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GEODESIC POLYHEDRA AND NETS[1]

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Abstract. Geodesics have a fundamental role in geometry of curved surfaces and manifolds, as well as in discrete geometry. We are going to expand some known facts about geodesics in regular differential geometry to the discrete geometry. We present a discrete analogy of the smooth surfaces parameterized by geodesics. The goal of our consideration is the definition of discrete surfaces generated by geodesics, studying of their properties and finding the algorithm for the generation of these surfaces.

1. INTRODUCTION

Geodesics on smooth surfaces generalize the idea of straight lines. They are the straightest and locally shortest curves. The concept of discrete geodesics differ from smooth one in that view. It is well–known that they cannot have both of these previously mentioned properties. Geodesics on polyhedral surfaces has been intensively studied. The authors in [1] define geodesics on polyhedral surfaces as locally shortest curves. Beside their important role in the study of (non–)regular differential geometry, shortest geodesics cannot be extended as locally shortest curves across spherical

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polyhedral vertex and, therefore, fail to solve the initial value problem for geodesics at the polyhedral vertices. The authors in [9] and [10] define polyhedral geodesics as straightest geodesics and that concept we shall use in our consideration.

In Section 2, we are going to give a review of polyhedral surfaces, straightest geodesics and discrete nets. In Section 3, we define a new class of polyhedral surfaces which we shall name *geodesic* (or G–)*polyhedra*. Then, we give some closer picture of them by determining theirs properties and giving several examples. In Section 4, we define G*–nets* (by the same analogy with the G–polyhedra). A reason for that is the possibility of giving some algorithm for the generation of G–nets. At the end, we present a method for *geodesation* of arbitrary discrete net. This method is based on numerical computation and theory of minimization of differentiable functions.

2. PRELIMINARIES

We recall basic definitions and statements which we use in further.

Definition 1. *A* ***polyhedral surface*** $\mathcal{P}$ *is a two–dimensional manifold consisting of finite or denumerable set* F *of topological triangles and intrinsic metric* ρ *such that*

1. *Any point* $p \in \mathcal{P}$ *lies in at least one triangle* $f \in F$.
2. *Each point* $p \in \mathcal{P}$ *has a neighborhood that intersect only finitely many triangles* $f \in F$.
3. *The intersection of any of two non–identical triangles* $f, g \in F$ *is either empty, or consists of a common vertex, or of a simple arc that is an edge of each of the two triangles.*
4. *The intrinsic metric* ρ *is flat on each triangle, i. e. each triangle is isometric to a triangle in* $\boldsymbol{R}^2$.

Remark 1. *The topological triangle* f *in a two–dimensional manifold* $\mathcal{M}$ *is a simple domain* $f \subset \mathcal{M}$ *whose boundary is split by three vertices into three edges with no common interior points. For simplification, we restrict our attention to polyhedral* *surfaces consisting of planar triangles. A class of polyhedra, also, consists of discrete surfaces which faces are not triangles. For example, cube is a polyhedral surface (in sense of previous definition) because each of its faces can be divided on two triangles by diagonal.*

Each polyhedral edge, incident to exactly one face, we call **boundary edge**. Each vertex, incident to boundary edge, is **boundary vertex** and all other vertices are **inner vertices**.

Definition 2. *Let $\boldsymbol{\alpha} \in \mathcal{P}$ be a polyhedral curve whose segments on each face are rectifiable. Then, the **length** of $\boldsymbol{\alpha}$ is given by*

$$l(\boldsymbol{\alpha}) = \sum_{f \in F} l(\boldsymbol{\alpha}|_f).$$

Definition 3. *Let $\mathcal{P}$ be a polyhedral surface and $v \in \mathcal{P}$ a vertex. Let G be the set of faces containing v as a vertex, and θ_i be the interior angle of the face $f_i \in G$ at the vertex v. Then, **total vertex angle** $\theta(v)$ is given by*

$$\theta(v) = \sum_{f_i \in G} \theta_i(v).$$

Interior points p of face or of an open edge have a neighborhood which is isometric to a planar euclidean domain and, in that case, we define $\theta(p) = 2\pi$.

