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ON NEGATIVELY SKEWED EXTENDED GENERALIZED LOGISTIC DISTRIBUTION

A. K. Olapade

Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria
(e-mail: akolapad@oauife.edu.ng)

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Abstract. In this paper, we considered a form of generalized logistic distribution which is call negatively skewed extended generalized logistic distribution or extended type II generalized logistic distribution. Some theorems that relate the distribution to some other statistical distributions are established. A possible application of one of the theorems is included.

1. INTRODUCTION

The probability density function of a random variable that has logistic distribution is

$$f_X(x) = \frac{e^{-x}}{(1+e^{-x})^2}, \quad -\infty < x < \infty, \tag{1.1}$$

the corresponding cumulative distribution function is given by

$$F_X(x) = (1+e^{-x})^{-1}, \quad -\infty < x < \infty, \tag{1.2}$$

while the characteristic function of a logistic random variable X is

$$\phi_X(t) = \Gamma(1+it)\Gamma(1-it).$$

The importance of the logistic distribution is already been felt in many areas of human endeavour. Verhulst (1845) used it in economic and demographic studies. Berkson (1944, 1951, 1953) used the distribution extensively in analysing bio-assay and quantal response data. George and Ojo (1980), Ojo (1989), Ojo (1997), Ojo (2002), Ojo (2003) are few of many publications on logistic distribution.

The simplicity of the logistic distribution and its importance as a growth curve have made it one of the many important statistical distributions. The shape of the logistic distribution that is similar to that of the normal distribution makes it simpler and also profitable on suitable occasions to replace the normal by the logistic distribution with negligible errors in the respective theories.

Balakrishnan and Leung (1988) show the probability density function of a random variable X that has a type II generalized logistic distribution. It is given by

$$f_X(x;b) = \frac{be^{-bx}}{(1+e^{-x})^{b+1}}, \quad -\infty < x < \infty, \quad b > 0. \tag{1.3}$$

The corresponding cumulative distribution function is

$$F_X(x;b) = 1 - \frac{e^{-bx}}{(1+e^{-x})^{b}}, \quad -\infty < x < \infty, \quad b > 0. \tag{1.4}$$

The distribution in (1.3) is known to be a family of negatively skewed generalized logistic distributions depending on the value of b. Wu, Hung and Lee (2000) proposed an extended form of the generalized logistic distribution which is referred to as the five parameter generalized logistic distribution. Its density function is given by

$$\begin{aligned} & f_X(x;\mu,\sigma,\lambda,\phi,m) \\ & \quad = \frac{\lambda^{\phi}}{\sigma B(\phi,m)}[exp(\frac{x-\mu}{\sigma})]^m[\lambda + exp(\frac{x-\mu}{\sigma})]^{-(\phi+m)}, -\infty < x < \infty, \end{aligned} \tag{1.5}$$

where $-\infty < \mu < \infty, \ \lambda > 0, \phi > 0, \sigma > 0, m > 0$. Several properties of this distribution such as moments are examined and some applications are discused in that paper.

In this paper, we derive a form of generalized logistic distribution function that generalizes the type II generalized logistic distribution of Balakrishnan and Leung (1988). The new function, which is a particular case of the general case considered in Wu, Hung and Lee (2000), is called negatively skewed extended generalized logistic distribution or extended type II generalized logistic distribution. Throughout the rest of this paper, we shall use the name "extended type II generalized logistic distribution".

2. EXTENDED TYPE II GENERALIZED LOGISTIC DISTRIBUTION

As mentioned above, Wu, *et al* (2000) present a generalized logistic distribution with density function (1.5). Putting $\mu = 0$ and $\sigma = 1$ and working with $-X$ instead of X, its density function can be written as

$$f_X(x;\lambda,\phi,m) = \frac{\lambda^\phi}{B(\phi,m)}\frac{e^{-mx}}{(\lambda+e^{-x})^{\phi+m}},\ -\infty < x < \infty,\ \lambda > 0,\ \phi > 0,\ m > 0. \quad (2.1)$$

In this section, we shall derive a form of generalized logistic distribution which is a special case of the one in the equation (2.1) as equation (1.3) is a special case of the generalized logistic distribution in George and Ojo (1980).

Let X be a continuously distributed random variable with two parameters Gumbel probability density function

$$f_X(x;\alpha,p) = \frac{\alpha^p}{\Gamma(p)}e^{-px}exp(-\alpha e^{-x}),\quad -\infty < x < \infty,\quad p > 0,\quad \alpha > 0. \quad (2.2)$$

Let us assume that the parameter α has an exponential distribution with probability density function

$$g(\alpha;\lambda) = \lambda e^{-\lambda\alpha},\quad \lambda > 0. \quad (2.3)$$

Then we obtain the probability density of the compound distribution based on (2.2) and (2.3) as

$$\begin{aligned} f_X(x;\lambda,p) &= \int_0^\infty f_X(x;\alpha)g(\alpha;\lambda,)d\alpha \\ &= \frac{\lambda p e^{-px}}{(\lambda+e^{-x})^{p+1}},\quad -\infty < x < \infty, \lambda > 0,\quad p > 0. \end{aligned} \quad (2.4)$$

This equation (2.4) corresponds to $m = p,\ \phi = 1$ in the equation (2.1).

The corresponding cumulative distribution function can be obtained as

$$F_X(x;\lambda,p) = 1 - \frac{e^{-px}}{(\lambda+e^{-x})^p},\quad -\infty < x < \infty, \lambda > 0,\quad p > 0. \quad (2.5)$$

We refer to the function in the equation (2.4) as the extended type II generalized logistic distribution. The case $\lambda = 1$ in the equation (2.4) corresponds to the type II generalized logistic distribution of Balakrishnan and Leung (1988) and Olapade (2002). As expected, the case $\lambda = p = 1$ corresponds to the standard logistic density function given in the equation (1.1).

