Kragujevac J. Math. 28 (2005) 173–183.

ON THE COMPACTNESS AND CONDENSING OF NONLINEAR SUPERPOSITION OPERATORS

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Abstract. In the present paper we give some propositions about conditions for compactness and condensation of the nonlinear superposition operator (1) in $l_{p,\sigma}$ spaces.

1. INTRODUCTION

Firstly, we recall some definitions and results. Nonlinear operator F generated by a real function $f(s,u)$:

$$F(u) = f(s,u) \quad , \ (u \in \mathbb{R}), \tag{1}$$

is called superposition operator. By $l_{p,\sigma}$ $(1 \leq p < \infty)$ we denote spaces of functions $x : \mathbb{N} \to \mathbb{R}$ (real sequences), for which norm

$$||x||_{p,\sigma} = \left(\sum_{s \in \mathbb{N}} |x(s)|^p \sigma(s) \right)^{\frac{1}{p}} \ , \ (1 \leq p < \infty) \ ,$$

makes sense and it is finite. Here σ is a weight function. Note that linear space $l_{p,\sigma}$ is not normed space if $0 < p < 1$. On the other hand the functional $[x]_p =$

$\sum_{s\in\mathbb{N}} |x(s)|^p \sigma(s)$ defines a p-norm at $l_{p,\sigma}$, what makes the space $l_{p,\sigma}$ a complete p-normed space. So that results given in this paper can be extended to the case of $0 < p < 1$.

We suppose here that weight function satisfies the next condition:

$$(\forall s \in \mathbb{N})\ \sigma(s) \geq 1$$

which is both necessary and sufficient condition that $p < q$ implies $l_{p,\sigma} \subset l_{q,\sigma}$. By

$$B_p(r) = \{x \in l_{p,\sigma} : ||x||_{p,\sigma} \leq r\}$$

we denote the ball of the radius r, which is a closed and convex set in $l_{p,\sigma}$. P_D is multiplication operator by characteristic function of the set D, i.e.,

$$P_D x(s) = \chi_D(s) x(s).$$

The *$\mathcal{L}$-characteristic* $\mathcal{L}(F, \mathcal{P})$ of the nonlinear superposition operator (1) is defined as a set of all pairs $(l_{p,\sigma}, l_{q,\tau})$ spaces, such that F has some property $\mathcal{P}$, as an operator from $l_{p,\sigma}$ into $l_{q,\tau}$, see [4].

Theorem A 1 *([3]) Nonlinear superposition operator (1) maps $l_{p,\sigma}$ into $l_{q,\tau}$ if and only if there exist a function $a \in l_{q,\tau}$ and constants $b \geq 0$ and $\delta > 0$ such that the estimate*

$$|f(s,u)| \leq a(s) + b\sigma^{\frac{1}{q}}(s)\tau^{-\frac{1}{q}}(s)|u|^{\frac{p}{q}} \tag{2}$$

holds for all pairs $(s,u) \in \mathbb{N} \times \mathbb{R}$, for which is $\sigma(s)|u|^p \leq \delta^p$.

Theorem B 1 *([3]) Nonlinear superposition operator F is bounded on $l_{p,\sigma}$ if and only if for any $r > 0$ there exist a function $a_r \in l_{q,\tau}$ and constants $b_r \geq 0$ such that holds*

$$|f(s,u)| \leq a_r(s) + b_r\sigma^{\frac{1}{q}}(s)\tau^{-\frac{1}{q}}(s)|u|^{\frac{p}{q}} \quad , \ (\sigma(s)|u|^p \leq r^p).$$

2. COMPACTNESS OF SUPERPOSITION OPERATOR ON $l_{p,\sigma}$

In [7] the property of absolute boundedness of sets in arbitrary ideal spaces has been studied in detail. But as it is well known, those results does no cover the case when assumption of measure continuity is missing. Clearly we are going to be held up on general conditions of absolute boundedness of sets in spaces $l_{p,\sigma}$ and considering that, on the conditions of complete continuity (compactness) of superposition operators (1).

