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INEQUALITIES OF GRÜSS TYPE FOR THE STIELTJES INTEGRAL AND APPLICATIONS

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Abstract. Sharp bounds of two Čebyšev functionals for the Stieltjes integrals and applications for quadrature rules are given.

1. INTRODUCTION

Consider the *weighted Čebyšev functional*

$$T_w(f, g) := \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) g(t) dt - \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) dt \cdot \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) g(t) dt \quad (1)$$

where $f, g, w : [a, b] \rightarrow \mathbf{R}$ and $w(t) \geq 0$ for a.e. $t \in [a, b]$ are measurable functions such that the involved integrals exist and $\int_a^b w(t) dt > 0$.

In [2], the authors obtained, among others, the following inequalities:

$$\begin{aligned}
& |T_w(f, g)| \\
& \leq \frac{1}{2} (M - m) \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b w(s) g(s) ds \right| dt \\
& \leq \frac{1}{2} (M - m) \left[\frac{1}{\int_a^b w(t) dt} \int_a^b w(t) \right. \\
& \quad \left. \times \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b w(s) g(s) ds \right|^p dt \right]^{\frac{1}{p}} \quad (p > 1) \\
& \leq \frac{1}{2} (M - m) \operatorname{ess\,sup}_{t \in [a, b]} \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b w(s) g(s) ds \right|
\end{aligned} \tag{2}$$

provided

$$-\infty < m \leq f(t) \leq M < \infty \quad \text{for a.e. } t \in [a, b] \tag{3}$$

and the corresponding integrals are finite. The constant $\frac{1}{2}$ is sharp in all the inequalities in (2) in the sense that it cannot be replaced by a smaller constant.

In addition, if

$$-\infty < n \leq g(t) \leq N < \infty \quad \text{for a.e. } t \in [a, b], \tag{4}$$

then the following refinement of the celebrated Grüss inequality is obtained:

$$\begin{aligned}
& |T_w(f, g)| \\
& \leq \frac{1}{2} (M - m) \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b w(s) g(s) ds \right| dt \\
& \leq \frac{1}{2} (M - m) \left[\frac{1}{\int_a^b w(t) dt} \int_a^b w(t) \right. \\
& \quad \left. \times \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b w(s) g(s) ds \right|^2 dt \right]^{\frac{1}{2}} \\
& \leq \frac{1}{4} (M - m) (N - n).
\end{aligned} \tag{5}$$

Here, the constants $\frac{1}{2}$ and $\frac{1}{4}$ are also sharp in the sense mentioned above.

In this paper, we extend the above results for Riemann-Stieltjes integrals. A quadrature formula is also considered.

For this purpose, we introduce the following Čebyšev functional for the Stieltjes

integral

$$T(f, g; u) := \frac{1}{u(b) - u(a)} \int_a^b f(t) g(t) du(t) - \frac{1}{u(b) - u(a)} \int_a^b f(t) du(t) \cdot \frac{1}{u(b) - u(a)} \int_a^b g(t) du(t), \quad (6)$$

where $f, g \in C[a, b]$ (are continuous on $[a, b]$) and $u \in BV[a, b]$ (is of bounded variation on $[a, b]$) with $u(b) \neq u(a)$.

For some recent inequalities for Stieltjes integral see [3]-[6].

2. SOME INEQUALITIES FOR THE GENERALISED ČEBYŠEV FUNCTIONAL

The following result holds [9].

Theorem 1 *Let $f, g : [a, b] \rightarrow \mathbf{R}$ be continuous on $[a, b]$ and $u : [a, b] \rightarrow \mathbf{R}$ with $u(a) \neq u(b)$. Assume also that there exists the real constants m, M such that*

$$m \leq f(t) \leq M \quad \text{for each } t \in [a, b]. \quad (7)$$

If u is of bounded variation on $[a, b]$, then we have the inequality

$$|T(f, g; u)| \leq \frac{1}{2} (M - m) \frac{1}{|u(b) - u(a)|} \times \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_{\infty} \bigvee_a^b(u), \quad (8)$$

where $\bigvee_a^b(u)$ denotes the total variation of u in $[a, b]$. The constant $\frac{1}{2}$ is sharp, in the sense that it cannot be replaced by a smaller constant.

Proof. It is easy to see, by simple computation with the Stieltjes integral, that the following equality

$$T(f, g; u) = \frac{1}{u(b) - u(a)} \int_a^b \left[f(t) - \frac{m + M}{2} \right] \times \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t) \quad (9)$$

holds.

