Fuzzy Pairwise Almost Strong Precontinuity

Biljana Krsteska

St. Cyril and Methodius University, Faculty of Natural Sciences and Mathematics, Institute of Mathematics, P.O. Box 162, Skopje, R. Macedonia
(e-mail: biljanak@iunona.pmf.ukim.edu.mk)

Abstract. The concept of fuzzy pairwise almost strongly precontinuous mappings has been introduced and studied. Their properties and relationships with other classes of early defined weaker forms of fuzzy pairwise continuous mappings has been investigated.

1. Introduction

The concept of fuzzy set was introduced by Zadeh in his classic paper [8]. Chang [2] first introduced the fuzzy topological spaces by using the fuzzy sets. Kandil [3] introduced and studied the notion of fuzzy bitopological spaces as a natural generalization of fuzzy topological spaces. Recently, Kumar [6,7] defined the $(\tau_i, \tau_j)$ – fuzzy semiopen (semiclosed) sets, $(\tau_i, \tau_j)$ – fuzzy preopen (preclosed) sets and $(\tau_i, \tau_j)$ – fuzzy strongly semiopen (semiclosed) sets. Continuing his work, Kumar [6,7] defined the fuzzy pairwise semicontinuous mappings, fuzzy pairwise precontinuous, fuzzy pairwise strongly semicontinuous mappings. The author [4] defined the concept of fuzzy pairwise strongly precontinuous mappings.
In this paper, we will define the concept of fuzzy pairwise almost strongly pre-continuous mappings. We will establish their properties and relationships with other classes of early defined weaker forms of fuzzy pairwise continuous mappings.

2. PRELIMINARIES

A triple $(X, \tau_1, \tau_2)$ consisting of a nonempty set $X$ with two fuzzy topologies $\tau_1$ and $\tau_2$ on $X$ is called a fuzzy bitopological spaces, shortly fbts. Throughout this paper, the indices $i$ and $j$ take values in $\{1, 2\}$ and $i \neq j$. For a fuzzy set $A$ of a fbts $(X, \tau_1, \tau_2)$, $\tau_i - \text{int} A$ and $\tau_j - \text{cl} A$ means, respectively, the interior and closure of $A$ with respect to the fuzzy topologies $\tau_i$ and $\tau_j$.

**Definition 2.1** [4, 6, 7] Let $A$ be a fuzzy set of a fbts $X$. Then $A$ is called

1. a $(\tau_i, \tau_j)$ – fuzzy semiopen set if and only if there exists $\tau_i$ – fuzzy open set $U$ such that $U \leq A \leq \tau_j - \text{cl} U$;
2. a $(\tau_i, \tau_j)$ – fuzzy preopen set if and only if $A \leq \tau_i - \text{int}(\tau_j - \text{cl} A)$;
3. a $(\tau_i, \tau_j)$ – fuzzy strongly semiopen set if and only if $A \leq \tau_i - \text{int}(\tau_j - \text{cl} (\tau_i - \text{int} A))$;
4. a $(\tau_i, \tau_j)$ – fuzzy semipreopen set if and only if $A \leq \tau_i - \text{cl} (\tau_j - \text{int} (\tau_i - \text{cl} A))$;
5. a $(\tau_i, \tau_j)$ – fuzzy strongly preopen set if and only if $A \leq \tau_i - \text{int}((\tau_j, \tau_i) - \text{pcl} A)$;
6. a $(\tau_i, \tau_j)$ – fuzzy regular open set if and only if $A = \tau_i - \text{int} (\tau_j - \text{cl} A)$.

The family of all $(\tau_i, \tau_j)$ – fuzzy semiopen sets, $(\tau_i, \tau_j)$ – fuzzy preopen sets, $(\tau_i, \tau_j)$ – fuzzy strongly semiopen sets, $(\tau_i, \tau_j)$ – fuzzy semipreopen sets, $(\tau_i, \tau_j)$ – fuzzy strongly preopen sets and $(\tau_i, \tau_j)$ – fuzzy regular open sets of a fbts $(X, \tau_1, \tau_2)$ will be denote by $(\tau_i, \tau_j)$ – FSO, $(\tau_i, \tau_j)$ – FPO, $(\tau_i, \tau_j)$ – FSSO, $(\tau_i, \tau_j)$ – FSPO, $(\tau_i, \tau_j)$ – FSEPO, $(\tau_i, \tau_j)$ – FRO respectively.

**Definition 2.2** [5, 7, 8] Let $A$ be a fuzzy set of a fbts $X$. Then $A$ is called

1. a $(\tau_i, \tau_j)$ – fuzzy semiclosed set if and only if $A^c$ is a $(\tau_i, \tau_j)$ – fuzzy semiopen set;
2. a $(\tau_i, \tau_j)$ – fuzzy preclosed set if and only if $A^c$ is a $(\tau_i, \tau_j)$ – fuzzy preopen set;
(3) a \((\tau_i, \tau_j)\) – fuzzy strongly semiclosed set if \(A^c\) is a \((\tau_i, \tau_j)\) – fuzzy strongly semiopen set;

(4) a \((\tau_i, \tau_j)\) – fuzzy semipreclosed set if \(A^c\) is a \((\tau_i, \tau_j)\) – fuzzy semipreopen set;

(5) a \((\tau_i, \tau_j)\) – fuzzy strongly preclosed set if and only if \(A^c\) is a \((\tau_i, \tau_j)\) – fuzzy strongly preopen set;

(6) a \((\tau_i, \tau_j)\) – fuzzy regular closed set if and only if \(A^c\) is a \((\tau_i, \tau_j)\) – fuzzy regular open set.