Set $M \subset l_{p,\sigma}$ is called absolutely bounded ([1]), if

$$\lim_{n\to\infty} \sup_{x\in M} ||P_n x||_{l_{p,\sigma}} = 0,$$

where is $P_n x = \chi_D x$ an operator of multiplying with characteristic function of set $D = \{n+1, n+2, \ldots\}$. We see that notions absolute boundedness of set and its precompactness are equivalent in our case. Consequently, an absolutely bounded operator, i.e. the operator which is mapping bounded sets into absolutely bounded sets, is nothing else but a completely continuous operator. We shall take later on an example of the bounded set in $l_{p,\sigma}$, but no absolute bounded. The general conditions of precompactness of sets in $l_{p,\sigma}$ spaces we will formulate as an analogous of well known Vallee-Poussin theorem which applies to $L_p([0,1])$ spaces ([1]), and it is given in following:

Lemma 1 *Let $1 \le p < \infty$ and $u_0(s)$ arbitrary positive function from $l_{p,\sigma}$. Set $M \subset l_{p,\sigma}$ is absolutely bounded if and only if there exists monotone increasing on $[0,\infty)$ function $\Phi(u)$ for which*

$$\lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty \ ,$$

and

$$\sup_{x\in M} \left|\left| \Phi\left(\frac{|x|}{u_0}\right) u_0 \right|\right|_{p,\sigma} < \infty \ . \tag{3}$$

Proof. Let set $M \subset l_{p,\sigma}$ is an absolutely bounded one. If we put

$$\psi(n) = \sup_{x\in M} \left(\sum_{s>n} |x(s)|^p \sigma(s) \right)^{\frac{1}{p}} ,$$

then $\lim_{n\to\infty}\psi(n)=0$. Let (n_k) is a subsequence of natural numbers $0<n_0<n_1<\ldots n_k<\ldots$, such that series $\sum_{k=0}^{\infty}\psi(n_k)$ is convergent. Denote by (δ_k) a monotone increasing to infinity sequence of numbers which satisfies the condition

$$\sum_{k=1}^{\infty}\delta_k\psi(n_{k-1})<\infty\ ,$$

and let ϕ be a monotone increasing function for which $\phi(n_k)=\delta_k$ $(k=1,2,\ldots)$. It is clear that the function $\Phi(u)=u\phi(u)$ $(0<u<\infty)$ satisfies Lemma's conditions. Further let

$$S_k=\{s\in\mathbb{N}: n_{k-1}u_0(s)<|x(s)|\le n_k u_0(s),\ u_0(s)>0\}\ \ ,\ k=1,2,\ldots$$

For $x\in M$, we have

$$\begin{aligned}\left(\sum_{s=1}^{\infty}\left|\Phi\left(\frac{|x(s)|}{u_0(s)}\right)u_0(s)\right|^p\sigma(s)\right)^{\frac{1}{p}} &\le \sum_{k=1}^{\infty}\left(\sum_{s\in S_k}\left|\Phi\left(\frac{|x(s)|}{u_0(s)}\right)u_0(s)\right|^p\sigma(s)\right)^{\frac{1}{p}}\\ &\le \sum_{k=1}^{\infty}\left(\sum_{s\in S_k}|\phi(n_k)|^p|x(s)|^p\sigma(s)\right)^{\frac{1}{p}}\\ &\le \sum_{k=1}^{\infty}\delta_k\left(\sum_{s\in S_k}|x(s)|^p\sigma(s)\right)^{\frac{1}{p}}\\ &= \sum_{k=1}^{\infty}\delta_k\psi(n_{k-1})<\infty\ .\end{aligned}$$