Using the known inequality

$$\left| \int_a^b p(t) dv(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \bigvee_a^b(v), \quad (10)$$

provided $p \in C[a, b]$ and $v \in BV[a, b]$, we have, by (9), that

$$\begin{aligned} |T(f, g; u)| &\leq \sup_{t \in [a, b]} \left| \left[f(t) - \frac{m+M}{2} \right] \left[g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right] \right| \\ &\quad \cdot \frac{1}{|u(b)-u(a)|} \bigvee_a^b(u) \\ &\quad \left(\text{since } \left| f(t) - \frac{m+M}{2} \right| \leq \frac{M-m}{2} \text{ for any } t \in [a, b] \right) \\ &\leq \frac{M-m}{2} \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right\|_{\infty} \cdot \frac{1}{|u(b)-u(a)|} \bigvee_a^b(u) \end{aligned}$$

and the inequality (8) is proved.

To prove the sharpness of the constant $\frac{1}{2}$ in the inequality (8), we assume that it holds with a constant $C > 0$, i.e.,

$$\begin{aligned} |T(f, g; u)| &\leq C(M-m) \frac{1}{|u(b)-u(a)|} \\ &\quad \times \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right\|_{\infty} \bigvee_a^b(u). \end{aligned} \quad (11)$$

Let us consider the functions $f = g$, $f : [a, b] \rightarrow \mathbf{R}$, $f(t) = t$, $t \in [a, b]$ and $u : [a, b] \rightarrow \mathbf{R}$ given by

$$u(t) = \begin{cases} -1 & \text{if } t = a, \\ 0 & \text{if } t \in (a, b), \\ 1 & \text{if } t = b. \end{cases} \quad (12)$$

Then f, g are continuous on $[a, b]$, u is of bounded variation on $[a, b]$ and

$$\begin{aligned} \frac{1}{u(b)-u(a)} \int_a^b f(t) g(t) du(t) &= \frac{b^2+a^2}{2}, \\ \frac{1}{u(b)-u(a)} \int_a^b f(t) du(t) &= \frac{b+a}{2}, \end{aligned}$$

$$\left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_{\infty} = \sup_{t \in [a, b]} \left| t - \frac{a+b}{2} \right| = \frac{b-a}{2}$$

and

$$\bigvee_a^b(u) = 2, \quad M = b, \quad m = a.$$

Inserting these values in (11), we get

$$\left| \frac{a^2 + b^2}{2} - \frac{(a+b)^2}{4} \right| \leq C(b-a) \cdot \frac{1}{2} \cdot \frac{(b-a)}{2} \cdot 2,$$

giving $C \geq \frac{1}{2}$, and the theorem is thus proved.

The corresponding result for monotonic function u is incorporated in the following theorem [9].

Theorem 2 *Assume that f and g are as in Theorem 1. If $u : [a, b] \rightarrow \mathbf{R}$ is monotonic nondecreasing on $[a, b]$, then one has the inequality:*

$$|T(f, g; u)| \leq \frac{1}{2} (M - m) \frac{1}{u(b) - u(a)} \times \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t). \quad (13)$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. Using the known inequality

$$\left| \int_a^b p(t) dv(t) \right| \leq \int_a^b |p(t)| dv(t), \quad (14)$$

provided $p \in C[a, b]$ and v is a monotonic nondecreasing function on $[a, b]$, we have (by the use of equality (9)) that

$$\begin{aligned} |T(f, g; u)| &\leq \frac{1}{u(b) - u(a)} \int_a^b \left| f(t) - \frac{m+M}{2} \right| \\ &\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t) \\ &\leq \frac{1}{2} (M - m) \frac{1}{u(b) - u(a)} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t). \end{aligned}$$

Now, assume that the inequality (13) holds with a constant $D > 0$, instead of $\frac{1}{2}$, i.e.,

$$|T(f, g; u)| \leq D(M - m) \frac{1}{u(b) - u(a)} \times \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t). \quad (15)$$

If we choose the same function as in the proof of Theorem 1, we observe that f, g are continuous and u is monotonic nondecreasing on $[a, b]$. Then, for these functions, we have

$$\begin{aligned} T(f, g; u) &= \frac{a^2 + b^2}{2} - \frac{(a + b)^2}{4} = \frac{(b - a)^2}{4}, \\ \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t) &= \int_a^b \left| t - \frac{a + b}{2} \right| du(t) \\ &= b - a, \end{aligned}$$

and then, by (15) we get

$$\frac{(b - a)^2}{4} \leq D(b - a) \frac{1}{2}(b - a)$$

giving $D \geq \frac{1}{2}$, and the theorem is completely proved.

The case when u is a Lipschitzian function is embodied in the following theorem [9].

Theorem 3 *Assume that $f, g : [a, b] \rightarrow \mathbf{R}$ are Riemann integrable functions on $[a, b]$ and f satisfies the condition (7). If $u : (a, b) \rightarrow \mathbf{R}$ ($u(b) \neq u(a)$) is Lipschitzian with the constant L , then we have the inequality*

$$|T(f, g; u)| \leq \frac{1}{2}L(M - m) \frac{1}{|u(b) - u(a)|} \times \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt. \quad (16)$$

The constant $\frac{1}{2}$ cannot be replaced by a smaller constant.

