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HYPERBOLIC ANGLE FUNCTION IN THE LORENTZIAN PLANE

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Abstract. In this paper, using definitions of oriented hyperbolic angles between non–null vectors, we prove some theorems related to the angles in a triangle in the Lorentzian plane.

1. INTRODUCTION

In the Lorentzian plane L^2, there exist two kinds of non–null vectors: the spacelike and the timelike vectors. The oriented hyperbolic angle between these vectors is not defined in the same way in all cases. In the first case, the hyperbolic angle between two timelike vectors is defined in [1, 2] where the authors studied the basic properties of the hyperbolic angle function. In the next two cases, the hyperbolic angle between two spacelike vectors, as well as between the spacelike and the timelike vector, is explicitely defined in [4], where also the measure of an unoriented hyperbolic angle is given. Some trigonometric relations and hyperbolic sine and cosine laws are obtained in [5].

In this paper, using definitions of oriented hyperbolic angles between non–null vectors, we prove some theorems related to the angles in a triangle in the Lorentzian plane.

2. PRELIMINARIES

The Lorentzian plane L^2 is the vector space R^2 provided with the Lorentzian scalar product given by

$$g(X,Y) := x_1y_1 - x_2y_2,$$

where $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ are two vectors in L^2. Since g is indefinite metric, an arbitrary vector X in L^2 can have one of three causal characters: it can be spacelike if $g(X,X) > 0$ or $X = 0$, timelike if $g(X,X) < 0$, and null if $g(X,X) = 0$, whereby $X \neq 0$. The norm of a vector is given by $||X|| = \sqrt{|g(X,X)|}$. Two vectors X and Y are said to be orthogonal if $g(X,Y) = 0$. The time–orientation in L^2 is defined in the following way. Let $E = (0,1)$ be the unit timelike vector and let $X = (x_1, x_2)$ be non–null vector. Then X is future–pointing if $g(X,E) < 0$, and past–pointing if $g(X,E) > 0$. If X and Y are both future–pointing or past–pointing vectors, then they have the same time–orientation. Recall that the matrix of the hyperbolic rotation in L^2, for some hyperbolic angle $u \in R$, is given by

$$A(u) = \begin{bmatrix} \cosh(u) & \sinh(u) \\ \sinh(u) & \cosh(u) \end{bmatrix}.$$

The matrix of the reflection with respect to the straight line $x_1 = x_2$ in L^2 has the form

$$S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is clear that if X is spacelike (timelike) vector, then $S(X)$ is timelike (spacelike) vector. Next we briefly recall definitions of the oriented hyperbolic angles, obtained in [1, 2, 4]. Let $\angle(X,Y)$ denotes the oriented hyperbolic angle from X to Y.

Definition 1. *Let X and Y be two time-like unit vectors with the same (different) time-orientations. Then $u = \angle(X,Y)$, if $A(u)X = Y$ $(A(u)X = -Y)$.*

This definition implies respectively the following formulae:

$$\cosh(u) = -g(X,Y), \qquad \sinh(u) = -g(X,S(Y)),$$
$$\cosh(u) = g(X,Y), \qquad \sinh(u) = g(X,S(Y)).$$

Definition 2. *Let $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ be two spacelike unit vectors with $sgn\, x_1 = sgn\, y_1$ ($sgn\, x_1 \neq sgn\, y_1$). Then $u = \angle(X,Y)$, if $A(u)X = Y$ ($A(u)X = -Y$).*

From this definition, we obtain similar formulae for $\cosh(u)$ and $\sinh(u)$.

Definition 3. *Let $X = (x_1, x_2)$ be the spacelike unit vector and and $Y = (y_1, y_2)$ be the time-like unit vector, with $sgn\ x_1 = sgn\ y_2$ ($sgn\ x_1 \neq sgn\ y_2$). Then $u = \angle(X,Y)$, if $SA(u)X = Y$ ($SA(u)X = -Y$).*

From this definition, we respectively obtain the formulae:

$$\cosh(u) = g(X,S(Y)), \qquad \sinh(u) = g(X,Y),$$
$$\cosh(u) = -g(X,S(Y)) \qquad \sinh(u) = -g(X,Y).$$

3. SOME RELATIONS BETWEEN ANGLES IN THE TRIANGLE

Recall that the oriented hyperbolic angle function $\angle(\,\cdot\,,\,\cdot\,)$ has the following properties:

(1) $\angle(X,X) = \angle(X,-X) = 0$;

(2) $\angle(X,Y) = -\angle(Y,X)$;

(3) $\angle(X,Y) = \angle(-X,Y) = \angle(X,-Y) = \angle(-X,-Y)$;

(4) $\angle(X,Y) + \angle(Y,Z) = \angle(X,Z)$.

Let us denote the unoriented hyperbolic angle between non-null vectors X and Y by $[X,Y]$. Then the angle $[\cdot,\cdot]$ is defined by $[X,Y] = |\angle(X,Y)|$, where $|\cdot|$ is the absolute value. In [4], the measure of unoriented hyperbolic angle is given by:

$$m[X,Y] = \ln\Big(\frac{|g(X,Y)| + |g(X,S(Y))|}{||X||\,||Y||}\Big);$$

The aim of this paper is to prove the following three theorems.

