SOME RESULTS ABOUT BANACH COMPACT ALGEBRAS

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(Received September 20, 2003)

Abstract. In this paper, we prove that (i) if $A$ is a quasi-complete locally $m$–convex algebra on which the operator $x \mapsto yxy$ ($x \in A$) is Banach compact for all elements $y$ in a sequentially dense subset of $A$, then $A$ is a Banach compact locally $m$–convex algebra and (ii) that every Montel algebra is Banach compact.

Preliminary Definitions. Let $A$ be a linear associative algebra over the field of complex numbers $\mathbb{C}$. Suppose $A$ is also a topological vector space with respect to a Hausdorff topology $\tau$. Then $A$ is a topological algebra if, in addition, the maps $x \mapsto xy$ and $x \mapsto yx$ are continuous on $A$ for each $y \in A$. The topological algebra $A$ is a locally convex algebra if and only if $A$ is a locally convex space. A topological vector space $A$ with respect to a Hausdorff topology $\tau$ is quasi-complete if every bounded, Cauchy net in $A$ converges.

A barrel in a locally convex topological vector space is a subset which is radial, convex, circled and closed. Every locally convex topological vector space has a zero neighborhood base consisting of barrels. A barrelled space is a locally convex topological vector space in which the family of all barrels forms a neighborhood base at
zero. Every Banach space and every Fréchet space is barrelled.

A barrelled space with the further property that its closed bounded subsets are compact is called a Montel space. A locally convex algebra is said to be a Montel algebra or $(M)-$algebra, if its underlying locally convex topological vector space is a Montel space.

A locally convex algebra $A$ is said to be locally $m-$convex if the topology of $A$ is defined by a family $\{p_\alpha : \alpha \in \Gamma\}$ of seminorms satisfying the multiplicative condition:

$$p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$$

for all $x, y \in A$ and $\alpha \in \Gamma$. We note that every normed algebra is a locally $m-$convex algebra.

A $B_o-$algebra is a complete, metrizable, locally convex algebra. If $A$ is a $B_o-$algebra, the multiplication in $A$ is automatically jointly continuous (i.e. the map $(x, y) \mapsto xy : A \times A \to A$ is continuous). Then the topology $\tau$ of $A$ can be defined by means of increasing sequences $\{p_i : i \in \mathbb{N}\}$ of seminorms such that

$$p_i(xy) \leq p_{i+1}(x)p_{i+1}(y)$$

for all $i$ and $x, y \in A$. A locally $m-$convex $B_o-$algebra is termed a Fréchet algebra.

We present some definitions from operator theory. Let $A$ be a locally convex algebra and let $L(A)$ denote the collection of all continuous linear maps on $A$. A map $T \in L(A)$ is said to be Banach compact if $TB$ is relatively compact for every bounded subset $B$ of $A$. $T$ is said to be finite dimensional if it has a finite dimensional range. A finite dimensional map is Banach compact.

Let $y$ be a fixed element of a locally convex algebra $A$. Then $y$ is said to be left Banach compact (resp. right Banach compact) if the map $T_y := x \mapsto xy$ (resp. $T_{xy} := x \mapsto xy$) is Banach compact on $A$. $y$ is said to be (just) Banach compact if the map $T_{y,y} := x \mapsto yxy$ is Banach compact on $A$. If every element $y \in A$ is Banach compact, then $A$ is said to be a Banach compact locally convex algebra.
Theorem 1. Let $A$ be a quasi-complete locally $m-$convex algebra on which the operator $T_{y,y} := x \mapsto yxy : A \to A$ is Banach compact for all elements $y$ in a sequentially dense subset of $A$. Then $A$ is a Banach compact locally $m-$convex algebra.

Proof. Let $B$ be a sequentially dense subset of $A$. For any fixed element $y$ in $A$, there exists a bounded sequence $\{y_n\}$ in $B$ such that $\{y_n\}$ converges to $y$. Define the operators $T$ and $T_n(n = 1, 2, 3, \ldots)$ on $A$ by

$$T_{y,y} := x \mapsto yxy$$

and

$$T_{y_n,y_n} := x \mapsto y_nxy_n$$

respectively.

Let $q_\alpha : \alpha \in \Gamma$ be a family of continuous seminorms generating the topology of $A$. For each $q_\alpha \in \{q_\alpha : \alpha \in \Gamma\}$ we have

$$q_\alpha(T_{y_n,y_n}x - T_{y,y}x) = q_\alpha(y_nxy_n - yxy)$$

$$= q_\alpha(y_nxy_n - y_nxy + y_nxy - yxy)$$

$$= q_\alpha([y_nx(y_n - y) + (y_n - y)xy]$$

$$= q_\alpha[(y_n - y)(yn + y)x]$$

$$\leq q_\alpha(y_n - y)[q_\alpha(yn) + q_\alpha(y)]q_\alpha(x).$$

Let $x \in D$, a bounded subset of $A$, then there exists $\lambda > 0$ such that $q_\alpha(x) \leq \lambda$. As $\{y_n\}$ is bounded, then there exists $\mu > 0$ such that $q_\alpha(y_n) \leq \mu$ for all $n \in \mathbb{N}$. Therefore,

$$q_\alpha(T_{y_n,y_n}x - T_{y,y}x) \leq \lambda q_\alpha(y_n - y)[\mu + q_\alpha(y)].$$