Similarly, the family of all \((\tau_i, \tau_j)\) – fuzzy semiclosed sets, \((\tau_i, \tau_j)\) – fuzzy preclosed sets, \((\tau_i, \tau_j)\) – fuzzy strongly semiclosed sets, \((\tau_i, \tau_j)\) – fuzzy semipreclosed sets, \((\tau_i, \tau_j)\) – fuzzy strongly preclosed sets and \((\tau_i, \tau_j)\) – fuzzy regular closed sets will be denote by \((\tau_i, \tau_j)\) – FSC, \((\tau_i, \tau_j)\) – FPC, \((\tau_i, \tau_j)\) – FSSC, \((\tau_i, \tau_j)\) – FSEPC, \((\tau_i, \tau_j)\) – FSPC and \((\tau_i, \tau_j)\) – FRC, respectively.

**Definition 2.3** [4, 6, 7] A mapping \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \eta_1, \eta_2)\) from a fbts \(X\) into a fbts \(Y\) is called

(1) a fuzzy pairwise semicontinuous if \(f^{-1}(B)\) is a \((\tau_i, \tau_j)\) – fuzzy semiopen set of \(X\) for each \(\eta_i\) – fuzzy open set \(B\) of \(Y\);

(2) a fuzzy pairwise precontinuous if \(f^{-1}(B)\) is a \((\tau_i, \tau_j)\) – fuzzy preopen set of \(X\) for each \(\eta_i\) – fuzzy open set \(B\) of \(Y\);

(3) a fuzzy pairwise strongly semicontinuous if \(f^{-1}(B)\) is a \((\tau_i, \tau_j)\) – fuzzy strongly semiopen set of \(X\), for each \(\eta_i\) – fuzzy open set \(B\) of \(Y\);

(4) a pairwise semiprecontinuous if \(f^{-1}(B)\) is a \((\tau_i, \tau_j)\) fuzzy semipreopen set of \(X\), for each \(\eta_i\) – fuzzy open set \(B\) of \(Y\);

(5) a fuzzy pairwise strongly precontinuous if \(f^{-1}(B)\) is a \((\tau_i, \tau_j)\) fuzzy strongly preopen set of \(X\), for each fuzzy \(\eta_i\) – open set \(B\) of \(Y\);

(6) a fuzzy pairwise strong precontinuous irresolution if \(f^{-1}(B)\) is a \((\tau_i, \tau_j)\) – fuzzy strongly preopen set of \(X\), for each \(\tau_i, \tau_j\) – fuzzy strongly preopen set \(B\) of \(Y\).

**Definition 2.4** [1] A fuzzy set \(A\) of a fbts \(X\) is called fuzzy \((\tau_i, \tau_j)\) - \(\delta\) – open if and only if there exists fuzzy \((\tau_i, \tau_j)\) – regular open sets \(A_k, k \in I\) such that \(A = \bigvee_{k \in I} A_k\).
Definition 2.5 [1] A fuzzy set \( A \) of a fbts \( X \) is called fuzzy \((\tau_i, \tau_j)\) - \( \delta \) - closed if and only if \( A^c \) is a fuzzy \((\tau_i, \tau_j)\) - \( \delta \) - open set.

Definition 2.6 Let \( A \) be a fuzzy set of a fbts \((X, \tau_1, \tau_2)\). Then,

\[
(\tau_i, \tau_j) - \text{int}_\delta A = \{ B \mid B \leq A, B \in (\tau_i, \tau_j) - FRO \}
\]

is called the fuzzy \((\tau_i, \tau_j)\) - \( \delta \) - interior of \( A \).

\[
(\tau_i, \tau_j) - \text{cl}_\delta A = \{ B \mid B \geq A, B \in (\tau_i, \tau_j) - FRC \}
\]

is called the fuzzy \((\tau_i, \tau_j)\) - \( \delta \) - closure of \( A \).

3. FUZZY PAIRWISE ALMOST STRONG PRECONTINUITY

Definition 3.1 A mapping \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \eta_1, \eta_2) \) from a fbts \( X \) into a fbts \( Y \) is called fuzzy pairwise almost strongly precontinuous if \( f^{-1}(B) \) is a \((\tau_i, \tau_j)\) - fuzzy strongly preopen set of \( X \) for each \((\eta_i, \eta_j)\) - fuzzy regular open set \( B \) of \( X \).

Remark 3.1. Let \( f : X \rightarrow Y \) be a mapping from a fbts \( X \) into a fbts \( Y \). If \( f \) is fuzzy pairwise strongly precontinuous, then \( f \) is a fuzzy pairwise almost strongly precontinuous mapping. The following example shows that the converse statement may not be true.

