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LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITLLEWOOD-PALEY OPERATOR

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Abstract. In this paper, we will study the continuity of multilinear commutator generated by Littlewood-Paley operator and the of function b, wich belongs to Lipschiz space, in the Triebel-Lizorkin, Hardy and Herz-Hardy space.

1. INTRODUCTION

Let T be a Calderón-Zygmund operator. Coifman, Rochberg and Weiss proved [4] that the commutator $[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x)$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ and $b \in BMO$. Chanillo proved a similiar result [2] when T is replaced by the fractional operators. Janson and Paluszynski study these result [7, 15] for the Triebel-Lizorkin spaces and when $b \in Lip_\beta$, where Lip_β is the homogeneous Lipschitz space. The main purpose of this paper is to discuss the boundedness of multilinear commutator generated by Littlewood-Paley operator and continuity of $b \in Lip_\beta$ in the Triebel-Lizorkin, Hardy and Herz-Hardy space.

$M(f)$ denotes the Hardy-Littlewood maximal function of f and $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. Q denotes a cube of R^n with side parallel to the axes. Let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q|dy$. Mark Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \leq 1)$ has the atomic decomposition [11, 16, 17]. For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta,\infty}$ be the homogeneous Tribel-Lizorkin space. The Lipschitz space $Lip_\beta(R^n)$ is the space of functions f such that

$$||f||_{Lip_\beta} = \sup_{\substack{x,y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty.$$

Lemma 1. [15] *For* $0 < \beta < 1$ *and* $1 < p < \infty$,

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left\| \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x) - f_Q|dx \right\|_{L^p} \\ &\approx \left\| \sup_{\cdot \in Q} \inf_c \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x) - c|dx \right\|_{L^p}. \end{aligned}$$

Lemma 2. [15] *For* $0 < \beta < 1$ *and* $1 \leq p \leq \infty$,

$$\begin{aligned} ||f||_{Lip_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x) - f_Q|dx \\ &\approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p}. \end{aligned}$$

Lemma 3. [2] *For* $1 \leq r < \infty$ *and* $\beta > 0$, *let*

$$M_{\beta,r}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\frac{\beta r}{n}}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

if we suppose that $r < p < \beta/n$, *and* $1/q = 1/p - \beta/n$, *then*

$$||M_{\beta,r}(f)||_{L^q} \leq C||f||_{L^p}.$$

Lemma 4. [5] *If* $Q_1 \subset Q_2$ *then*

$$|f_{Q_1} - f_{Q_2}| \leq C||f||_{\dot{\Lambda}_\beta}|Q_2|^{\beta/n}.$$

Lemma 5. [10] *Let* $0 < \beta \leq 1$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ *and* $b \in Lip_\beta(R^n)$. *Then* g_ψ^b *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$.

Definition 1. *Let* $0 < p,\ q < \infty$, $\alpha \in R$, $B_k = \{x \in R^n, |x| \leq 2^k\}$, $A_k = B_k \backslash B_{k-1}$ *and* $\chi_k = \chi_{Ak}$ *for* $k \in \mathbf{Z}$.

1) The homogeneous Herz space is defined by

$$\dot{K}_q^{\alpha,p} = \{f \in L^q_{Loc}(R^n \backslash \{0\}), ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q}\right]^{1/p}.$$

2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f \in L^q_{Loc}(R^n), ||f||_{K_q^{\alpha,q}(R^n)} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}(R^n)} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||^p_{L^q} + ||f\chi_{B_0}||^p_{L^q}\right]^{1/p}.$$

Definition 2. *Let* $\alpha \in R$ *and* $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}}$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the characterization of the atomic decomposition.

Definition 3. *Let $\alpha \in R$ and $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if*

1) $Supp a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),

2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,

3) $\int_{R^n} a(x)x^\eta dx = 0$ for $|\eta| \leq [\alpha - n(1 - 1/q)]$.

Lemma 6. [6, 14] *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1 - 1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$(or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms(or central (α, q)-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$)in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}}(or\ ||f||_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Definition 4. *Let $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:*

1) $\int_{R^n} \psi(x)dx = 0$,

2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1)}$,

3) $|\psi(x + y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon)}$ when $2|y| < |x|$.

