \{n_1 + \ldots + n_k + 1, \ldots, m\} ,\\
Q &= Q_1 \cup \ldots \cup Q_k \cup Q_{k+1} ,\\
Q^2 &= (Q_1 \times Q_1) \cup \ldots \cup (Q_k \times Q_k) \cup (Q_{k+1} \times Q_{k+1}) ,\\
\nabla^2 &= (Q \times Q)/Q^2 ,
\end{aligned}$$

we have

$$\tau(L_1) + \ldots + \tau(L_k) \geq \frac{1}{2}\phi + \frac{1}{2}\widetilde{\varepsilon} \sum_{(\alpha,\beta)\in\nabla^2} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2 \ . \tag{7}$$

The signature of the embedding space E_n is chosen to be $(m, 1)$ such that $\widetilde{\varepsilon} = 1$ and the condition of causal-type preserving ensures that the terms with possible minus signs appearing on the righthand side vanish. We have

$$\tau(L_1) + \ldots + \tau(L_k) \geq \frac{1}{2}\phi \ .$$

This holds for all mutually orthogonal subspaces L_j, in particular for the infimum,

$$\|H\|_\perp^2 \geq c(n_1, \ldots, n_k)\, \Lambda(n_1, \ldots, n_k) - \frac{1}{2}c(n_1, \ldots, n_k) \left\{ \sum_{\alpha=1}^{m} \varepsilon_\alpha \, \lambda_\alpha - \lambda_{m+1} \right\} \ . \tag{8}$$

The case when $\vec{H}$ is timelike is analogous and we find instead of (7),

$$\tau(L_1) + \ldots + \tau(L_k) \leq \frac{1}{2}\phi + \frac{1}{2}\widetilde{\varepsilon} \sum_{(\alpha,\beta)\in\nabla^2} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2 \ .$$

We choose the signature of the embedding space to be $(m - 1, 2)$, i.e., the normal direction is timelike. We find

$$\tau(L_1) + \ldots + \tau(L_k) \leq \frac{1}{2}\phi \ .$$

This holds again for all mutually orthogonal subspaces, in particular for the supremum,

$$\|H\|_\perp^2 \leq c(n_1, \ldots, n_k)\, \hat{\Lambda}(n_1, \ldots, n_k) - \frac{1}{2}c(n_1, \ldots, n_k) \left\{ \sum_{\alpha=1}^{m} \varepsilon_\alpha \, \lambda_\alpha - \lambda_{m+1} \right\} \ . \tag{9}$$

It remains to show the inequality when $k = 0$. Starting from (3) and again choosing e_{m+1} along $\vec{H}$ we find

$$2\tau = \widetilde{\varepsilon} m^2 \|H\|_\perp^2 - \widetilde{\varepsilon} \sum_{\alpha=1}^{m} (a_\alpha)^2 - \widetilde{\varepsilon} \sum_{\alpha\neq\beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2 + \sum_{\alpha=1}^{m} \varepsilon_\alpha \, \lambda_\alpha - \widetilde{\varepsilon}\, \lambda_{m+1} \ , \tag{10}$$

with $a_\alpha = \varepsilon_\alpha \Omega_{\alpha\alpha}$. We have

$$\begin{aligned}
\sum_{\alpha=1}^{m} (a_\alpha)^2 &= \left(\sum_{\alpha=1}^{m} a_\alpha\right)^2 - 2\sum_{\alpha<\beta=1}^{m} a_\alpha a_\beta \\
&= \widetilde{\varepsilon}\, m^2 \|H\|_\perp^2 + \sum_{\alpha<\beta=1}^{m} (a_\alpha - a_\beta)^2 - (m-1)\sum_{\alpha=1}^{m}(a_\alpha)^2 \\
m\sum_{\alpha=1}^{m} (a_\alpha)^2 &= \widetilde{\varepsilon}\, m^2 \|H\|_\perp^2 + \sum_{\alpha<\beta=1}^{m} (a_\alpha - a_\beta)^2 \\
&\geq \widetilde{\varepsilon}\, m^2 \|H\|_\perp^2 \ .
\end{aligned}$$