In other words, we have shown (3), or that Lemma's condition is necessary.
The condition is sufficient: let $\varepsilon>0$. For $u\ge\lambda$ $(0<\lambda<\infty)$ is $u\le\varepsilon\Phi(u)$ and if $u_0\in l_{p,\sigma}$ $(u_0(s)>0)$, we have:

$$\begin{aligned}\left(\sum_{s>n}|x(s)|^p\sigma(s)\right)^{\frac{1}{p}} &\le \left(\sum_{\substack{s>n\\ |x(s)|>\lambda u_0(s)}}\frac{|x(s)|^p}{u_0(s)}\sigma(s)\right)^{\frac{1}{p}}+\left(\sum_{\substack{s>n\\ |x(s)|\le\lambda u_0(s)}}|x(s)|^p\sigma(s)\right)^{\frac{1}{p}}\\ &\le \left(\lambda^p\sum_{s>n}|u_0(s)|^p\sigma(s)\right)^{\frac{1}{p}}+\left(\varepsilon^p\sum_{s>n}\left|\Phi\left(\frac{x(s)}{u_0(s)}\right)u_0(s)\right|^p\sigma(s)\right)^{\frac{1}{p}}\end{aligned}$$

$$\leq \ \lambda \left(\sum_{s>n} |u_0(s)|^p \sigma(s) \right)^{\frac{1}{p}} + \varepsilon m \leq \varepsilon(\lambda + m) \ ,$$

where is with m denoted the left side in (3). From the last sequence of inequations, it is easy to conclude that

$$\lim_{n\to\infty} \sup_{x\in M} \left(\sum_{s>n} |x(s)|^p \sigma(s) \right)^{\frac{1}{p}} = 0 \ ,$$

or that set M is absolutely bounded, by which the Lemma is completely proved. $\square$

As a direct consequence of previous Lemma, we have:

Lemma 2 *Let F is a superposition operator generated by function $f(s,u)$, observed as an operator between $l_{p,\sigma}$ and $l_{q,\tau}$, bounded on set $A \subset l_{p,\sigma}$. For set $FA \subset l_{q,\tau}$ to be absolutely bounded, it is necessary and sufficient that operator $\tilde{F}$ generated by function*

$$\tilde{f}(s,u(s)) = u_0(s)\Phi\left(\frac{|f(s,u(s))|}{u_0(s)}\right)$$

is bounded on A, where $u_0(s)$ is an arbitrary function from $l_{q,\tau}$ and Φ increasing on $[0,+\infty)$ function which satisfies the condition $\lim_{u\to+\infty} \frac{\Phi(u)}{u} = \infty$.

From Lemma 1 and Theorem A 1, follows:

Theorem 1 *Let $1 \leq p, q < \infty$ and the operator (1) generated by function $f(s,u)$ is mapping $l_{p,\sigma}$ into $l_{q,\tau}$. Then this operator is completely continuous (compact) if and only if for any $r > 0$ there exists a monotone increasing on $[0,\infty)$ function $\Phi_r(u)$, for which is*

$$\lim_{u\to\infty} \frac{\Phi_r(u)}{u} = \infty \ ,$$

and

$$\Phi_r\left(\frac{|f(s,u)|}{u_0(s)}\right) u_0(s) \leq a_r(s) + b_r \sigma^{\frac{1}{q}}(s)\tau^{-\frac{1}{q}}(s)|u|^{\frac{p}{q}} \ \ , \ (\sigma(s)^{\frac{1}{p}}|u| \leq r) \ , \qquad (4)$$

where $a_r(s) \in l_{q,\tau}$, $b_r \geq 0$.

Proof. The condition is sufficient. Let there for every $r > 0$ exists the function $\Phi_r(u)$, with previously pointed properties and (4) is valid. Assume also that x belongs to the set $M \subset B(r) = \{x : ||x|| \leq r\}$; then $|x(s)| \leq r$ $(s \in \mathbb{N})$ and from (4), easily follows ([3])

$$\left|\left|\Phi_r\left(\frac{|f(s,x(s))|}{u_0(s)}\right)u_0(s)\right|\right|_{q,\tau} \leq ||a_r||_{q,\tau} + b_r\sigma^{\frac{1}{q}}(s)\tau^{-\frac{1}{q}}(s)r^{\frac{p}{q}} \quad , \ (u_0(s) > 0) \ ,$$