Let m be a positive integer and $b_j (1 \leq j \leq m)$ be locally integrable functions and $\vec{b} = (b_1, \cdots, b_m)$. The multilinear commutator of Littlewood-Paley operator is defined *by*

$$g_\psi^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(x)|^2 \frac{dt}{t}\right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y))\psi_t(x-y)f(y)dy,$$

and $\psi_t(x) = t^{-n}\psi(x/t)$ *for* $t > 0$. *Let* $F_t(f) = \psi_t * f$. *Define Littlewood-Paley g function by [17]*

$$g_\psi(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t}\right)^{1/2}.$$

Let H be the space, $H = \{h : ||h|| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2} < \infty\}$. For each fixed $x \in R^n$, $F_t(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_\psi(f)(x) = ||F_t(f)(x)|| \text{ and } g_\psi^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $g_\psi^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors ([1]-[4][7]-[10][12][15]). Our main purpose is to establish the boundedness of the multilinear commutator on Triebel-Lizorkin, Hardy and Herz-Hardy space.

Let m be a positive integer, $1 \le j \le m$, $||\vec{b}||_{Lip_\beta} = \prod_{j=1}^m ||b_j||_{Lip_\beta}$ and C_j^m be the family of all finite subsets $\sigma = \{\sigma(1), \ldots, \sigma(j)\}$ of $\{1, \ldots, m\}$ of j different elements. For $\sigma \in C_j^m$, let $\sigma^c = \{1, \ldots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \ldots, b_m)$ and $\sigma = \{\sigma(1), \ldots, \sigma(j)\} \in C_j^m$, let $\vec{b}_\sigma = (b_{\sigma(1)}, \ldots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{Lip_\beta} = ||b_{\sigma(1)}||_{Lip_\beta} \cdots ||b_{\sigma(j)}||_{Lip_\beta}$.

3. THEOREMS AND PROOFS

Theorem 1. *Let* $0 < \beta < min(1, \varepsilon/m)$, $1 < p < \infty$, $\vec{b} = (b_1, \ldots, b_m)$ *where* $b_j \in Lip_\beta(R^n)$ *for* $1 \le j \le m$ *and* $g_\psi^{\vec{b}}$ *be the multilinear commutator of Littlewood-Paley operator. Then* *(a)* $g_\psi^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $\dot{F}_p^{m\beta,\infty}(R^n)$.

(b) $g_\psi^{\vec{b}}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for* $1/p-1/q=m\beta/n$ *and* $1/p>m\beta/n$.

Proof. (a) Fix a cube $Q=(x_0,l)$ and $x\in Q$, when $m=1$ [10]. Now consider the case when $m\geq 2$. Let $\vec{b}_Q=((b_1)_Q,\ldots,(b_m)_Q)$, where $(b_j)_Q=|Q|^{-1}\int_Q b_j(y)dy$ and $1\leq j\leq m$. If $f=f_1+f_2$ where $f_1=f\chi_{2Q}$ and $f_2=f\chi_{R^n\setminus 2Q}$, then

$$
\begin{aligned}
F_t^{\vec{b}}(f)(x) &= \int_{R^n}(b_1(x)-b_1(y))\cdots(b_m(x)-b_m(y))\psi_t(x-y)f(y)dy\\
&= (b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(x)\\
&\quad+(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f)(x)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-\vec{b}_Q)_\sigma\int_{R^n}(b(y)-\vec{b}_Q)_{\sigma^c}\psi_t(x-y)f(y)dy\\
&= (b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(x)\\
&\quad+(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)\\
&\quad+(-1)^m F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-\vec{b}_Q)_\sigma F_t((b-\vec{b}_Q)_{\sigma^c}f)(x).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&|g_\psi^{\vec{b}}(f)(x)-g_\psi(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)|\\
&\leq\ ||F_t^{\vec{b}}(f)(x)-F_t(((b_1)_Q-b_1)\cdots((b_m)_Q-b_m)f_2)(x_0)||\\
&\leq\ ||(b_1(x)-(b_1)_Q)\cdots(b_m(x)-(b_m)_Q)F_t(f)(x)||\\
&\quad+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||(b(x)-\vec{b}_Q)_\sigma F_t((b-\vec{b}_Q)_{\sigma^c}f)(x)||\\
&\quad+||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)||\\
&\quad+||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)\\
&\qquad\qquad-F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x_0)||\\
&=\ I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q |g_\psi^{\vec{b}}(f)(x) - g_\psi((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)|dx \\
&\quad\le \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_1(x)dx + \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_2(x)dx + \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_3(x)dx \\
&\qquad + \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_4(x)dx \\
&\quad= I + II + III + IV.
\end{aligned}
$$