Proof. It is well known that if $p : [a, b] \rightarrow \mathbf{R}$ is Riemann integrable on $[a, b]$ and $v : [a, b] \rightarrow \mathbf{R}$ is Lipschitzian with the constant L , then the Riemann-Stieltjes integral

$\int_a^b p(t) dv(t)$ exists and

$$\left| \int_a^b p(t) dv(t) \right| \leq L \int_a^b |p(t)| dt. \quad (17)$$

Using this fact and the identity (9), we deduce

$$\begin{aligned} |T(f, g; u)| &\leq \frac{L}{|u(b) - u(a)|} \int_a^b \left| f(t) - \frac{m+M}{2} \right| \\ &\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\ &\leq \frac{1}{2} (M - m) \frac{L}{|u(b) - u(a)|} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \end{aligned}$$

and the inequality (16) is proved.

Now, assume that (16) holds with a constant $E > 0$ instead of $\frac{1}{2}$, i.e.,

$$\begin{aligned} |T(f, g; u)| &\leq EL(M - m) \frac{1}{|u(b) - u(a)|} \\ &\quad \times \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt. \end{aligned} \quad (18)$$

Consider the function $f = g$, $f : [a, b] \rightarrow \mathbf{R}$ with

$$f(t) = \begin{cases} -1 & \text{if } t \in \left[a, \frac{a+b}{2} \right] \\ 1 & \text{if } t \in \left(\frac{a+b}{2}, b \right] \end{cases}$$

and $u : [a, b] \rightarrow \mathbf{R}$, $u(t) = t$. Then, obviously, f and g are Riemann integrable on $[a, b]$ and u is Lipschitzian with the constant $L = 1$.

Since

$$\begin{aligned} \frac{1}{u(b) - u(a)} \int_a^b f(t) g(t) du(t) &= \frac{1}{b - a} \int_a^b dt = 1, \\ \frac{1}{u(b) - u(a)} \int_a^b f(t) du(t) &= \frac{1}{u(b) - u(a)} \int_a^b g(t) du(t) = 0, \\ \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt &= \int_a^b dt = b - a \end{aligned}$$

and

$$M = 1, \quad m = 1$$

then, by (18), we deduce $E \geq \frac{1}{2}$, and the theorem is completely proved.

The following result holds [10].

Theorem 4 *Let $f, g : [a, b] \rightarrow \mathbf{R}$ be such that f is of r -Hölder type on $[a, b]$, i.e.,*

$$|f(t) - f(s)| \leq H |t - s|^r \quad \text{for any } t, s \in [a, b], \quad (19)$$

and g is continuous on $[a, b]$. If $u : [a, b] \rightarrow \mathbf{R}$ is of bounded variation on $[a, b]$ with $u(a) \neq u(b)$, then we have the inequality

$$|T(f, g; u)| \leq \frac{H(b-a)^r}{2^r} \cdot \frac{1}{|u(b) - u(a)|} \times \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_{\infty} \bigvee_a^b(u), \quad (20)$$

where $\bigvee_a^b(u)$ denotes the total variation of u on $[a, b]$.

Proof. It is easy to see, by simple computation with the Stieltjes integral, that the following equality

$$T(f, g; u) = \frac{1}{u(b) - u(a)} \int_a^b \left[f(t) - f\left(\frac{a+b}{2}\right) \right] \times \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] du(t) \quad (21)$$

holds.

Using the known inequality

$$\left| \int_a^b p(t) dv(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \bigvee_a^b(v) \quad (22)$$

provided $p \in C[a, b]$ and $v \in BV[a, b]$, we have, by (21), that

$$\begin{aligned} |T(f, g; u)| &\leq \sup_{t \in [a, b]} \left| \left[f(t) - f\left(\frac{a+b}{2}\right) \right] \left[g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right] \right| \\ &\quad \times \frac{1}{|u(b) - u(a)|} \bigvee_a^b(u) \\ &\leq \sup_{t \in [a, b]} \left| f(t) - f\left(\frac{a+b}{2}\right) \right| \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_{\infty} \end{aligned}$$

$$\begin{aligned}
& \times \frac{1}{|u(b) - u(a)|} \bigvee_a^b(u) \\
\leq & L \left(\frac{b-a}{2} \right)^r \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_\infty \\
& \times \frac{1}{|u(b) - u(a)|} \bigvee_a^b(u),
\end{aligned}$$

and the inequality (20) is proved.

The following corollary may be useful in applications [10].

