Kragujevac J. Math. 28 (2005) 155–164.

BASIC POLYHEDRA IN KNOT THEORY[1]

Slavik V. Jablan[1], Ljiljana M. Radović[2], Radmila Sazdanović[3]

[1] *The Mathematical Institute, Knez Mihailova 35, P.O. Box 367, 11001 Belgrade, Serbia and Montenegro*
(e-mail: jablans@mi.sanu.ac.yu)

[2] *University of Niš, Faculty of Mechanical Engineering, Beogradska 14, Niš, Serbia and Montenegro*
(e-mail: liki@masfak.ni.ac.yu)

[3] *The Mathematical Institute, Knez Mihailova 35, P.O. Box 367, 11001 Belgrade, Serbia and Montenegro*
(e-mail: radmilas@gmail.com)

Abstract. Using the table of four-regular 3- and 2-connected planar graphs computed by Brendan McKay, the complete list of basic polyhedra with $n \leq 16$ crossings is derived. For all the basic polyhedra, with the exception of four of them, the number of source links which could be derived from them by adding single digons in their vertices is computed.

1. INTRODUCTION

In the history of knot theory, one of fundamental problems was the derivation of basic polyhedra: prime knots or links without digons. Considered from the graph

[1]The author was supported in part by the Serbian Ministry of Science and Environmental Protection (Project #2005: Geometry, education and visualization with applications).

theory point of view, basic polyhedra are 4-regular 3- or 2-vertex-connected planar graphs that are at least 3-edge connected. Namely, every knot or link without digons can be represented by a 4-regular planar 3- or 2-connected graph. The edge connectivity condition excludes from the definition knots or links that are not prime. AS knots are 1-component links we use the term "links" for both knots and links.

The problem for $n \leq 12$ was solved by Kirkman [1], where the basic polyhedra with $n \leq 12$ vertices were obtained by introducing new triangular faces in link diagrams, in order to eliminate all digons. For that, it is necessary that a link diagram contains at most three digons, and that all digons belong to the same face. In such a face we inscribe a triangle, with the vertices belonging to the face edges (e.g., in their midpoints), and each digon must contain a vertex of the triangle.

In the following table all diagrams of links for $3 \leq n \leq 9$ that satisfy that necessary condition and the list of basic polyhedra with $(n+3)$ vertices derived from them (Fig. 1, Fig. 2) are given in Dowker notation.

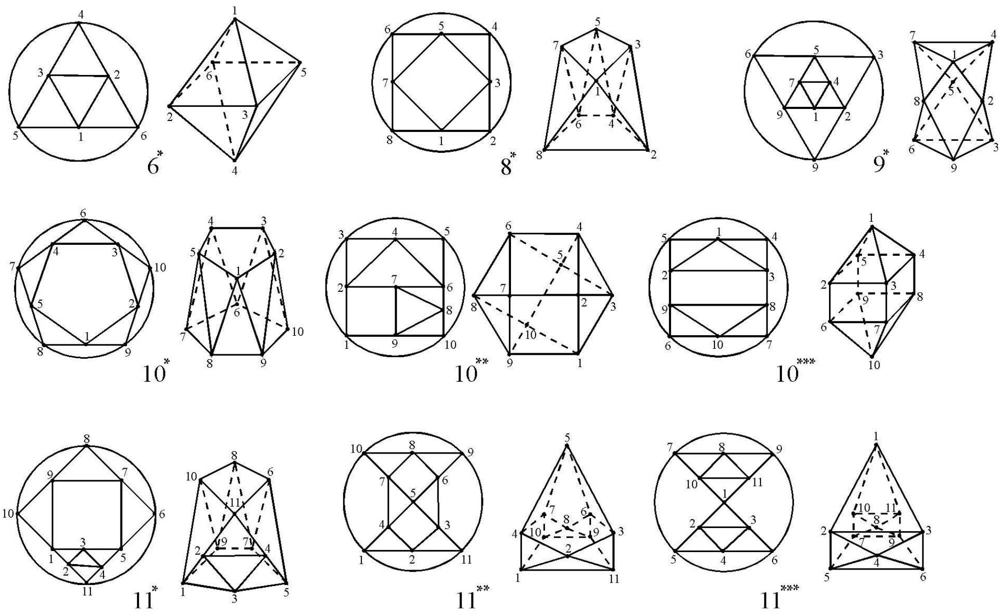

Figure 1.