If $\vec{H}$ is spacelike, (10) with the above inequality becomes

$$2\tau \leq m(m-1)\|H\|_\perp^2 - \sum_{\alpha\neq\beta=1}^{m} \varepsilon_{\alpha\beta}(\Omega_{\alpha\beta})^2 + \sum_{\alpha=1}^{m} \varepsilon_\alpha\, \lambda_\alpha - \widetilde{\varepsilon}\, \lambda_{m+1} \ .$$

The signature of the embedding space is chosen to be $(m,1)$ and because of the condition of causal-type preserving, we find

$$\begin{aligned}
\|H\|_\perp^2 &\geq \frac{2}{m(m-1)}\tau - \frac{1}{m(m-1)}\left(\sum_{\alpha=1}^{m} \varepsilon_\alpha \lambda_\alpha - \lambda_{m+1}\right) \\
&\geq \Lambda(0) - \frac{1}{m(m-1)}\left(\sum_{\alpha=1}^{m} \varepsilon_\alpha \lambda_\alpha - \lambda_{m+1}\right) \ . \qquad (11)
\end{aligned}$$

The proof for the timelike case is similar. $\square$

Corollary 1 *Equality holds in (1) or (2) if and only if the second fundamental form has the following form with respect to the eigenframe of the Ricci tensor $\widetilde{S}$,*

$$(\Omega_{\alpha\beta}) = \begin{pmatrix} A_{n_1} & & 0 & 0 \\ & \ddots & & \\ 0 & & A_{n_k} & \\ 0 & & & \mu I_s \end{pmatrix} ,$$

with $s = m - \sum_{j=1}^{k} n_j$, A_{n_j} is a symmetric $n_j \times n_j$-matrix with $Tr(A_{n_j}) = \mu$.

Definition 4 *If the equality is satisfied in (1), the mean curvature is minimal, i.e., the hypersurface receives the least amount of tension from the surrouding space, and we call the embedding ideal. We apply the same name if equality holds in (2), although the situation there is far from ideal.*

3. APPLICATIONS IN GENERAL RELATIVITY

3.A. WHY MORE THAN FOUR DIMENSIONS?

In 1916 Einstein and Hilbert independently constructed the equations for the pure gravitational field [19, 24],

$$G_{\alpha\beta} := S_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} S = \kappa T_{\alpha\beta} \ ,$$

with S the trace of the Ricci tensor, $T_{\alpha\beta}$ the energy-momentum tensor and κ some dimension-transposing parameter.

The remaining part of his scientific career Eistein searched for an unification between his description of gravity and the other known forces, in particular electromagnetism [13]. Several attempts were made by using e.g., a non-symmetric metric, a connection with torsion, etc.

In 1921 Kaluza proposed to unify gravity and electromagnetism into a theory of five dimensions. To avoid the question why no fifth dimension was ever observed, Kaluza demanded that all derivatives with respect to the fifth coordinate would vanish. In other words, physics was to take place on a four-dimensional hypersurface in a five-dimensional universe (= Kaluza's cylinder condition). With this assumption one is able to obtain the field equations of both gravity and electromagnetism from a single five-dimensional theory, i.e., from ${}^5G_{ab} = 0$, with $a, b = 0, \ldots, 4$, we find ${}^4G_{\alpha\beta} = \kappa T^{EM}_{\alpha\beta}$, $\alpha, \beta = 0, \ldots, 3$, together with Maxwell's laws.

Klein in 1926 proposed to compactify the fifth dimension, i.e.,

1) one assumes a circular topology in which case physical fields would depend periodically on the fifth coordinate and could be Fourier expanded,
2) assume a small enough scale (= compactified) in which case only the zero mode in the Fourier expansion is physically interesting.

Although the theory was later abandonned because it gives the wrong mass for the electron with discrepancy of 22-orders of magnitude, the ideas of Kaluza and

Klein came to dominate later attempts at unification in physics leading to the eleven-dimensional superstring theories and the recent membrane theory [20].

