Kragujevac J. Math. 26 (2004) 123–128.

SOME RESULTS ABOUT BANACH COMPACT ALGEBRAS

B. M. Ramadisha and V. A. Babalola

School of Computational and Mathematical Sciences
University of the North, Private Bag X1106, Sovenga, 0727, South Africa

(Received September 20, 2003)

Abstract. In this paper, we prove that (i) if A is a quasi-complete locally $m-$convex algebra on which the operator $x \longmapsto yxy (x \in A)$ is Banach compact for all elements y in a sequentially dense subset of A, then A is a Banach compact locally $m-$convex algebra and (ii) that every *Montel* algebra is Banach compact.

Preliminary Definitions. Let A be a linear associative algebra over the field of complex numbers $\mathbb{C}$. Suppose A is also a *topological vector space* with respect to a Hausdorff topology τ. Then A is a *topological algebra* if, in addition, the maps $x \longmapsto xy$ and $x \longmapsto yx$ are continuous on A for each $y \in A$. The topological algebra A is a locally convex algebra if and only if A is a locally convex space. A topological vector space A with respect to a Hausdorff topology τ is *quasi-complete* if every bounded, Cauchy net in A converges.

A *barrel* in a locally convex topological vector space is a subset which is radial, convex, circled and closed. Every locally convex topological vector space has a zero neighborhood base consisting of barrels. A *barrelled space* is a locally convex topological vector space in which the family of all barrels forms a neighborhood base at zero. Every *Banach space* and every *Fréchet space* is barrelled.

A barrelled space with the further property that its closed bounded subsets are compact is called a *Montel space*. A locally convex algebra is said to be a *Montel algebra* or $(M)-$*algebra*, if its underlying locally convex topological vector space is a Montel space.

A locally convex algebra A is said to be locally $m-$convex if the topology of A is defined by a family $\{p_\alpha : \alpha \in \Gamma\}$ of seminorms satisfying the multiplicative condition:

$$p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$$

for all $x, y \in A$ and $\alpha \in \Gamma$. We note that every normed algebra is a locally $m-$convex algebra.

A B_o-*algebra* is a complete, metrizable, locally convex algebra. If A is a B_o-algebra, the multiplication in A is automatically jointly continuous (i.e. the map $(x, y) \longmapsto xy : A \times A \longrightarrow A$ is continuous). Then the topology τ of A can be defined by means of increasing sequences $\{p_i : i \in \mathbb{N}\}$ of seminorms such that

$$p_i(xy) \leq p_{i+1}(x)p_{i+1}(y)$$

for all i and $x, y \in A$. A locally $m-$convex B_o-algebra is termed a *Fréchet algebra*.

We present some definitions from operator theory. Let A be a locally convex algebra and let $L(A)$ denote the collection of all continuous linear maps on A. A map $T \in L(A)$ is said to be *Banach compact* if TB is relatively compact for every bounded subset B of A. T is said to be *finite dimensional* if it has a finite dimensional range. A finite dimensional map is Banach compact.

Let y be a fixed element of a locally convex algebra A. Then y is said to be *left Banach compact* (resp. *right Banach compact*) if the map $T_y := x \longmapsto yx$ (resp. $T_{,y} := x \longmapsto xy$) is Banach compact on A. y is said to be (just) *Banach compact* if the map $T_{y,y} := x \longmapsto yxy$ is Banach compact on A. If every element $y \in A$ is Banach compact, then A is said to be a *Banach compact locally convex algebra*.

Theorem 1. *Let A be a quasi-complete locally $m-$convex algebra on which the operator $T_{y,y} := x \longmapsto yxy : A \longrightarrow A$ is Banach compact for all elements y in a sequentially dense subset of A. Then A is a Banach compact locally $m-$convex algebra.*

Proof. Let B be a sequentially dense subset of A. For any fixed element y in A, there exists a bounded sequence $\{y_n\}$ in B such that $\{y_n\}$ converges to y. Define the operators T and $T_n (n = 1, 2, 3, \ldots)$ on A by

$$T_{y,y} := x \longmapsto yxy$$

and

$$T_{y_n,y_n} := x \longmapsto y_n x y_n$$

respectively.