By using Lemma 2, we have for I,

$$
\begin{aligned}
I &\le \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\sup_{x\in Q}|b_1(x) - (b_1)_Q|\cdots|b_m(x) - (b_m)_Q|\int_Q |g_\psi(f)(x)|dx \\
&\le C||\vec{b}||_{Lip_\beta}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{\frac{m\beta}{n}}\int_Q |g_\psi(f)(x)|dx \\
&\le C||\vec{b}||_{Lip_\beta}M(g_\psi(f))(x).
\end{aligned}
$$

Fix r, such that $1 < r < p$. Let μ, μ' be the integers such that $\mu + \mu' = m$, $0 \le \mu < m$ and $0 < \mu' \le m$. By using Hölder's inequality, the boundedness of g_ψ on L^r and Lemma 2, we get

$$
\begin{aligned}
II &\le \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma||g_\psi((\vec{b} - \vec{b}_Q)_{\sigma^c}f)(x)|dx \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\Big(\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{r'}dx\Big)^{1/r'}\Big(\int_Q |g_\psi((\vec{b} - \vec{b}_Q)_{\sigma^c}f)(x)|^r dx\Big)^{1/r} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\Big(\int_Q |(\vec{b}(x) - \vec{b}_Q)_\sigma|^{r'}dx\Big)^{1/r'}\Big(\int_Q |(\vec{b}(x) - \vec{b}_Q)_{\sigma^c}f(x)|^r dx\Big)^{1/r} \\
&\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{\frac{1}{r'}}||\vec{b}_\sigma||_{Lip_\beta}|Q|^{\frac{\mu\beta}{n}}||\vec{b}_{\sigma^c}||_{Lip_\beta}|Q|^{\frac{\mu'\beta}{n}}\Big(\int_Q |f(x)|^r dx\Big)^{1/r} \\
&\le C||\vec{b}||_{Lip_\beta}M_r(f)(x);
\end{aligned}
$$

By Hölder's inequality, we have for III

$$
III = \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q |g_\psi((b_1 - (b_1)_Q)\cdots(b_m - (b_m)_Q)f_1)(x)|dx
$$

$$
\begin{aligned}
&\leq C\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\left(\int_{R^n}|g_\psi(\prod_{j=1}^m(b_j-(b_j)_Q)f_1)(x)|dx\right)^{1/r}|Q|^{1-1/r}\\
&\leq C\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{1-1/r}\left(\int_{2Q}|\prod_{j=1}^m(b_j(x)-(b_j)_Q)f(x)|^r dx\right)^{1/r}\\
&\leq C\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{1-1/r}||\vec{b}||_{Lip_\beta}|Q|^{\frac{m\beta}{n}}\left(\int_{2Q}|f(x)|^r dx\right)^{1/r}\\
&\leq C||\vec{b}||_{Lip_\beta}M_r(f)(x);
\end{aligned}
$$

Since $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$, using by Lemma 4 and the condition of ψ, we have for IV,

$$
\begin{aligned}
&||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)-F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x_0)||\\
&\leq \left[\int_0^\infty(\int_{(2Q)^c}|\psi_t(x-y)-\psi_t(x_0-y)||f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy)^2\frac{dt}{t}\right]^{1/2}\\
&\leq C\left[\int_0^\infty\left(\int_{(2Q)^c}\frac{t|x-x_0|^\varepsilon}{(t+|x_0-y|)^{n+1+\varepsilon}}|f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy\right)^2\frac{dt}{t}\right]^{1/2}\\
&\leq C\int_{(2Q)^c}|x_0-x|^\varepsilon|x_0-y|^{-(n+\varepsilon)}|f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy\\
&\leq C\sum_{k=1}^\infty\int_{2^{k+1}Q\setminus 2^kQ}|x_0-x|^\varepsilon|x_0-y|^{-(n+\varepsilon)}|f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy\\
&\leq C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{-1}\int_{2^{k+1}Q}|f(y)|\prod_{j=1}^m(|b_j(y)-(b_j)_{2^{k+1}Q}|+|(b_j)_{2^{k+1}}-(b_j)_Q|)dy\\
&\leq C\sum_{k=1}^\infty 2^{-k\varepsilon}|2^{k+1}Q|^{\frac{m\beta}{n}}||\vec{b}||_{Lip_\beta}M(f)\\
&\leq C||\vec{b}||_{Lip_\beta}|Q|^{\frac{m\beta}{n}}M(f)\sum_{k=1}^\infty 2^{(m\beta-\varepsilon)k}\\
&\leq C||\vec{b}||_{Lip_\beta}|Q|^{\frac{m\beta}{n}}M(f).
\end{aligned}
$$

The following holds

$$IV\leq C||\vec{b}||_{Lip_\beta}M(f).$$

Putting these estimates together, taking the supremum over all Q such that $x\in Q$

and by using Lemma 1, we obtain

$$||g_\psi^{\vec{b}}(f)(x)||_{\dot{F}_p^{m\beta,\infty}} \le C||\vec{b}||_{Lip_\beta}||f||_{L^p}.$$

This complete the proof of (a).