Hence,

$$\lim_n q_{D,\alpha}(T_{y_n,y_n}x - T_{y,y}x) = \lim_n \sup_{x \in D} q_\alpha(T_{y_n,y_n}x - T_{y,y}x) = 0.$$ 

Therefore $T_{y_n,y_n} \to T_{y,y}$ in the topology of bounded convergence on $L(A)$. Since the space of all Banach compact operators on $A$ is closed in $L(A)$ and since the operators $\{T_n : n \in \mathbb{N}\}$ are Banach compact, it follows that $T$ is Banach compact. Thus $A$ is Banach compact.
**Theorem 2.** Every Montel algebra is Banach compact.

**Proof.** Let $A$ be a Montel algebra. Let $y$ be any element of $A$. Consider the operator $T_{y,y} := x \mapsto xy : A \rightarrow A$. Let $B$ be a bounded subset of $A$. $T_{y,y}$ is continuous, therefore $T_{y,y}B$ is again a bounded subset of $A$. Since every bounded subset of a Montel algebra $A$ is relatively compact, we have that $T_{y,y}B$ is relatively compact in $A$. Therefore for any element $y$ in $A$, $T_{y,y}$ is Banach compact on $A$. Thus $A$ is Banach compact.

**Example.** Let $A = \mathbb{R}^\infty$ denote the product of countably, infinitely many copies of $\mathbb{R}$, the real line. Let addition, scalar multiplication and vector multiplication in $\mathbb{R}^\infty$ be defined co-ordinate wise. For example, for $x = (\lambda_n), y = (\mu_n) \in \mathbb{R}^\infty$, let the multiplication of $x$ and $y$ be defined by $xy = (\lambda_n \mu_n)$. With these operations, $\mathbb{R}^\infty$ becomes an algebra. For any $n \in \mathbb{N}$, let

$$q_n(x) = |\lambda_n|.$$

Then the family of seminorms $\{q_n : n \in \mathbb{N}\}$ generates a locally convex Hausdorff topology on $\mathbb{R}^\infty$ with respect to which $\mathbb{R}^\infty$ is complete. This topology is metrizable because it is defined by a countable system of seminorms. Furthermore, for each $n \in \mathbb{N}$ and for every $x, y \in \mathbb{R}^\infty$, we have

$$q_n(xy) = |\lambda_n \mu_n| = |\lambda_n| |\mu_n| = q_n(x)q_n(y).$$

Therefore $q_n(xy) \leq q_n(x)q_n(y)$ for all $x, y \in \mathbb{R}^\infty$; $n \in \mathbb{N}$. Thus $A$ is a Fréchet algebra.

Now consider the subspace $\Psi$ of $\mathbb{R}^\infty$ consisting of those elements $x \in \mathbb{R}^\infty$ with only finitely many nonzero co-ordinates. Let $\Psi$ have the topology induced from $\mathbb{R}^\infty$ and multiplication consisting of co-ordinate wise multiplication. Then $\Psi$ is a locally $m-$convex algebra. Let $y = (\mu_n) \in \Psi$ be arbitrary and consider the multiplication operator

$$T_{y,y} := x \mapsto xy : \Psi \rightarrow \Psi.$$
For any $y \in \Psi$, there exists $n_0(y) > 0$ such that $\mu_n = 0$ for all $n \geq n_0(y)$. Therefore $T_{y,y}x = yxy \in \mathbb{R}^{n_0(y)}$. This shows that $\dim T_{y,y}\Psi < \infty$. Therefore, the operator

$$T_{y,y} := x \mapsto yxy$$

is Banach compact on $\Psi$. Thus $\Psi$ is a Banach compact locally $m$-convex algebra.

We note that every Banach space and, more generally, every Fréchet space is barrelled. Thus the space $A = \mathbb{R}^\infty$ is barrelled.

The locally $m$-convex algebra $A = \mathbb{R}^\infty$ is a Montel algebra. Therefore by theorem 2, it is Banach compact.

We also realize that $A = \mathbb{R}^\infty$ is a quasi-complete locally $m$-convex algebra. Furthermore $A = \mathbb{R}^\infty$ contains a sequentially dense subset $\Psi$ on which the operator $x \mapsto yxy$ ($x \in A$) is Banach compact for every $y \in \Psi$. Therefore, by theorem 1, $A$ is Banach compact.

Acknowledgement. We would like to thank Prof O. D. Makinde for his encouragement and his helpful remarks.

References