Example 3.1. Let \( X = \{a, b, c\} \) and \( A, B, C \) and be fuzzy sets of \( X \) defined as follows:

\[
\begin{align*}
A(a) &= 0.5 & A(b) &= 0.3 & A(c) &= 0.6; \\
B(a) &= 0.3 & B(b) &= 0.4 & B(c) &= 0.3; \\
C(a) &= 0.5 & C(b) &= 0.5 & C(c) &= 0.6.
\end{align*}
\]

If we put \( \tau_1 = \tau_2 = \{0, B, A \lor B, 1\} \), \( \eta_1 = \eta_2 = \{0, A, B, A \land B, A \lor B, 1\} \) and \( f = id : (X, \tau_1, \tau_2) \rightarrow (Y, \eta_1, \eta_2) \) we conclude that \( f \) is fuzzy pairwise almost strongly precontinuous but \( f \) is not fuzzy pairwise strong precontinuous mapping.

Theorem 3.1 Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \eta_1, \eta_2) \) be a mapping from a fbts \( X \) into a fbts \( Y \). Then the following statements are equivalent:
(i) \( f \) is a fuzzy pairwise almost strongly precontinuous mapping;

(ii) \( f^{-1}(B) \) is a \((\tau_i, \tau_j)\)–fuzzy strongly preclosed set of \( X \) for each \((\eta_i, \eta_j)\)–fuzzy regular closed set \( B \) of \( Y \);

(iii) \((\tau_i, \tau_j) - \text{spcl} f^{-1}(\eta_i - \text{cl}(\eta_j - \text{int} B)) \leq f^{-1}(B) \) for each \( \eta_i \)–fuzzy closed set \( B \) of \( Y \);

(iv) \( f^{-1}(B) \leq (\tau_i, \tau_j) - \text{spint} f^{-1}(\eta_i - \text{int}(\eta_j - \text{cl} B)), \) for each \( \eta_i \)–fuzzy open set \( B \) of \( Y \);

(v) \( f^{-1}(B) \leq (\tau_i, \tau_j) - \text{spint} f^{-1}(\eta_i - \text{int}(\eta_j - \text{cl} B)) \) for each \((\tau_i, \tau_j)\) fuzzy strongly semiopen set \( B \) of \( Y \);

(vi) \((\tau_i, \tau_j) - \text{spcl} f^{-1}(\eta_i - \text{cl}(\eta_j - \text{int} B)) \leq f^{-1}(B), \) for each \((\tau_i, \tau_j)\)–fuzzy strongly semiclosed set \( B \) of \( Y \);

(vii) \((\tau_i, \tau_j) - \text{spcl} f^{-1}(\eta_i - \text{cl} B) \leq f^{-1}(B), \) for each \((\tau_i, \tau_j)\)–fuzzy preclosed set \( B \) of \( Y \);

(viii) \( f^{-1}(B) \leq (\tau_i, \tau_j) - \text{spint} f^{-1}(\eta_i - \text{int}(\eta_j - \text{cl} B)), \) for each \((\tau_i, \tau_j)\)–fuzzy preopen set \( B \) of \( Y \);

(ix) \((\tau_i, \tau_j) - \text{spcl} f^{-1}(\eta_i - \text{cl}(\eta_j - \text{int} B)) \leq f^{-1}(\eta_i - \text{cl} B), \) for each fuzzy set \( B \) of \( Y \);

(x) \( f^{-1}(\eta_i - \text{int} B) \leq (\tau_i, \tau_j) - \text{spint} f^{-1}(\eta_i - \text{int}(\eta_j - \text{cl}(\eta_i - \text{int} B))), \) for each fuzzy set \( B \) of \( Y \);

**Proof.** (i) \( \Rightarrow \) (ii) Let \( B \) be any \((\eta_i, \eta_j)\)–fuzzy regular closed set of fbts \( Y \). Then \( B^c \) is a \((\eta_i, \eta_j)\)–fuzzy open set of fbts \( Y \). By assumption and the Definition 3.1 we obtain that \( f^{-1}(B^c) \) is a \((\tau_i, \tau_j)\)–fuzzy strongly preopen set of fbts \( X \). From \( f^{-1}(B^c) = (f^{-1}(B))^c \) it follows that \( f^{-1}(B) \) is a \((\tau_i, \tau_j)\)–fuzzy strongly preclosed set of fbts \( X \).

(ii) \( \Rightarrow \) (iii) Let \( B \) be any \( \eta_i \)–fuzzy closed set of fbts \( Y \). Then \( \eta_i - \text{cl}(\eta_j - \text{int} B) \leq B, \) so \( f^{-1}(\eta_i - \text{cl}(\eta_j - \text{int} B)) \leq f^{-1}(B) \). Since \( \eta_i - \text{cl}(\eta_j - \text{int} B) \) is a \((\eta_i, \eta_j)\)–fuzzy regular closed set, by assumptions it follows that \( f^{-1}(B) \) is a \((\tau_i, \tau_j)\)–strongly preclosed set. Hence \((\tau_i, \tau_j) - \text{spcl} f^{-1}(\eta_i - \text{cl}(\eta_j - \text{int} B)) \leq f^{-1}(B) \).