(b) Like in the proof of (a), we have

$$\begin{aligned}
& \frac{1}{|Q|}\int_Q |g_\psi^{\vec{b}}(f)(x) - g_\psi(((b_1)_Q - b_1)\cdots((b_m)_Q - b_m)f_2)(x_0)|dx \\
\le\ & \frac{1}{|Q|}\int_Q I_1(x)dx + \frac{1}{|Q|}\int_Q I_2(x)dx + \frac{1}{|Q|}\int_Q I_3(x)dx + \frac{1}{|Q|}\int_Q I_4(x)dx \\
\le\ & C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(g_\psi(f)) + M_{m\beta,r}(f) + M_{m\beta,r}(f) + M_{m\beta,1}(f)).
\end{aligned}$$

Thus

$$(g_\psi^{\vec{b}}(f))^\# \le C||\vec{b}||_{Lip_\beta}(M_{m\beta,1}(g_\psi(f)) + M_{m\beta,r}(f) + M_{m\beta,1}(f)).$$

By using Lemma 3 and the boundedness of g_ψ, we have

$$\begin{aligned}
||g_\psi^{\vec{b}}(f)||_{L^q} &\le C||(g_\psi^{\vec{b}}(f))^\#||_{L^q} \\
&\le C||\vec{b}||_{Lip_\beta}(||M_{m\beta,1}(g_\psi(f))||_{L^q} + ||M_{m\beta,r}(f)||_{L^q} + ||M_{m\beta,1}(f)||_{L^q}) \\
&\le C||f||_{L^p}.
\end{aligned}$$

This completes the proof of (b).

Theorem 2. *Let $0 < \beta \le 1$, $\max(n/(n+m\beta), n/(n+m\varepsilon)) < p \le 1$, $1/q = 1/p - m\beta/n$ and $\vec{b} = (b_1, \ldots, b_m)$ where $b_j \in Lip_\beta(R^n)$ for $1 \le j \le m$. Then $g_\psi^{\vec{b}}$ is bounded from $H^p(R^n)$ to $L^q(R^n)$.*

Proof. It is enough to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||g_\psi^{\vec{b}}(a)||_{L^q} \le C.$$

Let a be a H^p-atom supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \le |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \le [n(1/p-1)]$.

When $m = 1$ see [10]. Now consider the case $m \ge 2$.

$$\begin{aligned}
||g_\psi^{\vec{b}}(a)(x)||_{L^q} &\le \left(\int_{|x-x_0|\le 2r} |g_\psi^{\vec{b}}(a)(x)|^q dx\right)^{1/q} + \left(\int_{|x-x_0|>2r} |g_\psi^{\vec{b}}(a)(x)dx|^q\right)^{1/q} \\
&= I + II.
\end{aligned}$$

Choose $1 < p_1 < 1/\beta$ and q_1 such that $1/q_1 = 1/p_1 - \beta/n$. By the boundednss of $g_\psi^{\vec{b}}$ from $L^{p_1}(R^n)$ to $L^{q_1}(R^n)$(see Lemma 5), we get for I

$$I \leq C||g_\psi^{\vec{b}}(a)||_{L^{q_1}} r^{n(\frac{1}{q}-\frac{1}{q_1})} \leq C||a||_{L^{q_1}} r^{n(\frac{1}{q}-\frac{1}{q_1})} \leq C.$$