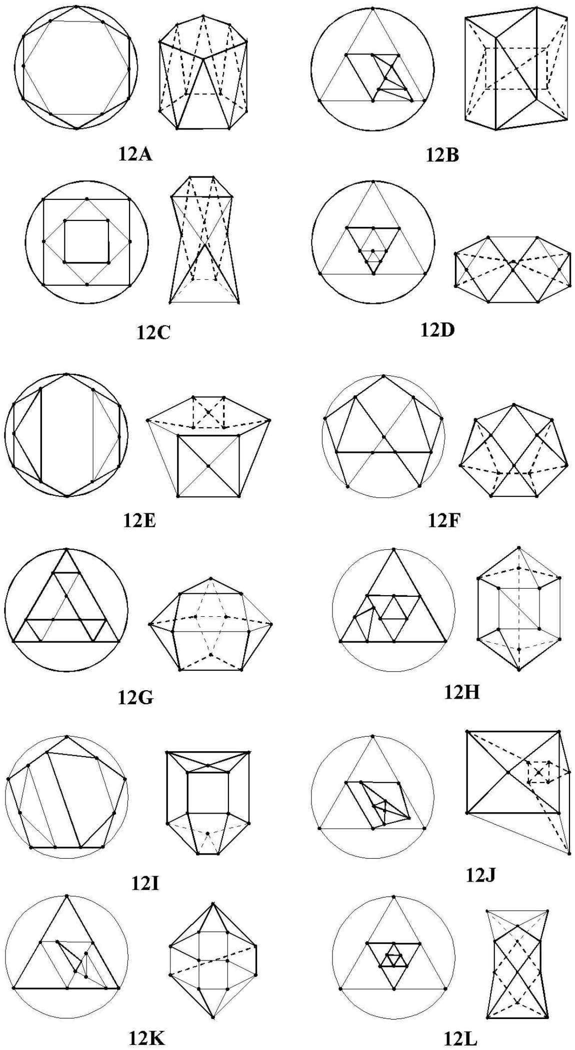

Figure 2.

Table 1

$n = 3$	3	4 6 2	6^*
$n = 5$	212	6 8|2 10 4	8^*
$n = 6$	312	4 8 10 12 2 6	9^*
	6^*	6 8|10 12|2 4	9^*
$n = 7$	$21112'$	4 8 10 12 2 14 6	$10^*, 10^{**}$
	.2	6 8|10 12 14 2 4	$10^{**}, 10^{***}$

$n = 8$	31112	4 10 12 14 2 16 8 6	11^*
	$21212'$	4 10 12 14|8 2 16 6	11^{**}
	$21212'''$	8 10 14|2 16 4 6 12	11^*
	.3	6 8|12 14 16|10 2 4	$11^*, 11^{**}$
	.21	6 8|10 14 12 16 2 4	$11^*, 11^{**}$
	.2.20	6 8 14 12 4 16 2 10	$11^*, 11^{***}$
	8^*	6 8 10 12 14 16 2 4	$11^*, 11^{**}$
	3#212		11^{**}
$n = 9$	31212	4 12 10 16 14 2 18 6 8	12D
	$21312'$	4 10 12 14 18 2 16 6 8	12D
	2111112	4 10 12 14 2 18 16 8 6	12A, 12B, 12F
	$2111112'''$	4 12 10 16 18 2 8 6 14	12B, 12F
	21, 21, 21	8 12 16|2 18 4 10 6 14	12G
	.4	6 8|12 14 16 18 2 4 10	12E
	.31	6 8|10 14 16 18 2 4 12	12J, 12L
	.22	6 8|16 14 12 18 2 4 10	12E
	.211	6 8|12 14 18|16 2 4 10	12B, 12H, 12I, 12J, 12K
	.3.20	8 10 12|14 2 16 18 6 4	12D
	$.21.2'$	4 8 14 12 2 16 18 10 6	12B, 12F, 12H
	2 : 2 : 2	8 12 16|2 14 4 18 6 10	12C
	.(2, 2)	10 12|14 18|6 16 8 2 4	12I
	8^*2	8 10 12|6 14 16 18 2 4	12B, 12F, 12G, 12H, 12I
	8^*20	6 8 10 16 14 18 4 2 12	12F, 12I, 12K
	9^*	6 16 14 12 4 2 18 10 8	12D, 12H, 12L
	212#1#3		12E
	6^*#3		12J