The existence of (polyhedral) faces will limit us in further consideration. So, we shall, also, deal with discrete nets (which consist only of vertices and edges). First of all, we need the notion of **poly–graph**. Poly–graph consists of vertices and (straight) edges which bound at most denumerable set of simple polygons (see Figure 1).

Definition 4. *Let G be (connected) poly–graph. A **discrete net** is a map*

$$\mathcal{S} : G \mapsto \boldsymbol{R}^3.$$

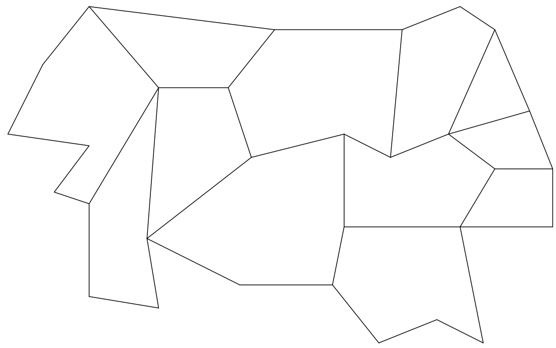

Figure 1. Poly–graph.

Each (connected) planar graph determines the partition of plane into the set of open regions and exactly one of them is unbounded. That region we call outer region. All graph vertices that belong to the border of outer region are outer vertices and all of them are mapped by $\mathcal{S}$ into **outer vertices** of discrete net. All other discrete net vertices are **inner**.

Definition 5. *A **degree** of (polyhedral or net) vertex is a number of edges incident to that vertex.*

We shall deal only with polyhedral surfaces or discrete nets whose all inner vertices have even degree. In that case, we can naturally define opposite edges, angles and (in case of polyhedra) faces (compare Figure 2).

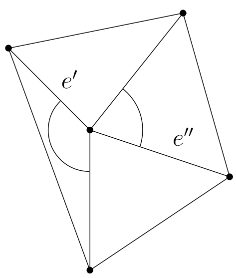

Figure 2. Opposite edges, angles and faces at inner vertex of even degree.

Definition 6. *Let $\mathcal{P}$ be a polyhedral surface and $\boldsymbol{\gamma} \subset \mathcal{P}$ a curve. Then $\boldsymbol{\gamma}$ is a*

straightest geodesic *on* $\mathcal{P}$ *if for each point* $p \in \boldsymbol{\gamma}$ *the left and right curve angles* θ_l *and* θ_r *at point* p *are equal (compare Figure 3).*

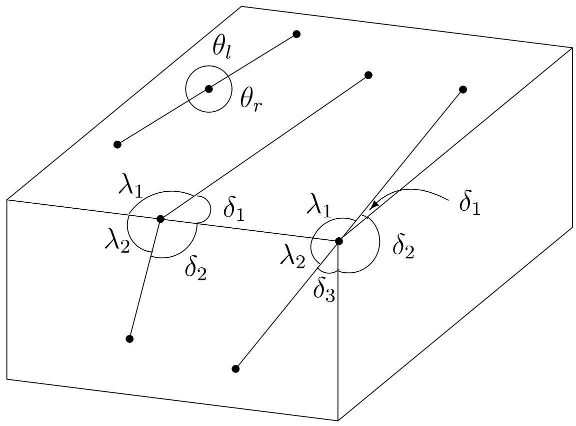

Figure 3. Left $\theta_l = \sum \lambda_i$ and right $\theta_r = \sum \delta_j$ curve angles. If $\theta_l = \theta_r$, curve is straightest geodesic.

Definition 7. *Let* $\boldsymbol{\gamma}$ *be a curve on a polyhedral surface* $\mathcal{P}$. *Let* θ *be the total vertex angle and* φ *one of two curve angles of* $\boldsymbol{\gamma}$ *at* p. *Then the* ***discrete geodesic curvature*** *of* $\boldsymbol{\gamma}$ *at* p *is given by*

$$k_g = \frac{2\pi}{\theta}\left(\frac{\theta}{2} - \varphi\right).$$

If we choose the other curve angle, geodesic curvature k_g *will change the sign.*

For more details about straightest geodesics one can see [9] and [10].