Let $\tau, \tau' \in N$ such that $\tau + \tau' = m$, and $\tau' \neq 0$. We get for II

$$\begin{aligned}
|F_t^{\vec{b}}(a)(x)| &\leq |(b_1(x) - b_1(x_0)) \cdots (b_m(x) - b_m(x_0)) \int_B (\psi_t(x-y) - \psi_t(x-x_0))a(y)dy| \\
&\quad + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} |(b(x) - b(x_0))_{\sigma^c} \int_B (b(y) - b(x_0))_\sigma \psi_t(x-y)a(y)dy| \\
&\leq C||\vec{b}||_{Lip_\beta} |x - x_0|^{m\beta} \cdot \int_B |\psi_t(x-y) - \psi_t(x-x_0)||a(y)|dy \\
&\quad + C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x - x_0|^{\tau\beta} \int_B |y - x_0|^{\tau'\beta} |\psi_t(x-y)||a(y)|dy \\
&\leq C||\vec{b}||_{Lip_\beta} \frac{|x - x_0|^{m\beta} t}{(t + |x - x_0|)^{n+1+\varepsilon}} \int_B |x_0 - y|^\varepsilon |a(y)|dy \\
&\quad + C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x - x_0|^{\tau\beta} \frac{t}{(t + |x - x_0|)^{n+1}} \int_B |y - x_0|^{\tau'\beta} |a(y)|dy \\
&\leq C||\vec{b}||_{Lip_\beta} \frac{t}{(t + |x - x_0|)^{n+1+\varepsilon}} \cdot r^{m\beta+\varepsilon+n(1-\frac{1}{p})} \\
&\quad + C||\vec{b}||_{Lip_\beta} \frac{t}{(t + |x - x_0|)^{n+1}} \cdot r^{m\beta+n(1-\frac{1}{p})}.
\end{aligned}$$

Thus

$$\begin{aligned}
|g_\psi^{\vec{b}}(a)(x)| &\leq C||\vec{b}||_{Lip_\beta} \left(\int_0^\infty \left(\frac{t}{(t + |x - x_0|)^{n+1+\varepsilon}} \right)^2 \frac{dt}{t} \right)^{1/2} \cdot r^{m\beta+\varepsilon+n(1-\frac{1}{p})} \\
&\quad + C||\vec{b}||_{Lip_\beta} \left(\int_0^\infty \left(\frac{t}{(t + |x - x_0|)^{n+1}} \right)^2 \frac{dt}{t} \right)^{1/2} \cdot r^{m\beta+n(1-\frac{1}{p})} \\
&\leq C||\vec{b}||_{Lip_\beta} |x - x_0|^{-n} \cdot r^{m\beta+n(1-\frac{1}{p})},
\end{aligned}$$

so

$$\begin{aligned}
II &\leq C||\vec{b}||_{Lip_\beta} \cdot r^{m\beta+n(1-\frac{1}{p})} \left(\int_{|x-x_0|>2r} |x - x_0|^{-nq} dx \right)^{1/q} \\
&\leq C||\vec{b}||_{Lip_\beta}.
\end{aligned}$$

This completes the proof of Theorem 2.

Theorem 3. *Let $0<\beta\le 1$, $0<p<\infty$, $1<q_1,q_2<\infty$, $1/q_1-1/q_2=m\beta/n$, $n(1-1/q_1)\le\alpha<n(1-1/q_1)+m\beta$ and $\vec{b}=(b_1,\dots,b_m)$ where $b_j\in Lip_\beta(R^n)$ for $1\le j\le m$. Then $g_\psi^{\vec{b}}$ is bounded from $H\dot{K}_{q_1}^{\alpha,p}(R^n)$ to $\dot{K}_{q_2}^{\alpha,p}$.*

Proof. Let $f\in H\dot{K}_{q_1}^{\alpha,p}(R^n)$ and $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j,\ suppa_j\subset B_j=B(0,2^j),\ a_j$ be a central $(\alpha,q)-$atom, and $\sum_{j=-\infty}^{\infty}|\lambda_j|^p<\infty$ (Lemma 6).

When $m=1$, we have

$$\begin{aligned}||g_\psi^{b_1}||^p_{\dot{K}_{q_2}^{\alpha,p}} &\le C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\cdot||g_\psi^{b_1}(a_j)\chi_k||_{L^{q_2}}\right)^p\\ &\quad + C\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\cdot||g_\psi^{b_1}(a_j)\chi_k||_{L^{q_2}}\right)^p\\ &= I_1+II_2.\end{aligned}$$