(iii) \( \Rightarrow \) (iv) It can be proved by using the complement.
(iv) ⇒ (v) Let \( B \) be any \((\eta_i, \eta_j)\) – fuzzy strongly semiopen set of fbts \( Y \). Then \( B \leq \eta_i - \text{int} (\eta_j - \text{cl}(\eta_i - \text{int} B)) \), so \( f^{-1}B \leq f^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}(\eta_i - \text{int} B))) \). Since \( \eta_i - \text{int} (\eta_j - \text{cl}(\eta_i - \text{int} B)) \) is any \( \eta_i \) – fuzzy open set, by the assumption it follows that

\[
f^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}(\eta_i - \text{int} B))) \leq (\tau_i, \tau_j) - \text{spint} f^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}(\eta_i - \text{int} B))) = (\tau_i, \tau_j) - \text{spint} f^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}(\eta_i - \text{int} B))).
\]

(v) ⇒ (vi) It can be proved by using the complement.

(vi) ⇒ (vii) Let \( B \) be any \((\eta_i, \eta_j)\) – fuzzy preclosed set of fbts \( Y \). Then \( B \geq \eta_i - \text{cl}(\eta_j - \text{int} B) \), so \( f^{-1}(B) \geq f^{-1}(\eta_i - \text{cl}(\eta_j - \text{int} B)) \). Since \( \eta_i - \text{cl}(\eta_j - \text{int} B) \) is a \((\eta_i, \eta_j)\) – fuzzy strongly semiclosed set, by the assumptions it follows that

\[
f^{-1}(B) \geq f^{-1}(\text{cl}(\text{int} B)) \geq (\tau_i, \tau_j) - \text{spcl} f^{-1}(\eta_i - \text{cl}(\eta_j - \text{cl}(\eta_i - \text{int} B)))) \geq (\tau_i, \tau_j) - \text{spcl} f^{-1}(\eta_i - \text{cl}(\eta_j - \text{cl}(\eta_i - \text{int} B)))).
\]

(vii) ⇒ (viii) It can be proved by using the complement.

(viii) ⇒ (ix) Let \( B \) be any fuzzy set of fbts \( Y \). Then \( \eta_i - \text{int} B^c \) is a \((\tau_i, \tau_j)\) – fuzzy preopen set, so

\[
f^{-1}(\eta_i - \text{int} B^c) \leq f^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}(\eta_i - \text{int} B^c))) \leq f^{-1}(\eta_i - \text{cl} B).
\]

(ix) ⇒ (x) It can be proved by using the complement.

(x) ⇒ (i) Let \( B \) be \( \eta_i \) – regular open set of fbts \( Y \). Then

\[
f^{-1}(B) = f^{-1}(\eta_i - \text{int} B) \leq \text{spint} f^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}(\eta_i - \text{int} B))) = (\tau_i, \tau_j) - \text{spint} f^{-1}(B),
\]

so \( f^{-1}(B) \) is a \((\tau_i, \tau_j)\) – fuzzy strongly preopen set of \( X \). Hence \( f \) is fuzzy pairwise almost strongly precontinuous. \( \square \)

**Theorem 3.2** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \eta_1, \eta_2) \) be a mapping from a fbts \( X \) into a fbts \( Y \). Then the following statements are equivalent:
(i) \( f \) is a fuzzy pairwise almost strongly precontinuous mapping.

(ii) \((\tau_i, \tau_j) - \text{spcl } f^{-1}(B) \leq f^{-1}(\eta_i - \text{cl}B), \) for each \((\eta_1, \eta_2) - \text{fuzzy semiopen set } B \) of \( Y \).

(iii) \( f^{-1}(\eta_i - \text{int } B) \leq (\tau_i, \tau_j) - \text{spint } f^{-1}(B), \) for each \((\eta_i, \eta_j) - \text{fuzzy semiclosed set } B \) of \( Y \).

(iv) \( f^{-1}(\eta_i - \text{int } B) \leq (\tau_i, \tau_j) - \text{spint } f^{-1}(B), \) for each \((\eta_i, \eta_j) - \text{fuzzy semipreclosed set } B \) of \( Y \).

(v) \((\tau_i, \tau_j) - \text{spcl } f^{-1}(B) \leq f^{-1}(\eta_i - \text{cl}B), \) for each \((\eta_i, \eta_j) - \text{fuzzy semipreopen set } B \) of \( Y \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( B \) be any \((\eta_i, \eta_j) - \text{fuzzy semiopen set } \) of \( \text{fbts } Y \). Then \( B \leq \eta_i - \text{cl}(\eta_j - \text{int } B) \), so \( f^{-1}(B) \leq f^{-1}(\eta_i - \text{cl}(\eta_j - \text{int } B)). \) Since \( \eta_i - \text{cl}(\eta_j - \text{int } B) \) is a \((\eta_i, \eta_j) - \text{fuzzy regular closed set } \), by the assumptions it follows that \( f^{-1}(B) \) is a \((\tau_i, \tau_j) - \text{strongly preclosed set } \). Hence \((\tau_i, \tau_j) - \text{spcl } f^{-1}(\eta_i - \text{cl}(\eta_j - \text{int } B)) \leq f^{-1}(\eta_i - \text{cl}B) \).