3. G–POLYHEDRA

We use Definition 6. to give a simple definition of polyhedra generated by geodesics. Before that, we need notion of polyhedral edges angle.

Definition 8. *Left and right angle between two edges incident to the same polyhedral vertex* v *are angles respectively equal to the left and right angle, at the vertex* v, *of curve which contains those edges.*

Definition 9. ***Geodesic*** *(or G–)* ***polyhedral surface*** *is a polyhedral surface whose left and right angles between pairs of opposite edges are equal.*

Obviously, the curve which coincides with the sequence of opposite polyhedral edges is geodesic.

Property 1. *Opposite angles of G–polyhedron are equal.*

Property 2. *If some inner polyhedral vertex has degree $2k$, there are $2k$ discrete geodesics which start at that vertex and consists only of polyhedral edges.*

We can give several simple examples to illustrate previously mentioned statements.

Example 1. Trivial example of G–polyhedron is octahedron (see Figure 4). Then, there are two G–polyhedra. The degree of all inner vertices is four, in the first case, while the degree of the only inner vertex is six, in the second one. The last example represents a rotational G–polyhedron.

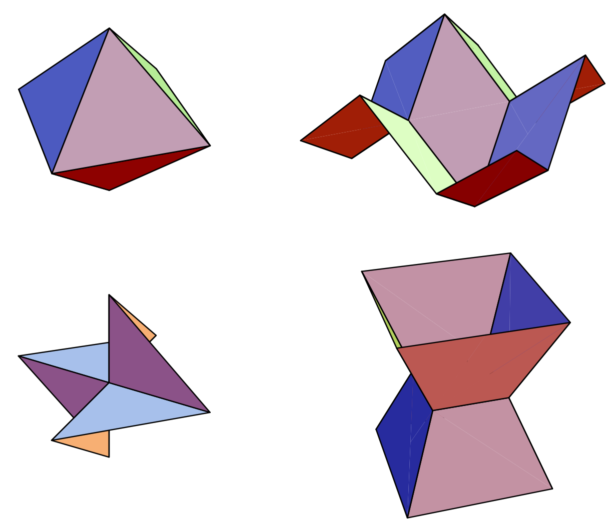

Figure 4. Examples of G–polyhedra.

4. G–NETS

In the previous section we presented a several simple G-polyhedra, which we obtained by using their basic properties. A natural question would be: is there any method for obtaining a complicated geodesic polyhedral surfaces? On the other hand, the existence of (polyhedral) faces includes limiting factor in trying to obtain algorithm which would generate G–polyhedra. Therefore, in this section, we shall deal with G–nets.

4.1. BASIC PROPERTIES AND GENERATION OF G–NETS

Definition 10. *If opposite angles at each inner vertex (which has even degree) of discrete net are equal, then we say that this net is* ***geodesic*** *(or G–)* ***net****.*

Next property is an immediate consequence of the previous definition.

Property 3. *Each conformal map of ambient space $\boldsymbol{R}^3$ preserves G–nets.*

From now–on we consider to G–nets whose all inner vertices are four–degrees. Generalization of further results on arbitrary G–nets (with even vertex degree) is simple.

Vertices of four degree G–net we denote by $p_{i,j}$, $i, j = 0, 1, 2, ...$, such that $p_{i-1,j}$, $p_{i+1,j}$, $p_{i,j-1}$, $p_{i,j+1}$ are vertices adjacent to $p_{i,j}$, where the first and the second pair are incident to opposite edges.

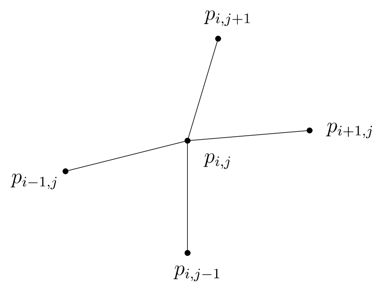

Figure 5. Vertices of four–degree G–net.

