*Kragujevac J. Math.* 26 (2004) 133–137.

# A NOTE ON INDECOMPOSABILITY OF BOOLEAN MATRICES

**Bo Zhou**

*Department of Mathematics, South China Normal University,*
*Guangzhou 510631, P. R. China*
(e-mail: zhoubo@scnu.edu.cn)

*(Received January 20, 2004)*

**Abstract.** We disscuss the properties of $n \times n$ $r$-indecomposable Boolean matrices with $-n < r < n$, and determine all the numbers attainable as the $k$-th upper generalized exponents of $n \times n$ $r$-indecomposable Boolean matrices with $1 \le r, k < n$.

Let $B_n$ be the set of all $n \times n$ Boolean matrices whose entries are 0 or 1 and the arithmetic underlying the matrix multiplication and addition is the usual integer arithmetic except that $1 + 1 = 1$. A set of rows and columns is said to cover the 1's if every 1 in the matrix belongs to one of these rows or columns. The term rank of $A$ is the minimum number of rows and columns to cover the 1's of $A$, denoted by $\lambda(A)$. Let $r$ be an integer with $-n < r < n$. A matrix $A \in B_n$ is $r$-indecomposable if it contains no $k \times l$ zero submatrix with $1 \le k, l \le n$ and $k + l = n - r + 1$. In particular, $A$ is $(1-n)$-indecomposable if and only if $A \ne O$, while $A$ is $(n-1)$-indecomposable if and only if $A = J_n$, the all-ones matrix in $B_n$. A 1-indecomposable matrix is also said to be fully indecomposable. A 0-indecomposable matrix is also called a Hall matrix.

Let $X \subseteq \{1, 2, \ldots, n\}$ and $R_t(A, X) = \{1 \le k \le n : a_{i_0 i_1} a_{i_1 i_2} \ldots a_{i_{t-1} k} = 1$ for some

$i_0 \in X\}$. Then $A \in B_n$ is $r$-indecomposable if and only if, for each $X \subseteq \{1, 2, \ldots, n\}$ with $\max\{1, 1-r\} \leq |X| \leq \min\{n, n-r\}$, $|R_1(A, X)| \geq |X| + r$.

For $A \in B_n$, a set of $s$ rows and $t$ columns of $A$ is called a proper mixture if $0 \leq s, t < n$ and $s + t > 0$. In [2], Dulmage and Mendelsohn defined $\alpha(A)$ as follows. If $A = O$, then $\alpha(A) = -n$ and if $A = J_n$, then $\alpha(A) = n$. If $A \neq O, J_n$, then $\alpha(A) = \min\{s+t\} - n = \lambda(A) - n$, the minimum being taken over all proper mixtures of $s$ rows and $t$ columns which cover the non-zeros of $A$. (Note that $\lambda(J_n) = n$.) $A$ is said to be cover-irreducible if $\alpha(A) > 0$, in this case $\alpha(A)$ is called the index of cover-irreducibility of $A$.

**Lemma 1.** *Let $A \in B_n$, $A \neq O, J_n$, and let $r$ be an integer with $-n < r < n$. Then $A$ is $r$-indecomposable if and only if $\alpha(A) \geq r$.*

**Proof.**

$A$ is $r$-indecomposable

$\Leftrightarrow$ $A$ contains no $k \times l$ zero submatrix with $1 \leq k, l \leq n$ and $k + l = n - r + 1$

$\Leftrightarrow$ any proper mixture of $s$ rows and $t$ columns to cover the nonzeros of $A$ satisfies $n - s + n - t \leq n - r$

$\Leftrightarrow$ $\alpha(A) \geq r$.

It follows from the definition of $r$-indecomposability that an $r$-indecomposable matrix is also $p$-indecomposable for all $p$, $-n < p \leq r$. By Lemma 1, if $A \neq O, J_n$, then $\alpha(A)$ is the maximum $r$ such that $A$ is $r$-indecomposable (which is called the index of indecomposability of $A$ in [3]).

If $-n < r \leq 0$, then $A \in B_n$ is $r$-indecomposable if and only if $\lambda(A) \geq n + r$. By König's theorem, $A$ is $r$-indecomposable for $-n < r \leq 0$ if and only if it dominates a permutation matrix of order $n + r$.

A matrix $A \in B_n$ is primitive if $A^m = J_n$ for some positive integer $m$; the least such $m$ is called the exponent of $A$, denoted by $\exp(A)$. It is known that $A \in B_n$ is primitive if $A$ is 1-indecomposable. It folows from Lemma 1 that $A \in B_n$ is cover-irreducible if and only if $A$ is 1-indecomposable.

Let $P \in B_n$ the be the permutation matrix with $p_{i,i+1} = p_{n1} = 1$ for $i = 1, 2, \ldots, n-1$ and let $C_s = I + P + \cdots + P^s$ if $s \geq 0$. It is easy to see that $C_s$ is $s$-indecomposable. $\lceil a \rceil$ denotes the smallest integer not less that $a$.