\( (ii) \Rightarrow (iii) \) It can be proved by using the complement.

\( (iii) \Rightarrow (iv) \) Let \( B \) be any \((\eta_i, \eta_j) - \text{fuzzy semipreclosed open set } \) of \( \text{fbts } Y \). From \( B \geq \eta_i - \text{int } (\eta_j - \text{cl}(\eta_j - \text{int } B)) \) it follows that \( \eta_i - \text{int } B \geq \eta_i - \text{int } (\eta_j - \text{cl}(\eta_j - \text{int } B)) \), so \( \eta_i - \text{int } B \) is a \((\eta_i, \eta_j) - \text{fuzzy semiclosed set } \). According to the assumption we get

\[ f^{-1}(\eta_i - \text{int } B) \leq (\tau_i, \tau_j) - \text{spint } f^{-1}(\eta_i - \text{int } B) \leq (\tau_i, \tau_j) - \text{spint } f^{-1}(B). \]

\( (iv) \Rightarrow (v) \) It can be proved by using the complement.

\( (v) \Rightarrow (i) \) Let \( B \) be any \((\eta_i, \eta_j) - \text{fuzzy regular closed set } \) of \( \text{fbts } Y \). Then \( B \) is a \((\eta_i, \eta_j) - \text{fuzzy semipreopen set } \). From the assumption we get \( f^{-1}(\eta_i - \text{cl } B) \geq \text{spcl } f^{-1}(B), \) so \( f^{-1}(B) \) is a fuzzy strongly preclosed set. Hence \( f \) is a fuzzy pairwise almost strongly precontinuous. \( \square \)

**Corollary 3.3** Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \eta_1, \eta_2) \) be a fuzzy almost strongly precontinuous mapping from a fbts \( X \) into a fbts \( Y \). Then the following statements holds:

(i) \( (\tau_i, \tau_j) - \text{spcl } f^{-1}(B) \leq f^{-1}(\eta_i - \text{cl } B), \) for each \( \eta_i - \text{fuzzy open set } B \) of \( Y \).
(ii) \( f^{-1}(\eta_i - \text{int} B) \leq (\tau_i, \tau_j) - \text{spint} f^{-1}(B) \), for each \( \eta_i \)-fuzzy closed set \( B \) of \( Y \).

The following theorem gives some local characterizations of the fuzzy almost strongly precontinuous mappings.

**Theorem 3.4** Let \( f : (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2) \) be a mapping from a fbts \( X \) into an fbts \( Y \). Then the following statements are equivalent:

(i) \( f \) is a fuzzy pairwise almost strongly precontinuous mapping.

(ii) for each fuzzy singleton \( x_\alpha \) of \( X \) and \( \eta_i \)-fuzzy open set \( B \) contain \( f(x_\alpha) \), there exists fuzzy strongly preopen set \( A \) of \( X \) containing \( x_\alpha \) such that \( f(A) \leq \eta_i - \text{int} (\eta_j - cB) \).

(iii) for each fuzzy singleton \( x_\alpha \) of \( X \) and \( (\eta_i, \eta_j) \)-fuzzy regular open set \( B \) containing \( f(x_\alpha) \) there exists \( (\tau_i, \tau_j) \)-fuzzy strongly preopen set \( A \) of \( X \) containing \( x_\alpha \) such that \( f(A) \leq B \).

**Proof.** (i) \( \Rightarrow \) (iii) Let \( f \) be \( \eta_i \)-fuzzy pairwise almost strongly precontinuous, \( x_\alpha \) be a fuzzy singleton of \( X \) and let \( B \) be an \( \eta_i \)-fuzzy open set of \( Y \) such that \( f(x_\alpha) \leq B \). Then

\[
x_\alpha \leq f^{-1}(B) \leq (\tau_i, \tau_j) - \text{spint} f^{-1}(\eta_i - \text{int} (\eta_j - cB)).
\]

Let \[
A = (\tau_i, \tau_j) - \text{spint} f^{-1}(\eta_i - \text{int} (\eta_j - cB)).
\]

Then \( A \) is a \( (\tau_i, \tau_j) \)-fuzzy strongly preopen set and

\[
f(A) = f((\tau_i, \tau_j) - \text{spint} f^{-1}(\eta_i - \text{int} (\eta_j - cB))) \leq f f^{-1}(\eta_i - \text{int} (\eta_j - cB)) \leq \eta_i - \text{int} (\eta_j - cB)).
\]

(ii) \( \Rightarrow \) (iii) Let \( x_\alpha \) be a fuzzy singleton of \( X \) and let \( B \) be a \( (\eta_i, \eta_j) \)-fuzzy regular open set of \( Y \) containing \( f(x_\alpha) \). Then \( B \) is a \( \eta_i \)-fuzzy open set. According to the assumption there exists a \( (\tau_i, \tau_j) \)-fuzzy strongly preopen set \( A \) of \( X \) containing \( x_\alpha \) such that \( f(A) \leq \eta_i - \text{int} (\eta_j - cB) = B \).