The list of links derived by Kirkman was complete for $n = 8$ crossings. For $n = 9$, two of the links from which 12-vertex basic polyhedra can be derived have been omitted by Kirkman, but even that incomplete list is sufficient for the derivation of all basic polyhedra for $n = 12$. Namely, the missing polyhedron 12E is the only polyhedron that could be derived from the projection of the link .22, denoted by Kirkman as ${}_9\mathrm{B}n$, so the uncomplete result of Kirkman (11 from 12 basic polyhedra derived for $n = 12$) could be just an omission in the process of derivation. The other more probable reason for this omission is that the polyhedron 12E is, among the basic polyhedra, the only 2-connected graph, and all the others are 3-connected, so maybe Kirkman decided to exclude 12E from his list. Except for that omission, the list obtained by Kirkman coincides with the enumeration made by A. Caudron [2], where the complete list of the 12-vertex basic polyhedra was obtained by composing hyperbolic tangles, so we use that list and notation. All the basic polyhedra for $6 \le n \le 10$ and the basic polyhedron 11^* has also been derived by Conway [3].

If we generalize Kirkman's method, introducing not only new triangular, but also p-gonal faces ($p > 3$) in the link diagrams in order to eliminate digons, we could derive in that way all the polyhedra with $6 \le n \le 12$ vertices from the links **3**, **4**, **5**, **6**,... or from their direct products (Fig. 3). In this case we may describe the derivation by the corresponding partitions: 6^* =**3**+3, 8^* =**4**+4, 9^* = 6^*+3 =**3**+3+3, 10^* =**5**+5, 10^{**} =**4**+3+3, 10^{***} =**4**+3 + 3, 11^* = 8^* + 3 =**4**+3 + 3, 11^* = 8^* + 3 =**4**+3 + 3, 11^{**} = 8^* + 3 =**4**+3 + 3, 11^{***} =**3**#**3**+5, 12A=**6**+6, 12B=**3**#**3**+3 + 3, 12C= 8^* + 4 =**4**+4 + 4, 12D= 9^* + 3 =**3**+3 + 3 + 3, 12E=**6**+3 + 3, 12F=**5**+3 + 4, 12G=**5**+3 + 4, 12H= 9^* + 3 =**3**+3 + 3 + 3, 12I=**5**+3 + 4, 12J=**3**#**3**+3 + 3, 12K=**3**#**3**+3 + 3, 12L= 9^* + 3 =**3**+3 + 3 + 3. For the larger values of n, the completeness of such derivation is an open question.

2. BASIC POLYHEDRA WITH $n \le 16$ CROSSINGS

The basis of our computation was the table of 4-regular 3- and 2-connected planar graphs with $n \le 16$ vertices, derived by Brendan McKay [4], generated by the program "plantri.c" created by Gunnar Brinkmann and Brendan McKay (http://cs.anu.edu.au/ bdm/plantri/). From that table we excluded 2-edge connected graphs, and as a result we, obtained the complete list of basic polyhedra with $n \le 16$ crossings given in the form of graphs. Hence, for $6 \le n \le 16$ they are, respectively, 1, 0, 1, 1, 3, 3, 12, 19, 64, 155, 510 basic polyhedra. Among them, they are, respectively, 1, 0, 1, 1, 3, 3, 11, 18, 58, 139, 451 3-connected and 0, 0, 0, 0, 0, 0, 1, 1, 6, 16, 59 2-vertex-connected graphs. The first sequence was computed by M. Dillencourt and included in N. Sloane's *On-Line Encyclopedia of Integer Sequences* [5] as the sequence A007022, and the new sequences that we obtained are now included in the same *Encyclopedia* as A078666 and A078672.

3. NUMBER OF SOURCE LINKS DERIVED FROM BASIC POLYHEDRA WITH $n \leq 16$ CROSSINGS

Links obtained from any basic polyhedron by adding single digons in its vertices we will call *source links*. We propose that term because, from every such link, different infinite families of links can be derived when replacing digons by rational, arborescent, generalized arborescent links, *etc*.

For most of basic polyhedra the number of source links that can be derived is possible to compute by using Pólya Enumeration Theorem (*PET*) [6].