Property 4. *If $\mathcal{S}$ is four–degree G–net, then for each inner vertex $p_{i,j}$ we have*

$$\left(\frac{\mathbf{p_{i,j}p_{i-1,j}}}{\mid \mathbf{p_{i,j}p_{i-1,j}} \mid} + \frac{\mathbf{p_{i,j}p_{i+1,j}}}{\mid \mathbf{p_{i,j}p_{i+1,j}} \mid}\right) \cdot \left(\frac{\mathbf{p_{i,j}p_{i,j-1}}}{\mid \mathbf{p_{i,j}p_{i,j-1}} \mid} - \frac{\mathbf{p_{i,j}p_{i,j+1}}}{\mid \mathbf{p_{i,j}p_{i,j+1}} \mid}\right) = 0$$

and

$$\left(\frac{\mathbf{p_{i,j}p_{i-1,j}}}{\mid \mathbf{p_{i,j}p_{i-1,j}} \mid} + \frac{\mathbf{p_{i,j}p_{i+1,j}}}{\mid \mathbf{p_{i,j}p_{i+1,j}} \mid}\right) \cdot \left(\frac{\mathbf{p_{i,j}p_{i,j-1}}}{\mid \mathbf{p_{i,j}p_{i,j-1}} \mid} - \frac{\mathbf{p_{i,j}p_{i,j+1}}}{\mid \mathbf{p_{i,j}p_{i,j+1}} \mid}\right) = 0.$$

Property 5. *The angles determined by the opposite edges (considered in ambient space $\boldsymbol{R}^3$) have the same bisector.*

By using Property 5. we give an algorithm for generation of G–nets.

Algorithm: For given vertices $p_{i,j-1}$, $p_{i-1,j}$, $p_{i+1,j}$ and $p_{i,j}$ we distinguish two cases:

1. $\angle p_{i-1,j}p_{i,j}p_{i+1,j} = \pi$. Then, new vertex $p_{i,j+1}$ is, up to the length of edge $p_{i,j}p_{i,j+1}$, determined by condition that belongs to the ray which is symmetrical to ray $p_{i-1,j}p_{i,j}$ in respect to point $p_{i,j}$.

2. $\angle p_{i-1,j}p_{i,j}p_{i+1,j} \neq \pi$. Then, new vertex $p_{i,j+1}$ is, up to the length of edge $p_{i,j}p_{i,j+1}$, determined by condition that belongs to the ray which is symmetrical to ray $p_{i-1,j}p_{i,j}$ in respect to bisector of angle $\angle p_{i-1,j}p_{i,j}p_{i+1,j}$.

Example 2. Figure 6 shows the part of G–net and their extension which we have made by using the previous algorithm.

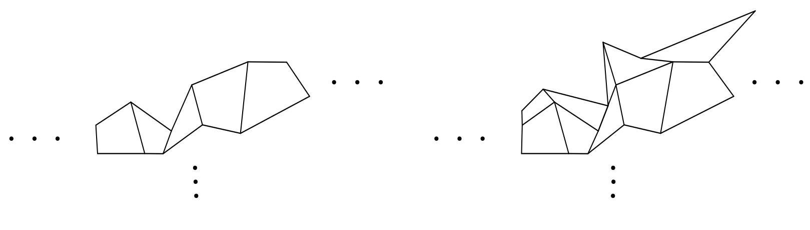

Figure 6. (a) Part of G–net. (b) Extension of G–net.

Let us consider the problem of obtaining G–net from an arbitrary discrete net.