By the boundedness of $g_\psi^{b_1}$ on (L^{q_1},L^{q_2}), we have for II_2

$$\begin{aligned}II_2 &\le C||b_1||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|\cdot||a_j||_{L^{q_1}}\right)^p\\ &\le C||b_1||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|^p\cdot 2^{-j\alpha}\right)^p\\ &\le C||b_1||^p_{Lip_\beta}\begin{cases}\sum_{k=-\infty}^{\infty}\sum_{j=k-1}^{\infty}|\lambda_j|^p\cdot 2^{(k-j)\alpha p}, & 0<p\le 1\\ \sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=k-1}^{\infty}|\lambda_j|^p\cdot 2^{-j\alpha p/2}\right)\left(\sum_{j=k-1}^{\infty}2^{-j\alpha p'/2}\right)^{p/p'}, & 1<p<\infty\end{cases}\\ &\le C||b_1||^p_{Lip_\beta}\begin{cases}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=-\infty}^{j+1}2^{(k-j)\alpha p}, & 0<p\le 1\\ \sum_{k=-\infty}^{\infty}\left(\sum_{j=k-1}^{\infty}|\lambda_j|^p\cdot 2^{\frac{1}{2}(k-j)\alpha p}\right)\left(\sum_{j=k-1}^{\infty}2^{\frac{p'}{2}(k-j)\alpha}\right)^{p/p'}, & 1<p<\infty\end{cases}\\ &\le C||b_1||^p_{Lip_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p.\end{aligned}$$

For I_1, we have

$$\begin{aligned}|F_t^{b_1}(a_j)(x)| &\le |(b_1(x)-b_1(0))\int_{B_j}(\psi_t(x-y)-\psi_t(x))a_j(y)dy|\\ &\quad +|\int_{B_j}\psi_t(b_1(y)-b_1(0))a_j(y)dy|\\ &\le C||b_1||_{Lip_\beta}\left[\int_{B_j}\frac{|x|^\beta|y|^\varepsilon t}{(t+|x|)^{n+1+\varepsilon}}\cdot|a_j(y)|dy\right.\\ &\quad \left.+\int_{B_j}\frac{t|y|^\beta}{(t+|x-y|)^{n+1}}\cdot|a_j(y)|dy\right]\end{aligned}$$

$$\begin{aligned}
&\leq C||b_1||_{Lip_\beta}\left[\frac{|x|^\beta t}{(t+|x|)^{n+1+\varepsilon}}\int_{B_j}|y|^\varepsilon|a_j(y)|dy\right.\\
&\quad\left.+\frac{t}{(t+|x|)^{n+1}}\int_{B_j}|y|^\varepsilon|a_j(y)|dy\right]\\
&\leq C||b_1||_{Lip_\beta}\left[\frac{|x|^\beta t}{(t+|x|)^{n+1+\varepsilon}}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\right.\\
&\quad\left.+\frac{t}{(t+|x|)^{n+1}}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right],
\end{aligned}$$

Thus

$$\begin{aligned}
g_\psi^{b_1}(a_j)(x) &\leq C||b_1||_{Lip_\beta}\left[\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1+\varepsilon}}\right)^2\right)^{1/2}\cdot|x|^\beta\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\right.\\
&\quad\left.+\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1}}\right)^2\frac{dt}{t}\right)^{1/2}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right]\\
&\leq C||b_1||_{Lip_\beta}\left[|x|^{-(n+\varepsilon)}\cdot|x|^\beta\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\right.\\
&\quad\left.\cdot|x|^{-n}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\right]\\
&\leq C||b_1||_{Lip_\beta}|x|^{-n}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}.
\end{aligned}$$

From that, we have

$$\begin{aligned}
||g_\psi^{b_1}(a_j)\chi_k||_{L^{q_2}} &\leq C||b_1||_{Lip_\beta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\left(\int B_k|x|^{-nq_2}dx\right)^{1/q_2}\\
&\leq C||b_1||_{Lip_\beta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot 2^{-kn(1-\frac{1}{q_2})}\\
&\leq C||b_1||_{Lip_\beta}\cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]},
\end{aligned}$$

so

$$\begin{aligned}
I_1 &\leq C||b_1||^p_{Lip_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|\cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right)^p\\
&\leq C||b_1||^p_{Lip_\beta}\begin{cases}\sum_{k=-\infty}^{\infty}\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\leq 1\\ \sum_{k=-\infty}^{\infty}2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{\frac{p}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right) & \\ \quad\times\left(\sum_{j=-\infty}^{k-2}2^{\frac{p'}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\right)^{p/p'}, & 1<p<\infty\end{cases}\\
&\leq C||b_1||^p_{Lip_\beta}\begin{cases}\sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\leq 1\\ \sum_{j=-\infty}^{\infty}|\lambda_j|^p\sum_{k=j+2}^{\infty}2^{\frac{p}{2}[(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)]}, & 1<p<\infty\end{cases}
\end{aligned}$$

$$\leq \; C||b_1||^p_{Lip_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$$