i.e., the condition (3) of Lemma 1, where naturally instead of M stands $F(M)$, and supremum is taken over all $Fx \in F(M)$, and F is a superposition operator (1). If we recall ourselves to the Lemma 1, we will get that $F(M)$ is a compact set (because $l_{q,\tau}$ is a Banach space), so that operator F is a compact one.
The condition is necessary. If operator F is a compact operator, then from Lemma 1, the existence of function $\Phi_r(u)$ follows, with all listed properties and the condition:

$$\sup_{Fx \in F(M)} \left|\left|\Phi_r\left(\frac{|f(s,x(s))|}{u_0(s)}\right)u_0(s)\right|\right|_{q,\tau} < \infty \ ,$$

where from we are concluding that the operator Φ_r generated by the function $\Phi_r\left(\frac{|f(s,u)|}{u_0(s)}\right)u_0(s)$ is bounded. Now the inequality (4) follows from Lemma 2, what proves the theorem. $\square$

Results obtained in [4], allow us to consider $\mathcal{L}(F, compact.)$ now as a part of the $\mathcal{L}(F, act.)$, namely we have:

Theorem 2 *Let F be a superposition operator generated by function $f(s,u)$ and compact operator from $l_{p_0,\sigma}$ to $l_{q_0,\tau}$ and at the same time from $l_{p_1,\sigma}$ to $l_{q_1,\tau}$. Then the operator F is compact operator from $l_{p,\sigma}$ to $l_{q,\tau}$ where are*

$$\frac{1}{p} = \frac{\alpha}{p_0} + \frac{1-\alpha}{p_1} \ , \ \frac{1}{q} = \frac{\alpha}{q_0} + \frac{1-\alpha}{q_1} \quad , \ (\alpha \in [0,1]) \ .$$

Proof. We will consider the case when $p_0 \leq p_1$, $q_0 \leq q_1$ i $\frac{p_0}{q_0} \leq \frac{p_1}{q_1}$. Let $p_0 \leq p \leq p_1$ and $q_0 \leq q \leq q_1$. Because of $p \leq p_1$ it is $l_{p,\sigma} \subset l_{p_1,\sigma}$ so for arbitrary bounded set $M \subset l_{p,\sigma}$, $M \subset l_{p_1,\sigma}$ and M is bounded in $l_{p_1,\sigma}$. As F is a compact operator, based on characterization of precompactness of set in Banach space $l_{p_1,\sigma}$ follows

$$(\exists \rho_s)(\forall x \in M) \ |f(s,x(s))| \leq \rho_s \quad , \ (s \in \mathbb{N}) \ . \tag{5}$$

On the other side, as $q \geq q_0$ it follows that $l_{q_0,\tau} \subset l_{q,\tau}$ i.e., for arbitrary y is $||y||_{q,\tau} \leq ||y||_{q_0,\tau}$, so that, for arbitrary $x \in M$ is

$$\begin{aligned} \sum_{s\in\mathbb{N}} |Fx(s)|^q \tau(s) &= \sum_{s\in\mathbb{N}} |f(s,x(s))|^q \tau(s) = ||Fx||_{q,\tau}^q \\ &\leq ||Fx||_{q_0,\tau}^q = \left(\sum_{s\in\mathbb{N}} |f(s,x(s))|^{q_0} \tau(s) \right)^q < \infty \ . \end{aligned} \tag{6}$$

Therefore the series $\sum_{s\in\mathbb{N}} |Fx(s)|^q$ is uniformly convergent on the set M. From (5) and (6), from characterization of precompactness of set in $l_{p,\sigma}$, we are concluding that FM is a precompact set in $l_{q,\tau}$, e.g. F is the compact operator from $l_{p,\sigma}$ into $l_{q,\tau}$ $\square$

As one can see Theorem 2. says that $\mathcal{L}(F, compact.)$ is a convex subset in the convex set $\mathcal{L}$(F,acts). Moreover, we have that $\mathcal{L}(F, compact.)$ is subset of the $\mathcal{L}(F, bound.)$.