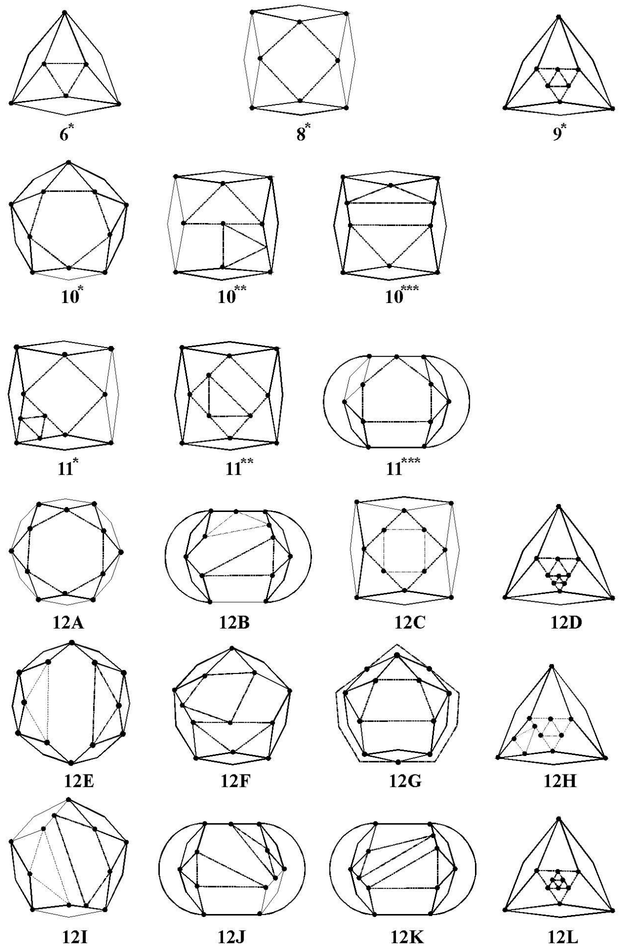

Figure 3.

In fact, it is possible to do so for all basic polyhedra without automorphisms (symmetries) of order 4 preserving invariant a vertex of the polyhedron. In that case, the number of source links derived from a basic polyhedron with n vertices by introducing, respectively, 0, 1, ..., n digons is given by the coefficients of the $Z_G(x,x,1)$, where Z_G is Pólya polynomial. From 769 basic polyhedra existing for $6 \leq n \leq 16$, there are only four of them that we have not been able to compute the number of source links by using *PET*. The basic polyhedra in question are 6^*, 10^{***}, and two basic polyhedra with $n = 14$ vertices (Fig. 4).

The computation of the number of source links with k digons ($0 \leq k \leq n$) derived from a basic polyhedron with n vertices by using *PET* will be illustrated by the example of the basic polyhedron 9^*. Its graph automorphism group $G = [2,3]$ of order 12 is generated by 3-rotation $S = (1,4,7)(2,5,8)(3,6,9)$ and by two reflections: $R = (1)(2,8)(3,6)(4,7)(5)(9)$ which contains the rotation axis, and $R_1 = (1,9)(2)(3,4)(5)(6,7)(8)$ which is perpendicular to it. Hence, $Z_G = \frac{1}{12}(t_1^9 + 4t_1^3t_2^3 + 3t_1t_2^4 + 2t_3^3 + 2t_3t_6)$, the coefficients of $Z_G(x,1) = 1 + 2x + 6x^2 + 12x^3 + 16x^4 + 16x^5 + 12x^6 + 6x^7 + 2x^8 + x^9$ represent, respectively, the number of different symmetry choices of k vertices for $0 \leq k \leq 9$, and the coefficients of $Z_G(x,x,1) = 1 + 4x + 20x^2 + 76x^3 + 202x^4 + 388x^5 + 509x^6 + 448x^7 + 228x^8 + 4x^9$ represent the number of source links with k digons derived from 9^* for $0 \leq k \leq 9$.

The number of all source links derived from the basic polyhedra with $n \leq 16$ vertices, except the four mentioned, is computed in the same way.

4. SOURCE LINKS DERIVED FROM BASIC POLYHEDRA WITH $n \leq 16$ CROSSINGS

The first of them is the regular octahedron 6^*, with the graph automorphism group $G = [3,4]$ of order 48, generated by 4-rotation $S = (1)(2,3,5,6)(4)$, 2-rotation $T = (1,3)(2,5)(4,6)$ and inversion $Z = (1,4)((2,5)(3,6)$. From 6^* we derive source links by substituting its vertices by digons. First we make all the different symmetry choices of k vertices ($0 \leq k \leq 6$), i.e. all different vertex bicolorings of the octahedron. We could find their number by using *PET*. For $G = [3,4]$, $Z_G = \frac{1}{48}(t_1^6 + 3t_1^4t_2 + 9t_1^2t_2^2 + 6t_1^2t_4 + 7t_2^3 + 6t_2t_4 + 8t_3^2 + 8t_6)$, and by the coefficients of $Z_G(x,1) = 1 + x + 2x^2 + 2x^3 + 2x^4 + x^5 + x^6$ is given, respectively, the number of different choices of k vertices for $0 \leq k \leq 6$. For $1 \leq k \leq 6$, that vertex bicolorings are: $\{1\}$; $\{1,2\}$, $\{1,4\}$; $\{1,2,3\}$, $\{1,2,4\}$; $\{1,2,4,5\}$, $\{1,2,3,4\}$; $\{1,2,3,4,5\}$; $\{1,2,3,4,5,6\}$, and to them correspond, respectively, source links of the form $.a$; $.a.b$, $.a:b$; $.a.b.c$, $a:b:c$; $.a.b.c.d$, $a.b.c.d$; $a.b.c.d.e$; $a.b.c.d.e.f$, given in Conway notation. After this is accomplished, we make one of two possible substitutions (2 or 2 0) in every chosen vertex, keeping in mind the symmetry of vertex bicolored octahedron. For $n \leq 12$, the source links obtained from 6^* by the vertex substitutions, are given in Table 2. Among them, for $n = 11$, there is a 3-component link 2.20.2.20.2, omitted in [2].