Then

$$||g^{b_1}_\psi(f)||_{\dot{K}^{\alpha,p}_{q_2}} \leq C||b_1||_{Lip_\beta}(\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p} \leq C||f||_{H\dot{K}^{\alpha,p}_{q_1}}.$$

When $m \geq 2$, we have

$$\begin{aligned} ||g^{\vec{b}}_\psi(f)||^p_{\dot{K}^{\alpha,p}_{q_2}} &\leq C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2} |\lambda_j|||g^{\vec{b}}_\psi(a_j)\chi_k||_{L^{q^2}}\right)^p \\ &\quad + C\sum_{k=-\infty}^{\infty} 2^{k\alpha p}\left(\sum_{j=k-1}^{\infty} |\lambda_j|||g^{\vec{b}}_\psi(a_j)\chi_k||_{L^{q_2}}\right)^p \\ &= I + II. \end{aligned}$$

By the boundedness of $g^{\vec{b}}_\psi$ on (L^{q_1}, L^{q_2}), we have for II

$$\begin{aligned} II &\leq C||\vec{b}||^p_{Lip_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p}(\sum_{j=k-1}^{\infty} |\lambda_j|||a_j||_{L^{q_1}})^p \\ &\leq C||\vec{b}||^p_{Lip_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p}(\sum_{j=k-1}^{\infty} |\lambda_j| \cdot 2^{-j\alpha})^p \\ &\leq C||\vec{b}||^p_{Lip_\beta} \begin{cases} \sum_{k=-\infty}^{\infty}\sum_{j=k-1}^{\infty} |\lambda_j|^p \cdot 2^{(k-j)\alpha p}, & 0 < p \leq 1 \\ \sum_{k=-\infty}^{\infty} 2^{k\alpha p}(\sum_{j=k-1}^{\infty} |\lambda_j|^p \cdot 2^{-j\alpha p/2})(\sum_{j=k-1}^{\infty} 2^{-j\alpha p'/2})^{p/p'}, & 1 < p < \infty \end{cases} \\ &\leq C||\vec{b}||^p_{Lip_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p. \end{aligned}$$

For I, we have

$$\begin{aligned} |F^{\vec{b}}_t(a_j))(x)| &\leq |(b_1(x) - b_1(0)) \cdots (b_m(x) - b_m(0)) \int_{B_j} (\psi_t(x-y) - \psi_t(x))a_j(y)dy| \\ &\quad + \sum_{j=1}^{\infty} \sum_{\sigma \in C^m_j} |(b(x) - b(0))_{\sigma^c} \int_{B_j} (b(y) - b(0))_\sigma \psi_t(x-y)a_j(y)dy| \\ &\leq C||\vec{b}||_{Lip_\beta}|x|^{m\beta} \int_{B_j} |\psi_t(x-y) - \psi_t(x)||a_j(y)|dy \\ &\quad + C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} \int_{B_j} |y|^{\tau'\beta}|\psi_t(x-y)||a_j(y)|dy \\ &\leq C||\vec{b}||_{Lip_\beta} \frac{|x|^{m\beta}t}{(t+|x|)^{n+1+\varepsilon}} \int_{B_j} |y|^\varepsilon |a_j(y)|dy \\ &\quad + C||\vec{b}||_{Lip_\beta} \sum_{\tau+\tau'=m} \frac{|x|^{\tau\beta}t}{(t+|x|)^{n+1}} \int_{B_j} |y|^{\tau'\beta}|a_j(y)|dy \end{aligned}$$

$$
\begin{aligned}
&\leq C\|\vec{b}\|_{Lip_\beta}\frac{|x|^{m\beta}t}{(t+|x|)^{n+1+\varepsilon}}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\\
&\quad + C\|\vec{b}\|_{Lip_\beta}\sum_{\tau+\tau'=m}\frac{|x|^{\tau\beta}t}{(t+|x|)^{n+1}}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
g_\psi^{\vec{b}}(a_j)(x) &= \left(\int_0^\infty |F_t^{\vec{b}}(a_j)(x)|^2\frac{dt}{t}\right)^{1/2}\\
&\leq C\|\vec{b}\|_{Lip_\beta}|x|^{m\beta}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\cdot\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1+\varepsilon}}\right)^2\frac{dt}{t}\right)^{1/2}\\
&\quad + C\|\vec{b}\|_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot\left(\int_0^\infty\left(\frac{t}{(t+|x|)^{n+1}}\right)^2\frac{dt}{t}\right)^{1/2}\\
&\leq C\|\vec{b}\|_{Lip_\beta}|x|^{m\beta}|x|^{-(n+\varepsilon)}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\\
&\quad + C\|\vec{b}\|_{Lip_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}|x|^{-n}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)}\\
&\leq C\|\vec{b}\|_{Lip_\beta}|x|^{-n}\cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\|g_\psi^{\vec{b}}(a_j)\chi_k\|_{L^{q_2}} &\leq C\|\vec{b}\|_{Lip_\beta}\cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot\left(\int_{B_j}|x|^{-nq_2}dx\right)^{1/q_2}\\
&\leq C\|\vec{b}\|_{Lip_\beta}\cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]},
\end{aligned}
$$