Let $q_\alpha : \alpha \in \Gamma$ be a family of continuous seminorms generating the topology of A. For each $q_\alpha \in \{q_\alpha : \alpha \in \Gamma\}$ we have

$$\begin{aligned} q_\alpha(T_{y_n,y_n}x - T_{y,y}x) &= q_\alpha(y_n x y_n - yxy) \\ &= q_\alpha(y_n x y_n - y_n xy + y_n xy - yxy) \\ &= q_\alpha[y_n x(y_n - y) + (y_n - y)xy] \\ &= q_\alpha[(y_n - y)(y_n + y)x] \\ &\leq q_\alpha(y_n - y)[q_\alpha(y_n) + q_\alpha(y)]q_\alpha(x). \end{aligned}$$

Let $x \in D$, a bounded subset of A, then there exists $\lambda > 0$ such that $q_\alpha(x) \leq \lambda$. As $\{y_n\}$ is bounded, then there exists $\mu > 0$ such that $q_\alpha(y_n) \leq \mu$ for all $n \in \mathbb{N}$. Therefore,

$$q_\alpha(T_{y_n,y_n}x - T_{y,y}x) \leq \lambda q_\alpha(y_n - y)[\mu + q_\alpha(y)].$$

Hence,

$$\lim_n \; q_{D,\alpha}(T_{y_n,y_n} - T_{y,y}) = \lim_n \sup_{x \in D} q_\alpha(T_{y_n,y_n}x - T_{y,y}x) = 0.$$

Therefore $T_{y_n,y_n} \longrightarrow T_{y,y}$ in the topology of bounded convergence on $L(A)$. Since the space of all Banach compact operators on A is closed in $L(A)$ and since the operators $\{T_n : n \in \mathbb{N}\}$ are Banach compact, it follows that T is Banach compact. Thus A is Banach compact.

Theorem 2. *Every Montel algebra is Banach compact.*

Proof. Let A be a Montel algebra. Let y be any element of A. Consider the operator $T_{y,y} := x \longmapsto yxy : A \longrightarrow A$. Let B be a bounded subset of A. $T_{y,y}$ is continuous, therefore $T_{y,y}B$ is again a bounded subset of A. Since every bounded subset of a Montel algebra A is relatively compact, we have that $T_{y,y}B$ is relatively compact in A. Therefore for any element y in A, $T_{y,y}$ is Banach compact on A. Thus A is Banach compact.

Example. Let $A = \mathbb{R}^\infty$ denote the product of countably, infinitely many copies of $\mathbb{R}$, the real line. Let addition, scalar multiplication and vector multiplication in $\mathbb{R}^\infty$ be defined co-ordinate wise. For example, for $x = (\lambda_n)$, $y = (\mu_n) \in \mathbb{R}^\infty$, let the multiplication of x and y be defined by $xy = (\lambda_n \mu_n)$. With these operations, $\mathbb{R}^\infty$ becomes an algebra. For any $n \in \mathbb{N}$, let

$$q_n(x) = |\lambda_n| .$$

Then the family of seminorms $\{q_n : n \in \mathbb{N}\}$ generates a locally convex Hausdorff topology on $\mathbb{R}^\infty$ with respect to which $\mathbb{R}^\infty$ is complete. This topology is metrizable because it is defined by a countable system of seminorms. Furthermore, for each $n \in \mathbb{N}$ and for every $x, y \in \mathbb{R}^\infty$, we have

$$q_n(xy) = |\lambda_n \mu_n| = |\lambda_n| \, |\mu_n| = q_n(x) q_n(y).$$

Therefore $q_n(xy) \leq q_n(x) q_n(y)$ for all $x, y \in \mathbb{R}^\infty$; $n \in \mathbb{N}$. Thus A is a Fréchet algebra.

Now consider the subspace Ψ of $\mathbb{R}^\infty$ consisting of those elements $x \in \mathbb{R}^\infty$ with only finitely many nonzero co-ordinates. Let Ψ have the topology induced from $\mathbb{R}^\infty$ and multiplication consisting of co-ordinate wise multiplication. Then Ψ is a locally $m-$convex algebra. Let $y = (\mu_n) \in \Psi$ be arbitrary and consider the multiplication operator

$$T_{y,y} := x \longmapsto yxy : \Psi \longrightarrow \Psi.$$

For any $y \in \Psi$, there exists $n_o(y) > 0$ such that $\mu_n = 0$ for all $n \geq n_o(y)$. Therefore $T_{y,y}x = yxy \in \mathbb{R}^{no(y)}$. This shows that $\dim T_{y,y}\Psi < \infty$. Therefore, the operator

$$T_{y,y} := x \longmapsto yxy$$

is Banach compact on Ψ. Thus Ψ is a Banach compact locally $m-$convex algebra.

We note that every Banach space and, more generally, every Fréchet space is barrelled. Thus the space $A = \mathbb{R}^{\infty}$ is barrelled.

The locally $m-$convex algebra $A = \mathbb{R}^{\infty}$ is a Montel algebra. Therefore by theorem 2, it is Banach compact.

We also realize that $A = \mathbb{R}^{\infty}$ is a quasi-complete locally $m-$convex algebra. Furthermore $A = \mathbb{R}^{\infty}$ contains a sequentially dense subset Ψ on which the operator $x \longmapsto yxy\,(x \in A)$ is Banach compact for every $y \in \Psi$. Therefore, by theorem 1, A is Banach compact.

Acknowledgement. We would like to thank Prof O. D. Makinde for his encouragement and his helpful remarks.

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