Table 2

$n = 7$	.2	
$n = 8$	.2.2	
	.2.20	
	.2 : 2	
	.2 : 20	
$n = 9$	.2.2.2	2 : 2 : 2
	.2.2.20	2 : 2 : 20
	.2.20.2	2 : 20 : 20
		20 : 20 : 20
$n = 10$	.2.2.2.2	2.2.2.2
	.2.2.2.20	2.2.2.20
	.2.2.20.20	2.2.20.2
	.2.20.2.20	2.2.20.20
		2.20.2.20
		20.2.2.20
		20.2.20.20
$n = 11$	2.2.2.2.2	
	2.2.2.2.20	
	2.2.2.20.2	
	2.2.2.20.20	
	2.20.2.2.20	
	20.2.2.2.20	
	2.20.2.20.2	
$n = 12$	2.2.2.2.2.2	
	2.2.2.2.2.20	
	2.2.2.2.20.20	
	2.2.2.20.2.20	
	2.2.20.2.2.20	
	2.2.2.20.20.20	
	2.20.2.20.2.20	

Because of the large number of the different choices of vertices and positions of digons placed in them (growing as the number of distinct permutations of the length n of three elements) further derivation "by hand" will be an almost impossible task. This holds for the three remaining basic polyhedra with a larger number of vertices (10, 14 and 14). It is a reason for the development of a computer algorithm which makes possible not only the calculation of the number of source links that can be derived from any basic polyhedron, but also the production of source links in their explicit form.

The theoretical background of that algorithm is the following: every basic link is given by a system of tangles placed in its vertices, this means, by external connections between the vertices of tangles and diagonal connections inside tangles. After that, two internal connections are introduced in every tangle: $(4k-3, 4k)$ and $(4k-2, 4k-1)$ in the case of tangle 2, and $(4k-3, 4k-2)$ and $(4k-1, 4k)$ in the case of the tangle 20. The list of mutually non-isomorphic graphs obtained that way is the list of source links derived from the basic polyhedron considered.

After writing the proposed computer program we hope to be able to complete the calculation of the number of source links derived from the basic polyhedra for $n \leq 16$, and to obtain the list of source links in their explicit form.

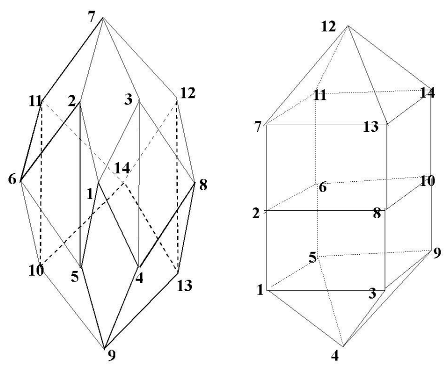

Figure 4.

Acknowledgements. We are very thankful to Brendan McKay for providing us the list of 4-regular 3- and 2-connected planar graphs, that was the basis of this work.

References

[1] T. P. Kirkman, *The enumeration, description and construction of knots of fewer than ten crossings*, Trans. Roy. Soc. Edinburgh **32** (1885), 281–309.

[2] A. Caudron, *Classification des nœuds et des enlancements*, Public. Math. d'Orsay 82. Orsay: Univ. Paris Sud, Dept. Math. (1982).

[3] J. Conway, *An enumeration of knots and links and some of their related properties*, Computational Problems in Abstract Algebra, Proc. Conf. Oxford 1967 (Ed. J. Leech), 329–358. New York: Pergamon Press (1970).

[4] B. McKay, *Personal communication* (2001).

[5] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, http://www.research.com/~njas/sequences/

[6] F. Harary, E. Palmer, *Graphical Enumeration*, New York, London: Academic Press (1973).