(iii) \( \Rightarrow \) (i) Let \( B \) be a \( (\eta_i, \eta_j) \)-fuzzy regularly open set of \( Y \) and let \( x_\alpha \) be a fuzzy singleton of \( X \) such that \( x_\alpha \leq f^{-1}(B) \). According to the assumption there exists a \( (\tau_i, \tau_j) \)-fuzzy strongly preopen set \( A \) of \( X \) such that \( x_\alpha \leq A \) and \( f(A) \leq B \). Hence

\[
x_\alpha \leq A \leq f^{-1} f(A) \leq f^{-1}(B)
\]
and

\[ x_\alpha \leq A = (\tau_i, \tau_j) - \text{spint } A \leq (\tau_i, \tau_j) - \text{spint } f^{-1}(B). \]

Since \( x_\alpha \) is arbitrary and \( f^{-1}(B) \) is the union of all fuzzy singletons of \( f^{-1}(B) \), \( f^{-1}(B) \leq (\tau_i, \tau_j) - \text{spint } f^{-1}(B) \). Thus \( f \) is fuzzy pairwise almost strongly precontinuous mapping. \( \square \)

**Theorem 3.5** Let \( f : (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2) \) be a mapping from a fbts \( X \) into a fbts \( Y \). Then the following statements are equivalent:

(i) \( f \) is a fuzzy pairwise almost strongly precontinuous mapping.

(ii) \( f^{-1}(B) \) is a \((\tau_i, \tau_j)\)–fuzzy strongly preopen set of \( X \), for each \((\eta_i, \eta_j)\)–fuzzy \( \delta \)–open set \( B \) of \( Y \).

(iii) \( f^{-1}(B) \) is a \((\tau_i, \tau_j)\)–fuzzy strongly preclosed set of \( X \), for each \((\eta_i, \eta_j)\)–fuzzy \( \delta \)–closed set \( Y \).

(iv) \( f((\tau_i, \tau_j) - \text{spcl } A) \leq (\eta_i, \eta_j) - \text{cl}_\delta f(A) \), for each fuzzy set \( A \) of \( X \).

(v) \((\tau_i, \tau_j) - \text{spcl } f^{-1}(B) \leq f^{-1}((\eta_i, \eta_j) - \text{cl}_\delta B) \), for each fuzzy set \( B \) of \( Y \).

(vi) \( f^{-1}((\eta_i, \eta_j) - \text{int}_\delta B) \leq (\tau_i, \tau_j) - \text{spint } f^{-1}(B) \), for each fuzzy set \( B \) of \( Y \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( B \) be any \((\eta_i, \eta_j)\)–fuzzy \( \delta \)–open set of \( Y \). Then \( B = \vee_{\alpha \in I} B_\alpha \), where \( B_\alpha \) is a \((\eta_i, \eta_j)\)–fuzzy regular open set of \( Y \), for each \( \alpha \in I \). From

\[ f^{-1}(B) = f^{-1}(\vee_{\alpha \in I} B_\alpha) = \vee_{\alpha \in I} f^{-1}(B_\alpha) \]

it follows that \( f^{-1}(B) \) is a \((\tau_i, \tau_j)\)–fuzzy strongly preopen set as a union of \((\tau_i, \tau_j)\)–fuzzy strongly preopen sets.

(ii) \( \Rightarrow \) (iii) Can be proved by using the complement.

(iii) \( \Rightarrow \) (iv) Let \( A \) be any fuzzy set of \( X \). Then \((\eta_i, \eta_j) - \text{cl}_\delta f(A) \) is a \((\eta_i, \eta_j)\)–fuzzy \( \delta \)–closed set of \( Y \). According to the assumption \( f^{-1}((\eta_i, \eta_j) - \text{cl}_\delta f(A)) \) is a \((\tau_i, \tau_j)\)–fuzzy strongly preclosed set of \( X \). Hence

\[
(\tau_i, \tau_j) - \text{spcl } A \leq (\tau_i, \tau_j) - \text{spcl } f^{-1}(f(A)) \\
\leq (\tau_i, \tau_j) - \text{spcl } f^{-1}((\eta_i, \eta_j) - \text{cl}_\delta f(A)) \\
= f^{-1}((\eta_i, \eta_j) - \text{cl}_\delta f(A)).
\]
so

\[ f((\tau_i, \tau_j) - \text{spcl} A) \leq (\eta_i, \eta_j) - \text{cl}_\delta f(A). \]

\((iv) \Rightarrow (v)\) Let \(B\) be any fuzzy set of \(Y\). From the assumption it follows that

\[(\tau_i, \tau_j) - \text{spcl} f^{-1}(B) \leq f^{-1}(\tau_i, \tau_j) - \text{spcl} f^{-1}(B) \leq f^{-1}(\eta_i, \eta_j) - \text{cl}_\delta B.\]

\((v) \Rightarrow (vi)\) Can be proved by using the complement.