If $f : \boldsymbol{R}^n \mapsto \boldsymbol{R}$ is differentiable function, consider the sequence $\{x_k\}$, $k = 0, 1, 2, ...,$ of points $x_k \in \boldsymbol{R}^n$, generated by formula $x_{k+1} = x_k + h_k \boldsymbol{s}_k$, $k = 0, 1, 2, ...,$ where point x_k moves to point x_{k+1} with the direction that is determined by vector $\boldsymbol{s}_k$, and a positive number h_k is the stepsize of $(k+1)$ iteration. By different choices of vector $\boldsymbol{s}_k$ we obtain different methods for minimization of differentiable functions and all of them satisfy condition:

$$\nabla f(x_k) \cdot \boldsymbol{s}_k \leq -\rho ||\nabla f(x_k)|| ||\boldsymbol{s}_k||, \quad \rho > 0,$$

which guarantees the decreasing of function f. If we choose $\boldsymbol{s}_k = -\nabla f(x_k)$, we get the **gradient method** which we shall use in further.

The most important properties of methods for minimization of differentiable functions are given in the following theorem (this is a preformulation of known facts that one can find in [4], [6] and [12]).

Theorem 1. *Let $f \in C^1(\boldsymbol{R}^n)$ and let method generate sequence $\{x_k\}$, $k = 0, 1, 2, ...$ Next conditions hold:*

1. *Sequence $f(x_k)$, $k = 0, 1, 2, ...,$ is decreasing.*
2. *For some $m \in \boldsymbol{N}$ it is fulfilled $\nabla f(x_m) = 0$ or each accumulating point $\widetilde{x}$ fulfills the condition $\nabla f(\widetilde{x}) = 0$.*
3. *If f is a convex function, then it has minimum at each accumulating point $\widetilde{x}$.*
4. *If f is a strictly convex function, sequence $\{x_k\}$ converges to the unique point of minimum.*

Let $\mathcal{S}$ be four–degree G–net. The set of inner vertices we shall denote by M_0. Let

$$\alpha_1^{p_{i,j}} \quad \text{and} \quad \alpha_0^{p_{i,j}}, \quad \text{i. e.} \quad \beta_1^{p_{i,j}} \quad \text{and} \quad \beta_0^{p_{i,j}}$$

be pairs of opposite angles at vertex $p_{i,j}$. We shall consider the problem of minimization of function

$$E(M_0) = \sum_{p_{i,j} \in M_0} ((\alpha_1^{p_{i,j}} - \alpha_0^{p_{i,j}})^2 + (\beta_1^{p_{i,j}} - \beta_0^{p_{i,j}})^2). \tag{1}$$

For the application of gradient method we need the gradient of angle. Let $\triangle pqr$ be arbitrary triangle with vertices $p(x,y,z)$, $q(q_1,q_2,q_3)$, $r(r_1,r_2,r_3)$. When we differentiate function

$$\alpha(x,y,z) = \arccos \frac{\mathbf{pq} \cdot \mathbf{pr}}{|\mathbf{pq}||\mathbf{pr}|},$$

by variables x, y and z, we obtain

$$\nabla_p \alpha = \frac{\sin \alpha^2}{\triangle} \mathbf{po},$$

where $\triangle$ is the area of triangle $\triangle pqr$ and point o is a circumcentre of the same triangle. Geometrical interpretation of the fact that gradient of angle and $\mathbf{po}$ are linearly dependent vectors runs like this: Gradient of angle determines the direction of moving vertex p so that angle α has the fastest increasing. Let us consider a sphere whose great circle is the circumcircle of triangle $\triangle pqr$. Choose coordinate system such that z–axis is determined by vector $\mathbf{po}$ and other two axes belong to the tangent plane at point p. Then, the gradient of angle α we can write as

$$\nabla_p \alpha = a\mathbf{x} + b\mathbf{y} + c\mathbf{z}.$$

But the linear combination of vectors $\mathbf{x}$ and $\mathbf{y}$ determine direction in which angle α decreases. Therefore, the fastest increasing we get for $a = b = 0$.