Also, the characteristic function for the extended type II generalized logistic distribution can be obtained as

$$\phi_X(t) = \frac{\lambda^{-it}\Gamma(1+it)\Gamma(p-it)}{\Gamma(p)}. \tag{2.6}$$

3. SOME THEOREMS THAT RELATE THE EXTENDED TYPE II GENERALIZED LOGISTIC TO SOME OTHER DISTRIBUTIONS

We state some theorems and prove them in this section. A similar work has been done in Olapade (2004) for the extended type I generalized logistic distribution.

Theorem 3.1 *Let Y be a continuously distributed random variable with probability density $f_Y(y)$. Then the random variable $X = -\ln[\frac{\lambda \sqrt[p]{Y}}{1-\sqrt[p]{Y}}]$ has an extended type II generalized logistic distribution with parameter p and λ if and only if Y has a uniform distribution over a unit range* $(0, 1)$.

Proof. If Y has a uniform distribution over a unit range (0,1), then the probability density function of Y is

$$f_Y(y) = 1, \quad 0 < y < 1. \tag{3.1}$$

Then $x = -\ln[\frac{\lambda \sqrt[p]{y}}{1-\sqrt[p]{y}}]$ implies that $y = \frac{e^{-px}}{(\lambda+e^{-x})^p}$. Therefore

$$f_X(x) = |\frac{dy}{dx}|f_Y(y) = \frac{\lambda p e^{-px}}{(\lambda + e^{-x})^{p+1}}, \quad -\infty < x < \infty, \tag{3.2}$$

which is the extended type II generalized logistic density function.

Conversely, if X is an extended type II generalized logistic random variable, then $x = -\ln[\frac{\lambda \sqrt[p]{y}}{1-\sqrt[p]{y}}]$ implies that

$$dx/dy = -1/py(1 - \sqrt[p]{y}) \tag{3.3}$$

and

$$f_Y(y) = |\frac{dx}{dy}|f_X(x) = 1, \quad 0 < y < 0. \tag{3.4}$$

Since this is the probability density function of a uniform random variable Y, the proof is complete.

Theorem 3.2 *Let Y be a continuously distributed random variable with probability density function $f_Y(y)$. Then the random variable $X = -\ln(\frac{\lambda e^{-Y}}{1-e^{-Y}})$ is an extended type II generalized logistic random variable if and only if Y follows an exponential distribution with parameters $p > 0$.*

Proof. Suppose Y has the exponential distribution with parameter p, then

$$f_Y(y;p) = pe^{-py}, \quad y > 0. \tag{3.5}$$

Then, $x = -\ln(\frac{\lambda e^{-y}}{1-e^{-y}})$ implies that $y = -\ln(\frac{e^{-x}}{\lambda+e^{-x}})$ and the Jacobian of the transformation is $|J| = \lambda/(\lambda + e^{-x})$.

Therefore,

$$f_X(x) = |J|f_Y(y) = \frac{\lambda pe^{-px}}{(\lambda + e^{-x})^{p+1}}, \quad -\infty < x < \infty, \tag{3.6}$$

which is the extended type II generalized logistic density function.

Conversely, if X is an extended type II generalized logistic random variable with probability distribution function shown in the equation (2.5), then

$$F_Y(y) = pr[Y \leq y] = pr[-\ln(\frac{e^{-x}}{\lambda + e^{-x}}) \leq y] \tag{3.7}$$

$$= F_X(\ln(\frac{1-e^{-y}}{\lambda e^{-y}}))$$

$$= 1 - e^{-py}, \quad y > 0. \tag{3.8}$$

Since the equation (3.8) is the cumulative distribution function for the exponential distribution with parameter p, the proof is complete.

Theorem 3.3. *The random variable X is extended type II generalized logistic with probability distribution function F given in the equation (2.5) if and only if F satisfies the homogeneous differential equation*

$$(\lambda + e^{-x})F' + \lambda pF - \lambda p = 0, \tag{3.9}$$

where prime denotes differentiation, F denotes $F(x)$ and F' denotes $F'(x)$.

Proof. Since

$$F = 1 - \frac{e^{-px}}{(\lambda + e^{-x})^p},$$

if the random variable X is an extended type II generalized logistic, it is easily shown that the F above satisfies the equation (3.9).

Conversely, let us assume that F satisfies the equation (3.9). Separating the variables in the equation (3.9) and integrating, we have

$$\ln(1 - F) = -px - p\ln(\lambda + e^{-x}) + \ln k, \tag{3.10}$$

where k is a constant. Obviously from the equation (3.10)

$$F = 1 - \frac{ke^{-px}}{(\lambda + e^{-x})^p}. \tag{3.11}$$

The value of k that makes F a distribution function is $k = 1$.

Possible Application of Theorem 3.3 From the equation (3.9), we have

$$x = -\ln(\frac{\lambda p(1 - F) - \lambda F'}{F'}). \tag{3.12}$$

Thus, the importance of the Theorem 3.3 lies in the linearising transformation (3.12). The transformation (3.12) which we call "extended type II generalized logit transform" can be regarded as an extended type II generalization of Berkson's logit transform in Berkson (1944) for the ordinary logistic model.

Therefore, in the analysis of bioassay and quantal response data, if model (2.4) is used, what Berkson's logit transform does for the ordinary logistic can be done for the extended type II generalized logistic model (2.4) by the transformation (3.12).

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