3. CONDENSING OF SUPERPOSITION OPERATOR ON $l_{p,\sigma}$

In continuation we will consider some facts in correlation with condensation of superposition operator (1), so firstly we ought to say that we will for the measure of noncompactness in spaces $l_{p,\sigma}$ (and $l_{q,\tau}$) take Hausdorff measure of noncompactness. Hausdorff measure of noncompactness $\alpha(M)$ which value on bounded set M is defined as an infimum of positive numbers ε, for which set M has a finite ε-net, as it is known ([2]) in spaces $l_{p,\sigma}$ is defined by formula:

$$\alpha(M) = \lim_{n\to\infty} \sup_{x\in M} ||P_n x|| \ . \tag{7}$$

Let us take now bounded set

$$B_p(1) = \{x \in l_{p,\sigma} : ||x||_{p,\sigma} \leq 1\}.$$

It is not hard to see that $B_p(1)$ is not absolutely bounded set. Indeed, give $(e_n)_{n\in\mathbb{N}} \subset l_{p,\sigma}$, $(e_n = (0, 0, ..., 1, 0, ...))$. While

$$||e_n - e_m||_{l_{p,\sigma}} = (\sigma(n) + \sigma(m))^{\frac{1}{p}} \ ,$$

this sequence isn't Cauchy sequence, and we can't give subsequence that is convergent, e.g. $B_p(1)$ is not precompact (in complete spaces absolute boundedness of set and precompactness are equal).

Remind ourselves that for the operator F, which acts from space $l_{p,\sigma}$ into space $l_{q,\tau}$, we say it is (k,α)-bounded or α-condensing , if for any set $M \subset B(x_0, r) \subset l_{p,\sigma}$ matters:

$$\alpha(FM) \leq k\alpha(M) \ . \tag{8}$$

Lemma 3 *Let the operators F i G, which are generated by functions $f(s,u)$ and $g(s,u)$ successively, act from $l_{p,\sigma}$ to $l_{q,\tau}$ $(1 \leq p, q < \infty)$. Then, if for every $x \in l_{p,\sigma}$ it matters that*

$$|f(s,x(s))| \leq |g(s,x(s))| \ , \tag{9}$$

it follows that

$$\alpha(FM) \leq \alpha(GM) \ . \tag{10}$$

Proof. From (9) and the definition of norm on $l_{q,\tau}$ it follows that for every $x \in l_{p,\sigma}$ and every $n \in \mathbb{N}$

$$||P_n F x||_{q,\tau} \leq ||P_n G x||_{q,\tau} \ ,$$

where from, considering (7), we easily get (10) $\square$

Theorem 3 *Let $1 \leq p, q < \infty$ and let operator superposition F, which is generated by function $f(s,u)$, acts from space $l_{p,\sigma}$ into space $l_{q,\tau}$. Then operator F is α-condensable, i.e. holds estimate*

$$\alpha(FM) \leq k(r)\alpha(M) \quad , \quad (M \subset B(x_0, r) \subset l_{p,\sigma}) \ , \tag{11}$$

where x_0 is an arbitrary point from $l_{p,\sigma}$, and

$$r^{\frac{q-p}{q}} k(r) = \inf\{b_\varepsilon : |f(s,u)| \leq a_\varepsilon(s) + b_\varepsilon \sigma^{\frac{1}{q}}(s)\tau^{-\frac{1}{q}}(s)|u|^{\frac{p}{q}} \ , \ \sigma^{\frac{1}{p}}(s)|u| \leq r, s > n\} \ .$$