The gradient of function (1) is given by

$$\begin{aligned} \nabla_{M_0} E(M_0) &= \nabla_{M_0} \textstyle\sum_{p_{i,j}} ((\alpha_1^{p_{i,j}} - \alpha_0^{p_{i,j}})^2 + (\beta_1^{p_{i,j}} - \beta_0^{p_{i,j}})^2) \\ &= \textstyle\sum_{p_{i,j}} 2((\alpha_1^{p_{i,j}} - \alpha_0^{p_{i,j}})(\nabla_{p_{i,j}} \alpha_1^{p_{i,j}} - \nabla_{p_{i,j}} \alpha_0^{p_{i,j}}) \\ &\qquad + (\beta_1^{p_{i,j}} - \beta_0^{p_{i,j}})(\nabla_{p_{i,j}} \beta_1^{p_{i,j}} - \nabla_{p_{i,j}} \beta_0^{p_{i,j}})) \end{aligned}$$

and iterative process we write as

$$M_{k+1} = M_k - h_k \nabla_{M_k} E(M_k)$$

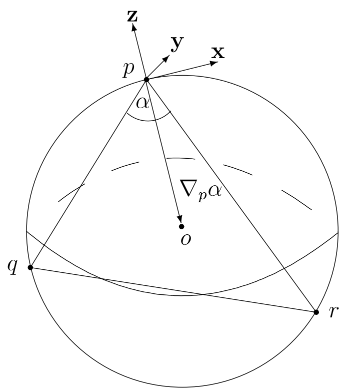

Figure 7. The gradient of angle.

where M_k is the set of inner vertices obtained in iteration k. Theorem 1. guarantees inequality

$$E(M_{k+1}) \leq E(M_k).$$

Clearly, if for some $m \in \boldsymbol{N}$ holds $E(M_m) = 0$, the obtained net is G–net. Generally, this should not happen. Namely, condition $\nabla E(M_n) = 0$ for some $n \in \boldsymbol{N}$ does not imply $E(M_n) = 0$. Most often, after a number of iterations, function E becomes near to zero. This just described iterative process we call the **geodesation process** of discrete net.

Example 3. Let $\mathcal{S}$ be discrete net with one inner vertex $p(1,0,1)$ and outer vertices $a(-1,-1,0)$, $b(1,-1,0)$, $c(1,1,0)$ and $d(-1,1,0)$. When we apply the geodesation process with different choices of stepsize, point p converges to different points

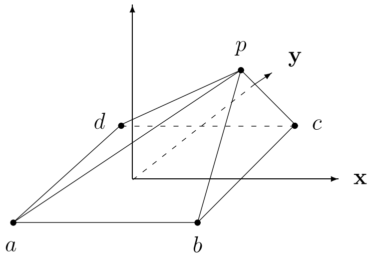

Figure 8. The geodesation process of discrete net with one inner vertex.

of minimum, which means that we do not have a unique resulting G–net. Namely, function E has the minimum at each point of z–axis.

The distance between points $p_{i,j}^{(k)}$ and $p_{i,j}^{(k+1)}$ obtained in consecutive iterations equals $h_k \|\nabla_{p_{i,j}^{(k)}} E(p_{i,j}^{(k)})\|$ and it is bounded only if the area of triangle determined by three corresponding vertices does not converge to zero. In that case, distance can be arbitrarily small if we use small enough stepsize.

Example 4. Figure 9 (a) shows discrete net determined by finite number of meridians and parallels of cylinder. Obviously, that net is G–net. Figure 9 (b) represents different net generated by other set of cylindric curves. When we apply geodesation process on that net, we get nets presented on Figure 10.

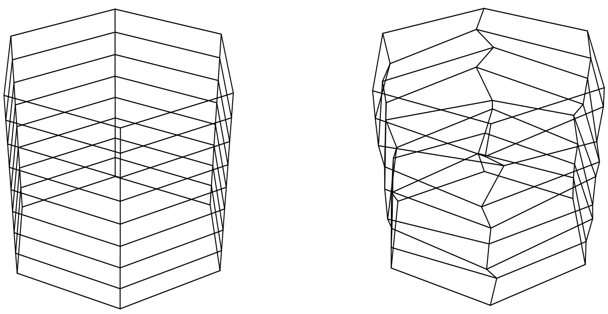

Figure 9. (a) G–net generated by meridians and parallels of cylinder. (b) Discrete net with the set of inner vertices M_0. Holds $E(M_0) \approx 96.7041$.