\((vi) \Rightarrow (i)\) Let \(B\) be any \((\eta_i, \eta_j)\) – fuzzy regular open set of \(Y\). Then \(B = (\eta_i, \eta_j) - \text{int}_\delta B\). According to the assumption

\[ f^{-1}(B) = f^{-1}((\eta_i, \eta_j) - \text{int}_\delta B) \leq (\tau_i, \tau_j) - \text{spint} f^{-1}(B) \leq f^{-1}(B). \]

Hence \(f^{-1}(B) = (\tau_i, \tau_j) - \text{spint} f^{-1}(B)\), so \(f^{-1}(B)\) is a \((\tau_i, \tau_j)\) – fuzzy strongly preopen set. Thus \(f\) is a fuzzy pairwise almost strongly precontinuous mapping. \(\Box\)

**Theorem 3.6** Let \(f : (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2)\) be a bijective mapping from a fbts \(X\) into an fbts \(Y\). The mapping \(f\) is a fuzzy pairwise almost strongly precontinuous if and only if

\[ (\eta_i, \eta_j) - \text{int}_\delta f(A) \leq f((\tau_i, \tau_j) - \text{spint} A), \]

for each fuzzy set \(A\) of \(X\).

**Proof.** We suppose that \(f\) is fuzzy pairwise almost strongly precontinuous. Then \(f^{-1}((\eta_i, \eta_j) - \text{int}_\delta f(A))\) is a \((\tau_i, \tau_j)\) – fuzzy strongly preopen set of \(X\), for any fuzzy set \(A\) of \(X\). Since \(f\) is injective, from Theorem 3.5 it follows that

\[ f^{-1}((\eta_i, \eta_j) - \text{int}_\delta f(A)) = (\tau_i, \tau_j) - \text{spint} f^{-1}((\eta_i, \eta_j) - \text{int}_\delta f(A)) \leq (\tau_i, \tau_j) - \text{spint} f^{-1}(A) = (\tau_i, \tau_j) - \text{spint} A. \]

Again, since \(f\) is surjective, we obtain

\[ (\eta_i, \eta_j) - \text{int}_\delta f(A) = (\eta_i, \eta_j) - ff^{-1}((\eta_i, \eta_j) - \text{int}_\delta f(A)) \leq f((\tau_i, \tau_j) - \text{spint} A). \]
Conversely, let $B$ be any $(\eta_i, \eta_j)$ – fuzzy $\delta$ – open set of $Y$. Then $(\eta_i, \eta_j)$ – int$_\delta B = B$. According to the assumption,

$$f((\tau_i, \tau_j) - \text{spint } f^{-1}(B)) \geq (\eta_i, \eta_j) - \text{int}_\delta f^{-1}(B)$$

$$= (\eta_i, \eta_j) - \text{int}_\delta B$$

$$= B.$$

Thus implies that

$$f^{-1}(f((\tau_i, \tau_j) - \text{spint } f^{-1}(B)) \geq f^{-1}(B).$$

Since $f$ is injective we obtain

$$(\tau_i, \tau_j) - \text{spint } f^{-1}(B) = f^{-1} f((\tau_i, \tau_j) - \text{spint } f^{-1}(B)) \geq f^{-1}(B).$$

Hence

$$(\tau_i, \tau_j) - \text{spint } f^{-1}(B) = f^{-1}(B),$$

so $f^{-1}(B)$ is a $(\tau_i, \tau_j)$ – fuzzy strongly preopen set.

Thus $f$ is a fuzzy pairwise almost strongly precontinuous mapping. \qed

**Definition 3.2** A mapping $f : (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2)$ from a fbts $X$ into a fbts $Y$ is called

(1) fuzzy pairwise almost precontinuity if $f^{-1}(B) \in (\tau_i, \tau_j)$ – FPO for each $B \in (\eta_i, \eta_j)$ – FRO

(2) fuzzy pairwise almost strong semicontinuity if $f^{-1}(B) \in (\tau_i, \tau_j)$ – FSSO for each $B \in (\eta_i, \eta_j)$ – FRO

(3) fuzzy pairwise almost semicontinuity if $f^{-1}(B) \in (\tau_i, \tau_j)$ – FSO for each $B \in (\eta_i, \eta_j)$ – FRO.

**Theorem 3.7** Let $(X_1, \tau_1, \tau_2), (X_2, \omega_1, \omega_2), (Y_1, \eta_1, \eta_2)$ and $(Y_2, \sigma_1, \sigma_2)$ be fbts’s such that $X_1$ is a product related to $X_2$. Then the product

$$f_1 \times f_2 : (X_1 \times X_2, \theta_1, \theta_2) \to Y_1 \times (Y_1 \times Y_2, \lambda_1, \lambda_2)$$

where $\theta_k$ (resp. $\lambda_k$) is the fuzzy product topology generated by $\sigma_k$ and $\omega_k$ (resp. $\eta_k$ and $\sigma_k$) (for $k = 1, 2$) of fuzzy almost strong precontinuous mappings $f_1 : (X_1, \tau_2, \tau_2) \to (X_2, \omega_1, \omega_2)$ and $f_2 : (Y_1, \eta_1, \eta_2) \to (Y_2, \sigma_1, \sigma_2)$ is a fuzzy pairwise almost precontinuous mapping.
Let \( B = \bigvee (U_\alpha \times V_\beta) \), where \( U_\alpha \) and \( V_\beta \) are \( \eta_i \)-fuzzy open sets of \( Y_1 \) and \( \sigma_i \)-fuzzy open sets of \( Y_2 \). From