Proof. First of all, the value $k(r)$ is correctly introduced because the operator F acts from $l_{p,\sigma}$ in $l_{q,\tau}$, so therefor the function $f(s,u)$, which generates this operator satisfies the conditions of Theorem A i.e. , the inequality (2). Let it now $f_1(s,u) = a_\varepsilon + b_\varepsilon \sigma^{\frac{1}{q}}(s)\tau^{-\frac{1}{q}}|u|^{\frac{p}{q}}$ and F_1 operator generated by this function. It is clear that in calculation of the value $\alpha(F_1)$, the function $a_\varepsilon(s)$ doesn't have an important role, so for $x \in M \subset B(x_0,r) \subset l_{p,\sigma}$, $(x \in l_{p,\sigma} \Rightarrow x^{\frac{p}{q}} \in l_{q,\sigma})$:

$$\alpha(F_1(M)) = \lim_{n\to\infty} \sup_{x\in M} b_\varepsilon \left(\sum_{s>n} |x(s)|^p \sigma(s) \right)^{\frac{1}{q}} \leq b_\varepsilon r^{\frac{p-q}{q}} \alpha(M) \ .$$

Since the last relation is valid for every b_ε, from Lemma 3, estimate (11) follows directly, and by that the Theorem 3, is proved. □

Considering that in practical application of the property of the condensation operator the basic role has a condensity coefficient $k(r)$, it is a motivation more for dealing with it a little bit more. Put

$$H(r,\delta) = \sup_{M\subset B(0,r),\alpha(M)\leq\delta} \alpha(F(M)) \ , \tag{12}$$

where r is a radius of ball B, α-Hausdorff measure of noncompactness. The function $H(r,\delta)$, defined by (12), is called the function of condensation of superposition operator F and its precisely calculation, as we can see, isn't easy at all. In the case which we are researching, it is possible for this function to give simple majorisations which are in the case of linear operators obviously becoming simplified and turning into equations.

Theorem 4 *Let $1 \leq p,q < \infty$ and the superposition operator F, which is generated by the function $f(s,u)$, acts from $l_{p,\sigma}$ in $l_{q,\sigma}$. Then following inequality is valid*

$$H(r;\delta) \leq \delta^{\frac{p}{q}} \inf W(f;r,\delta) \ , \tag{13}$$

where $W(f;r,\delta)$ is set of constants b, for which at suitable $a(s) \in l_{q,\sigma}$, $c \geq 0$ and $n \in \mathbb{N}$ the following inequality is valid

$$|f(s,u) - f(s,v)| \leq a(s) + b|u|^{\frac{p}{q}} + c|v|^{\frac{p}{q}} \ \ , \ (|u|,|v| \leq r,\ |u-v| \leq \delta) \ .$$

Proof. Let M is arbitrary set from the ball $B(0,r)$, $\delta > \alpha(M)$ and $b \in W(f;r,\delta)$. Denote by $\{x_j(s) : j = 1,2,\ldots,m\}$ finite δ-net of the set M. Let be now $x \in M$ and

$||x - x_j||_{p,\sigma} < \delta$. Then for big k we have:

$$\begin{aligned} ||P_k F x||_{q,\sigma} &\le ||P_k F x_j||_{q,\sigma} + ||P_k(Fx - Fx_j)||_{q,\sigma} \\ &\le ||P_k F x_j||_{q,\sigma} + ||P_k a||_{q,\sigma} + b||P_k x||_{p,\sigma}^{\frac{p}{q}} + c||P_k x_j||_{p,\sigma}^{\frac{p}{q}} \\ &\le \max_j \left(||P_k F x_j||_{q,\sigma} + ||P_k a||_{q,\sigma} + c||P_k x_j||_{p,\sigma}^{\frac{p}{q}} \right) + b\delta^{\frac{p}{q}} \ , \end{aligned}$$

where from we easily get (13). The theorem is proved. $\square$

Using (12), the coefficient of condensation of superposition operator (1) is defined by

$$k(r) = \sup_{0<\delta\le r} \frac{H(r,\delta)}{\delta} \ ,$$

and from Theorem 4, follow estimate

$$k(r) \le \sup_{0<\delta\le r} \inf \delta^{\frac{p-q}{q}} W(f; r, \delta) \ ,$$

which ultimately in the case of linear operator turns into equality.

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