Theorem 1. *Let $\vec{AB}$, $\vec{AC}$ and $\vec{BC}$ be three spacelike vectors and let $\angle\alpha = \angle(\vec{AB}, \vec{AC})$, $\angle\beta = \angle(\vec{AB}, \vec{BC})$ and $\angle\gamma = \angle(\vec{AC}, \vec{BC})$. Then $\angle\gamma = \angle\beta - \angle\alpha$.*

Proof. Let D be a point on line AB such that $\vec{CD}$ is timelike vector and $g(\vec{AB}, \vec{CD}) = 0$. Then in the triangle ADC there holds $\angle(\vec{AB}, \vec{AC}) = \angle(\vec{CD}, \vec{AC})$ ([5], page 222). Similarly, in the triangle BCD we have $\angle(\vec{AB}, \vec{BC}) = \angle(\vec{CD}, \vec{BC})$. Since

$$\angle(\vec{AC}, \vec{BC}) = \angle(\vec{AC}, \vec{CD}) + \angle(\vec{CD}, \vec{BC}),$$

it follows that

$$\angle\gamma = -\angle(\vec{CD}, \vec{AC}) + \angle(\vec{CD}, \vec{BC}).$$

Therefore,

$$\angle\gamma = \angle\beta - \angle\alpha.$$

Theorem 2. *There exist no triangle in the Lorentzian plane such that $m[\alpha] = m[\beta] = m[\gamma]$.*

Proof. Assume that $m[\alpha] = m[\beta] = m[\gamma]$. Since m is a measure, it follows that $[\alpha] = [\beta] = [\gamma]$, and thus $|\angle\alpha| = |\angle\beta| = |\angle\gamma|$. By theorem 1 we have $|\angle\gamma| = |\angle\beta - \angle\alpha|$. It follows that $|\angle\beta - \angle\alpha| = |\angle\alpha| = |\angle\beta|$. This implies that $\angle\alpha = 0$ or $\angle\beta = 0$, which is a contradiction.

Theorem 3. *Let $A = (a_1, a_2)$ and $B = (b_1, b_2)$ be two spacelike vectors with sgn $a_1 =$ sgn b_1 and let $C = A + B$, $\angle\alpha = \angle(C, B)$, $\angle\beta = \angle(A, C)$ and $\angle\gamma = \angle(A, B)$. Then $C = (c_1, c_2)$ is spacelike vector and $m[\alpha] = m[\beta]$ if and only if $||A|| = ||B||$.*

Proof. Since $C = A + B$, it follows that $c_1 = a_1 + b_1$, and consequently sgn $c_1 =$ sgn $a_1 =$ sgn b_1. If $\angle\gamma = \angle(A, B)$, by definition 2 we have $g(A, B) = ||A||||B|| \cosh(\angle\gamma)$ and hence

$$g(C, C) = g(A, A) + 2g(A, B) + g(B, B) > 0,$$

which means that C is spacelike vector. Let us first assume that $m[\alpha] = m[\beta]$. Since m is the measure, it follows that $[\alpha] = [\beta]$, and therefore $|\angle\alpha| = |\angle\beta|$. The last equation implies $\cosh(\angle\alpha) = \cosh(\angle\beta)$. From definition 2 we obtain that

$$\cosh(\angle\alpha) = \frac{g(B,C)}{||B||||C||}, \quad \cosh(\angle\beta) = \frac{g(A,C)}{||A||||C||},$$

and hence

$$\frac{g(B,C)}{||B||||C||} = \frac{g(A,C)}{||A||||C||}.$$

Putting $C = A + B$, we get

$$||A||g(B, A+B) = ||B||g(A, A+B),$$

and thus

$$||A||(g(A,B) + ||B||^2) = ||B||(g(A,B) + ||A||^2).$$

The last equation implies that

$$(||A|| - ||B||)(g(A,B) - ||A||||B||) = 0,$$

and since $g(A,B) = ||A||||B|| \cosh(\angle\gamma)$, it follows that $||A|| = ||B||$.

Conversely, let us suppose that $||A|| = ||B||$. Then the following equation is satisfied

$$(||A|| - ||B||)(g(A,B) - ||A||||B||) = 0.$$

The previous equation implies that

$$||A||g(B, A+B) = ||B||g(A, A+B).$$

Putting $C = A + B$, we obtain

$$||A||g(B,C) = ||B||g(A,C),$$

and consequently

$$\frac{g(A,C)}{||A||||C||} = \frac{g(B,C)}{||B||||C||}.$$

From this we find $\cosh(\angle\alpha) = \cosh(\angle\beta)$, so $|\angle\alpha| = |\angle\beta|$. Consequently, since $[\alpha] = [\beta]$, it follows that $m[\alpha] = m[\beta]$.

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