\[
(f_1 \times f_2)^{-1}(B) = \bigvee [f_1 \times f_2]^{-1}(U_\alpha \times V_\beta)
= \bigvee (f_1^{-1}(U_\alpha) \times f_2^{-1}(V_\beta))
\leq \bigvee ((\tau_i, \tau_j) - \text{spint} f_1^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}U_\alpha))
\quad \times (\omega_i, \omega_j) - \text{spint} f_2^{-1}(\eta_i - \text{spint} (\eta_j - \text{cl}V_\beta))
\leq \bigvee ((\tau_i, \tau_j) - \text{p int} f_1^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}U_\alpha))
\quad \times (\omega_i, \omega_j) - \text{p int} f_2^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}V_\beta))
\leq \bigvee ((\theta_i, \theta_j) - \text{p int} (f_1^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}U_\alpha)))
\quad \times f_2^{-1}(\eta_i - \text{int} (\eta_j - \text{cl}U_\beta))
\leq (\theta_i, \theta_j) - \text{p int} (f_1 \times f_2)^{-1}(\bigvee (\eta_i - \text{int} (\eta_j - \text{cl}U_\alpha))
\quad \times (\eta_i - \text{int} (\eta_j - \text{cl}U_\beta))
\leq (\theta_i, \theta_j) - \text{p int} (f_1 \times f_2)^{-1}(\bigvee (\lambda_i - \text{int} (\lambda_j - \text{cl}(U_\alpha \times V_\beta)))
\leq (\theta_i, \theta_j) - \text{p int} (f_1 \times f_2)^{-1}(\bigvee (\lambda_i - \text{int} (\lambda_j - \text{cl}(U_\alpha \times V_\beta)))
\leq (\theta_i, \theta_j) - \text{p int} (f_1 \times f_2)^{-1}(\lambda_i - \text{int} (\lambda_j - \text{cl}B),
\]

it follows that \( f_1 \times f_2 \) is fuzzy pairwise almost precontinuous mapping. \( \square \)

**Theorem 3.8** Let \( X, X_1, X_2 \) be fbts’s and \( p_k : X_1 \times X_2 \to X_i (k = 1, 2) \) are the projections of \( X_1 \times X_2 \) onto \( X_i \). If \( f : X \to X_1 \times X_2 \) is a fuzzy pairwise almost strongly precontinuous, then \( p_i f \) are also fuzzy pairwise almost strongly precontinuous mappings.

**Proof.** Follows from the fact that the projection mappings are fuzzy continuous \((p_k : (X_1 \times X_2) \to X_k, k = 1, 2)\) \( \square \)

**Theorem 3.9** Let \( f : (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2) \) be a mapping from a fbts \( X \) into a fbts \( Y \). If the graph \( g : (X, \tau_1, \tau_2) \to (X \times Y, \theta_1, \theta_2) \) of \( f \) is fuzzy pairwise almost strongly precontinuous, then \( f \) is fuzzy pairwise almost strongly precontinuous.
**Proof.** By Lemma 2.2 of [1] \( f^{-1}(B) = 1 \land f^{-1}(B) = g^{-1}(1 \times B) \), for each \((\eta_i, \eta_j)\) – fuzzy regular open set \(B\) of \(Y\). Since \(g\) is fuzzy pairwise almost precontinuous and \(1 \times B\) is a \((\theta_i, \theta_j)\) – fuzzy regular open set of \(X \times Y\), \(f^{-1}(B)\) is a \((\tau_i, \tau_j)\) – fuzzy strongly preopen set of \(X\), so \(f\) is fuzzy pairwise almost strongly precontinuous. \(\Box\)

**Theorem 3.10** Let \(f : (X, \tau_1, \tau_2) \to (Y, \eta_1, \eta_2)\) be a fuzzy pairwise strongly preopen and fuzzy pairwise strongly precontinuous irresolution from a fbts \(X\) into a fbts \(Y\) and let

\[
g : (Y, \eta_1, \eta_2) \to (Z, \sigma_1, \sigma_2)
\]

be a mapping from fbts \(Y\) into fbts \(Z\). The mapping \(gf\) is a fuzzy pairwise almost strongly precontinuous if and only if \(g\) is a fuzzy pairwise almost strongly precontinuous.

**Proof.** Let \(gf\) be a fuzzy pairwise almost strongly precontinuous. Then

\[
g^{-1}(C) = f(gf)^{-1}(C)
\]

is a \((\eta_i, \eta_j)\) – fuzzy strongly preopen set of \(Y\), for each \((\sigma_1, \sigma_2)\) – fuzzy regular open set \(C\) of \(Z\). Hence \(g\) is a fuzzy almost strongly precontinuous mapping.

Conversely, let \(g\) be a fuzzy pairwise almost strongly precontinuous mapping and let \(C\) be a \((\sigma_1, \sigma_2)\) – fuzzy regular open set of \(Z\). From \((gf)^{-1}(C) = f^{-1}(g^{-1}(C))\) it follows that \(gf\) is a fuzzy pairwise almost strongly precontinuous mapping. \(\Box\)

**References**