so

$$
\begin{aligned}
I &\leq C\|\vec{b}\|_{Lip_\beta}^p\sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|\cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\right)^p\\
&\leq C\|\vec{b}\|_{Lip_\beta}^p\begin{cases}\sum_{k=-\infty}^\infty\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{(j-k)(m\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\leq 1\\ \sum_{k=-\infty}^\infty 2^{k\alpha p}\left(\sum_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{\frac{p}{2}[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\right) & \\ \quad\times\left(\sum_{j=-\infty}^{k-2}2^{\frac{p'}{2}[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\right)^{p/p'} & ,1<p<\infty\end{cases}\\
&\leq C\|\vec{b}\|_{Lip_\beta}^p\sum_{j=-\infty}^\infty|\lambda_j|^p.
\end{aligned}
$$

From I and II, we have

$$
\|g_\psi^{\vec{b}}(f)\|\leq C\|\vec{b}\|_{Lip_\beta}\left(\sum_{j=-\infty}^\infty|\lambda_j|^p\right)^{1/p}\leq C\|f\|_{H\dot{K}_{q_1}^{\alpha,p}}.
$$

This completes the proof of Theorem 3.

References

[1] J. Alvarez, R. J. Babgy, D. S. Kurtz, C. Perez, *Weighted estimates for commutators of linear operators*, Studia Math., **104** (1993), 195-209.

[2] S. Chanillo, *A not on commutators*, Indiana Univ Math. J, **31** (1982), 7-16.

[3] W. G. Chen, *Besov estimates for a class of multilinear singular integrals*, Acta Math. Sinica, **16** (2000), 613-626.

[4] R. Coifman, R. Rochberg, G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), 611-635.

[5] R. A. Devore, R. C. Sharply, *Maximal functions measuring smoothness*, Mem. Amer. Math. Soc., **47** (1984).

[6] J. Garcia-Cuerva, M. J. L. Herrero, *A theory of Hardy spaces associated to Herz spaces*, Proc. London Math. Soc., **69** (1994), 605-628.

[7] S. Janson, *Mean Oscillation and commutators of singular integral operators*, Ark. Math., **16** (1978), 263-270.

[8] L. Z. Liu, *Boundedness of multilinear operator on Triebel-Lizorkin spaces*, Inter J. of Math. and Math. Sci., **5** (2004), 259-271.

[9] L. Z. Liu, *The continuity of commutators on Triebel-Lizorkin spaces*, Integral Equations and Operator Theory, **49** (2004), 65-76.

[10] L. Z. Liu, *Boundedness for multilinear Littlewood-Paley operators on Hardy and Herz-Hardy spaces*, Extracta Math., **19** (2)(2004), 243-255.

[11] S. Z. Lu, *Four lectures on real H^p spaces*, World Scientific, River Edge, NI, 1995.

[12] S. Z. Lu, Q. Wu, D. C. Yang, *Boundedness of commutators on Hardy type spaces*, Sci. in China (ser. A), **45** (2002), 984-997.

[13] S. Z. Lu, D. C. Yang, *The decomposition of the weighted Herz spaces and its applications*, Sci. in China (ser. A), **38** (1995), 147-158.

[14] S. Z. Lu, D. C. Yang, *The weighted Herz type Hardy spaces and its applications*, Sci. in China (ser. A), **38** (1995), 662-673.

[15] M. Paluszynski, *Characterization of the Besov spaces via the commutator operator of Coifman*, Rochbeg and Weiss, Indiana Univ. Math. J., **44** (1995), 1-17.

[16] E. M. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton: Princeton Univ Press (1993).

[17] A. Torchinsky, *The real variable methods in harmonic analysis*, Pure and Applied Math., **123**, Academic Press, New York (1986).