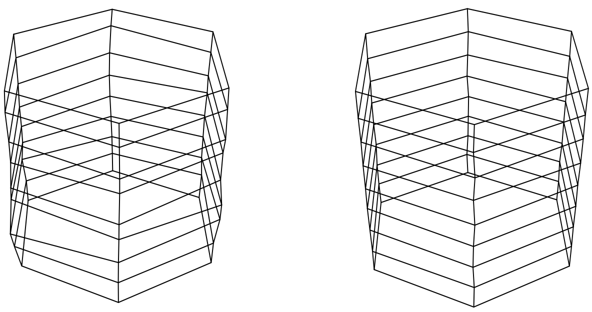

Figure 10. (a) G–net obtained after 1000 iterations with stepsize $h = 5 \cdot 10^{-6}$. Holds $E(M_{1000}) \approx 10.3960$. (b) G–net obtained after 5000 iterations with stepsize $h = 5 \cdot 10^{-6}$. Holds $E(M_{5000}) \approx 0.8277$.

Example 5. Geodesation process can be applied on each polyhedral surface whose faces are triangles. By using that process, we obtain a little bit complicated (quasi) G–polyhedron than the ones in Example 1.

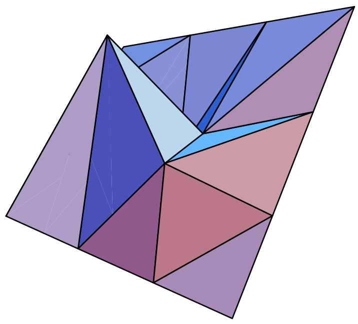

Figure 11. Polyhedral surface with four inner vertices. Holds $E(M_0) \approx 20.0930$.

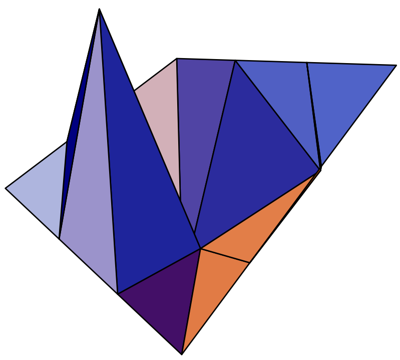

Figure 12. Quasi G–polyhedron obtained from the previous one.

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References

[1] A. D. Aleksandrov, V. A. Zalgaller, *Intrinsic Geometry of Surfaces*, Volume **15** of *Translation of Mathematical Monographs*, AMS, (1967).

[2] A. Bobenko, U. Pincall, *Discretization of Surfaces and Integrable Systems*, Calarendon Press, Oxford (1999).

[3] M. P. D. Carmo, *Differential Geometry of Curves and Surfaces*, Prentice–Hall (1976).

[4] B. P. Demidovich, I. A. Maron, *Computational Mathematics*, Mir Publishers, Moscow (1987).

[5] A. Gray, *Modern Differential Geometry of Curves and Surfaces with Mathematica*, CRC Press, Boca Raton (1997).

[6] F. S. Hillier, G. J. Liberman, *Introduction to Operations Research*, Holden–Day, San Francisco (1974).

[7] J. S. B. Mitchell, D. M. Mount, C. H. Papadimitriou, *The Discrete Geodesic Problem*, SIAM J. Comput. **16 (4)** (1987), 647–668.

[8] A. V. Pogorelov, *Quasigeodesic Lines on a Convex Surface*, Amer. Math. Soc., **6 (72)** (1952), 430–473.

[9] K. Polthier, M. Schmies, *Geodesic Flow on Polyhedral Surfaces*, Data Visualization, Springer–Verlag (1999).

[10] K. Polthier, M. Schmies, *Straightest Geodesics on Polyhedral Surfaces*, Mathematical Visualization pp. 135–150, Springer–Verlag, Heidelberg (1998).

[11] M. Sharir, A. Schorr, *On Shortest Paths in Polyhedral Space*, SIAM J. Comput. **15 (1)** (1986), 193–215.

[12] G. R. Walsh, *Methods of Optimization*, John Wiley and Sons, London (1975).