**Lemma 2.** [2] *Let $A \in B_n$ with $\alpha = \alpha(A) > 0$ and let $c = c(A)$ be the minimum number of nonzeros in a column of $A$. Then*

$$\exp(A) \leq \left\lceil \frac{n-c}{\alpha} \right\rceil + 1.$$

**Theorem 3.** *Suppose that $A \in B_n$ is $r$-indecomposable where $r \geq 1$. Then*

$$\exp(A) \leq \left\lceil \frac{n-1}{r} \right\rceil,$$

*and this bound is best possible.*

**Proof.** By Lemma 1, $\alpha(A) \geq r \geq 1$. Note that $c(A) \geq r+1$ by the definition of $r$-indecomposability. By Lemma 2, we obtain $\exp(A) \leq \lceil (n-1)/r \rceil$.

Note that there is an integer $k \geq 1$ such that $kr < n-1 \leq (k+1)r$, $C_r^k = I + P + \cdots + P^{kr} \neq J_n$, and $C_r^{k+1} = I + P + \cdots + P^{n-1} = J_n$. Then $\exp(C_r) = k+1 = \lceil (n-1)/r \rceil$.

**Remark 4.** This theorem has also been proved by Sachkov and Oshkin [3]. Actually a much general result is obtained there: if $A$ is $r_1$-indecomposable and $B$ is $r_2$-indecomposable, then $AB$ is $r$-indecomposable where $r_1, r_2 \geq 0$ and $r = \min\{r_1 + r_2, n-1\}$.

**Remark 5.** In fact we can derive the upper bound in Theorem 3 directly. If $A$ is $r$-indecomposable where $r \geq 1$, then for each $X \subseteq \{1, 2, \ldots, n\}$ with $\max\{1, 1-r\} \leq |X| \leq \min\{n, n-r\}$, we have $|R_1(A, X)| \geq |X| + r$. By induction, for any $m$ with $1 \leq m \leq t = \lfloor (n-1)/r \rfloor$, $R_1(A^m, X)| = |R_m(A, X)| = |R_1(A, R_{m-1}(A, X))| \geq |X| + mr$ and hence $|R_1(A^{t+1}, X)| = n$. This implies that $A^m$ is $mr$-indecomposable for $1 \leq m \leq t$ and if $t \neq (n-1)/r$, $A^{t+1}$ is $(n-1)$-indecomposable. It follows that $\exp(A) \leq \lceil (n-1)/r \rceil$.

For a primitive matrix $A \in B_n$, Brualdi and Liu [1] defined the $k$-upper generalized exponent, $F(A,k)$, to be the smallest positive integer $p$ such that there is no $k \times 1$ zero submatrix in $A^p$. Clearly $1 = F(A,n) \leq F(A,n-1) \leq \ldots \leq F(A,1) = \exp(A)$. The case $k = n$ is trivial. So we consider only the case $1 \leq k \leq n-1$.

We consider $F(A,k)$ for $r$-indecomposable matrix in $B_n$ with $r \geq 1$. For $n-r \leq k \leq n-1$ and every $r$-indecomposable matrix $A \in B_n$, clearly $F(A,k) = 1 = \lceil (n-k)/r \rceil$.

**Theorem 6.** *Suppose that $A \in B_n$ is $r$-indecomposable where $r \geq 1$, $1 \leq k \leq n-r-1$. Then*

$$F(A,k) \leq \left\lceil \frac{n-k}{r} \right\rceil,$$

*and this bound is best possible.*

**Proof.** For any $k$ with $1 \leq k \leq n-1$, there is an integer $m$ such that $n - mr \leq k < n-(m-1)r$. Note that $A^m$ is $mr$-indecomposable for $1 \leq m \leq t = \lfloor (n-1)/r \rfloor$, and $A^{t+1}$ is $(n-1)$-indecomposable. There is no zero $k \times 1$ submatrix in $A^m$, and hence $F(A,k) \leq m = \lceil (n-k)/r \rceil$.

Recall that $C_r$ is $r$-indecomposable. It is easy to see that $C_r^{m-1}$ has a $k \times 1$ zero submatrix indexed by the first $k$ rows and the last column, implying $F(C_r,k) \geq \lceil (n-k)/r \rceil$. Thus $F(C_r,k) = \lceil (n-k)/r \rceil$.

**Theorem 7.** *For any integer $p$ with $1 \leq p \leq \lceil (n-k)/r \rceil$ where $r \geq 1$, there is an $r$-indecomposable matrix in $B_n$ such that $F(A,k) = p$.*

**Proof.** If $n-r \leq k \leq n-1$, the theorem is obvious. Suppose $1 \leq k \leq n-r-1$. For any integer $p$ with $1 \leq p \leq \lceil (n-k)/r \rceil$, there is an integer $s$ such that $p = \lceil (n-k)/s \rceil$ with $r \leq s \leq n-1$. Take $C_s = I + P + \cdots + P^s$. Then $C_s$ is $s$-indecomposable and hence is $r$-indecomposable, and

$$F(C_s,k) = \left\lceil \frac{n-k}{s} \right\rceil = p.$$

**Acknowledgements:** The work was supported by the National Natural Science Foundation (No. 10201009) and the Guangdong Provincial Natural Science Foundation (No. 021072) of China.

# References

[1] R. A. Brualdi, B. Liu, *Generalized exponents of primitive directed graphs,* J. Graph Theory, **14** (1990), 483–499.

[2] A. L. Dulmage, N. S. Mendelsohn, *The exponents of incidence matrices,* Duke Math. J., **31** (1964), 575–584.

[3] V. N. Sachkov, I. B. Oshkin, *Exponents of classes of non-negative matrices,* Discrete Math. Appl., **3** (1993), 365–375.