An alternative to the compactified approach is to take the extra coordinate at face value, i.e., to follow the example of Minkowski's (1909) unification of time and space, and assume that the extra dimension, like time, is not lengthlike. In this case the explanation why we observe space-time as being four-dimensional is to be found in the physical interpretation of the extra coordinate, i.e., in the values of the dimension-transposing parameters (like c) needed to give it unit length. For example, in the space-time-mass theory of Wesson [26] it was suggested that a fifth dimension might be associated with rest mass of a particle via $x^4 = \frac{Gm}{c^2}$.

In these non-compactified Kaluza-Klein models physics is allowed to depend on the extra coordinates. The general theory, in which any part of the metric can depend on the fifth coordinate has recently been explored by Wesson and others [18, 27]. One starts from the five-dimensional vacuum field equations ${}^5G_{ab} = 0$ and obtains four-dimensional *general relativity*, ${}^4G_{\alpha\beta} = \kappa\, {}^4T_{\alpha\beta}$, with a general four-dimensional energy-momentum tensor constructed from the terms containing the fifth coordinate.

In this way, we arrive back at an old idea of Einstein, namely geometrizing matter. Matter is induced on the four-dimensional space-time by the properties of the fifth dimension.

The mathematical justification of this model is given by the Campbell-Magaard theorem [1] which states that every (semi-)Riemannian manifold with analytic metric can be locally and isometrically embedded as a hypersurface in a Ricci flat space. Recently, generalisations for embeddings into spaces with non-degenerate Ricci tensor were found [6].

Of course several problems remain. For example, there are several ways to embed a given four-dimensional space-time in a five-dimensional Ricci-flat manifold and vice versa, given a five-dimensional Ricci-flat manifold we can extract different four-dimensional space-times. The theory gives no criterion how to choose a particular embedding. In classical general relativity there exists a analogous problem. Namely, how to choose a particular solution of the field equations from the infinite solution set.

Using methods of submanifold theory we can give criterions to look for certain special solutions. For example, Pavšić [21, 22] and Chervon, Dahia and Romero [5] consider the space-time as a surface with $\vec{H} = 0$ in a higher-dimensional embedding space. As an extension, the above mentioned inequalities leading to the notion of ideal embeddings, in which the submanifold receives the least amount of tension from the surrounding space, are mathematically meaningfull criterions with a *physical* interpretation.

We can thus apply Theorem 1 on two different levels:

1) look within a class of solutions of the Einstein-Hilbert field equations in 4D for those in which the three-dimensional space is ideally embedded,
2) look for those space-times which are ideally embedded in a five-dimensional Ricci-flat space.

In [16] we considered ideally embedded space-times in a five-dimensional pseudo-Euclidean space. In the following we will present some examples of four-dimensional perfect fluid Bianchi models for which the three-dimensional spacelike hypersurface on which the group acts transitively, is ideally embedded.

3.B. PERFECT FLUID BIANCHI A MODELS

A Bianchi model is a space-time whose metric admits a three-dimensional group of isometries acting simply transitively on spacelike hypersurfaces [9, 11]. Bianchi cosmologies thus admit a Lie algebra of Killing vector fields with basis ζ_i, $i = 1, 2, 3$ and structure constants C^k_{ij},

$$[\zeta_i, \zeta_j] = C^k_{ij}\zeta_k \ .$$

The Killing vector fields ζ_i are tangent to the group orbits, i.e., the spacelike hypersurfaces.

The Bianchi cosmologies can be classified by classifying the Lie algebra of Killing vector fields. The problem then becomes that of classifying the structure constants

C^k_{ij}, which transform as a rank (1,2)-tensor under a change of basis of the Lie algebra and satisfy the Jacobi identities,

$$C^l_{s[i}C^s_{jk]} = 0 \ .$$

One can decompose the structure constants as

$$C^k_{ij} = \varepsilon_{ijl}n^{kl} + a_i\delta^k_j - a_j\delta^k_i \ ,$$

with $n^{ij} = n^{ji}$ and a_i constants. The Jacobi identities become

$$n^{ij}a_i = 0 \ .$$

Going over to the eigenframe of n^{ij}, with $a_1 \neq 0$, we can set

$$(n^{ij}) = \mathrm{diag}(n_1, n_2, n_3) \quad , \quad (a_i) = (a, 0, 0) \ .$$

There exists the following classification of the Bianchi cosmologies into ten groups using the eigenvalues of n^{ij}:

Group class	Group type	n_1	n_2	n_3
	I	0	0	0
	II	+	0	0
A ($a = 0$)	VI_0	0	+	-
	VII_0	0	+	+
	VIII	-	+	+
	IX	+	+	+
	V	0	0	0
B ($a \neq 0$)	IV	0	0	+
	VI_h	0	+	-
	VII_h	0	+	+

The constant h is defined by $a^2 = hn_2n_3$ if $n_2n_3 \neq 0$.

In the following we will consider Bianchi A non-tilted perfect fluid models, i.e., the four-velocity of the fluid is orthogonal to the group orbits, such that the spacelike hypersurface on which the group acts transitively, is ideally embedded in space-time. From Corollary 1 we have that the second fundamental form is either

$$\Omega_{\alpha\beta} = \phi g_{\alpha\beta} \ ,$$

i.e., the embedding is umbilical, or

$$\Omega_{\alpha\beta} = (\phi - \psi)e_{1\alpha}e_{1\beta} + \psi e_{2\alpha}e_{2\beta} + 2\nu e_{1\alpha}e_{2\beta} + \phi e_{3\alpha}e_{3\beta} \ . \tag{12}$$

In the umbilical case we find the following possible Bianchi models with ideally embedded spacelike hypersurface:

1) A conformally flat perfect fluid Bianchi I model,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (3\gamma t - c)^{4/3\gamma}\{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\} \ ,$$

with $\gamma \neq 0$. The constant γ is defined through the equation of state $p = (\gamma - 1)\mu$. The energy density of this metric is $\mu = \frac{12}{(3\gamma t - c)^2}$ with c an integration constant.

2) A conformally flat Einstein Bianchi I model,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{2Ht}\{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\} \ ,$$

with $S_{\alpha\beta} = -3H^2 g_{\alpha\beta}$, H a constant.

3) A conformally flat perfect fluid Bianchi IX model,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + e^{-2f(t)}\{\mathrm{d}x^2 + \mathrm{d}y^2 + 2\cos(x)\mathrm{d}y\mathrm{d}z + \mathrm{d}z^2\} \ , \tag{13}$$

with $f(t)$ a solution of

$$\ddot{f}(t) = \frac{3}{2}\gamma\dot{f}(t)^2 + \frac{1}{8}(3\gamma - 2)e^{2f(t)} \ .$$

The energy density is $\mu = 3(\dot{f})^2 + \frac{3}{4}e^{2f}$.

If the second fundamental form satisfies (12) we find the following Bianchi models:

1) A Kasner-type Bianchi I model,

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{\frac{3-c}{6}}\mathrm{d}x^2 + t^{\frac{3+c}{6}}\mathrm{d}y^2 + t\mathrm{d}z^2 \ ,$$

with $(\gamma - 2)\mu = 0$, $\mu = \frac{1}{4t^2}(45 - c^2)$ and c an integration constant. Remark that if $c = \sqrt{45}$, the model becomes the vacuum Kasner space-time.

2) Two Jacobs *stiff* ($\gamma = 2$) perfect fluid Bianchi I models [17],

$$ds^2 = -dt^2 + \sqrt{c-6t}\{dx^2 + dy^2\} + (c-6t)dz^2 \ ,$$

and

$$ds^2 = -dt^2 + dx^2 + (c-6t)\{dy^2 + dz^2\} \ .$$

3) The Ellis-MacCallum *dust* ($\gamma = 1$) perfect fluid Bianchi VI_0 model [10],

$$ds^2 = -dt^2 + t\{e^z dx^2 + e^{-z} dy^2\} + t^2 dz^2 \ ,$$

with $\mu = t^{-2}$ and $p = 0$.

4) A conformally flat perfect fluid Bianchi IX model, analogous as (13), with $f(t) = 0$, $\gamma = \frac{2}{3}$ and $\mu = \frac{3}{4}$.

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