Corollary 1 *Let f be Lipschitzian with the constant $L > 0$, i.e.,*

$$|f(t) - f(s)| \leq L|t - s| \quad \text{for any } t, s \in [a, b], \quad (23)$$

and u, g are as in Theorem 4. Then we have the inequality

$$\begin{aligned}
|T(f, g; u)| \leq & \frac{1}{2} \frac{L(b-a)}{|u(b) - u(a)|} \\
& \times \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_\infty \bigvee_a^b(u). \quad (24)
\end{aligned}$$

The constant $\frac{1}{2}$ cannot be replaced by a smaller constant.

Proof. The inequality (24) follows by (20) for $r = 1$. It remains to prove only the sharpness of the constant $\frac{1}{2}$.

Consider the functions $f = g$, where $f : [a, b] \rightarrow \mathbf{R}$, $f(t) = t$ and $u : [a, b] \rightarrow \mathbf{R}$, given by

$$u(t) = \begin{cases} -1 & \text{if } t = a, \\ 0 & \text{if } t \in (a, b), \\ 1 & \text{if } t = b. \end{cases} \quad (25)$$

Then, f is Lipschitzian with the constant $L = 1$, g is continuous and u is of bounded variation.

If we assume that the inequality (24) holds with a constant $C > 0$, i.e.,

$$|T(f, g; u)| \leq CL(b-a) \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_\infty \bigvee_a^b(u), \quad (26)$$

and since

$$\begin{aligned} \frac{1}{u(b) - u(a)} \int_a^b f(t) g(t) du(t) &= \frac{b^2 + a^2}{2}, \\ \frac{1}{u(b) - u(a)} \int_a^b f(t) du(t) &= \frac{1}{u(b) - u(a)} \int_a^b g(t) du(t) = \frac{b+a}{2}, \\ \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_{\infty} &= \sup_{t \in [a, b]} \left| t - \frac{a+b}{2} \right| = \frac{b-a}{2} \end{aligned}$$

and $V_a^b(u) = 2$, then, by (26), we have

$$\left| \frac{b^2 + a^2}{2} - \left(\frac{a+b}{2} \right)^2 \right| \leq C \frac{(b-a)}{2} \frac{b-a}{2} \cdot 2,$$

giving $C \geq \frac{1}{2}$.

The following result concerning monotonic function $u : [a, b] \rightarrow \mathbf{R}$ also holds [10].

Theorem 5 *Assume that f and g are as in Theorem 4. If $u : [a, b] \rightarrow \mathbf{R}$ is monotonic nondecreasing on $[a, b]$ with $u(b) > u(a)$, then we have the inequalities:*

$$\begin{aligned} |T(f, g; u)| &\leq \frac{H}{u(b) - u(a)} \int_a^b \left| t - \frac{a+b}{2} \right|^r \\ &\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t) \\ &\leq \frac{H(b-a)^r}{2^r [u(b) - u(a)]} \\ &\quad \times \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t). \end{aligned} \tag{27}$$

Proof. Using the known inequality

$$\left| \int_a^b p(t) dv(t) \right| \leq \int_a^b |p(t)| dv(t), \tag{28}$$

provided $p \in C[a, b]$ and v is monotonic nondecreasing on $[a, b]$, we have, by (21), the following estimate:

$$\begin{aligned} |T(f, g; u)| &\leq \frac{1}{u(b) - u(a)} \int_a^b \left| \left(f(t) - f\left(\frac{a+b}{2}\right) \right) \right. \\ &\quad \left. \times \left(g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right) \right| du(t) \end{aligned}$$

$$\begin{aligned}
&\leq \frac{H}{u(b) - u(a)} \int_a^b \left| t - \frac{a+b}{2} \right|^r \\
&\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t) \\
&\leq \frac{H}{u(b) - u(a)} \sup_{t \in [a,b]} \left| t - \frac{a+b}{2} \right|^r \\
&\quad \times \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t)
\end{aligned}$$

which simply provides (27).

The particular case of Lipschitzian functions that is relevant for applications is embodied in the following corollary [10].

Corollary 2 *Assume that f is L -Lipschitzian, g is continuous and u is monotonic nondecreasing on $[a, b]$ with $u(b) > u(a)$. Then we have the inequalities*

$$\begin{aligned}
|T(f, g; u)| &\leq \frac{L}{u(b) - u(a)} \int_a^b \left| t - \frac{a+b}{2} \right| \\
&\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t) \\
&\leq \frac{1}{2} \cdot \frac{L(b-a)}{u(b) - u(a)} \\
&\quad \times \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t).
\end{aligned} \tag{29}$$

The first inequality is sharp. The constant $\frac{1}{2}$ in the second inequality cannot be replaced by a smaller constant.

Proof. The inequality (29) follows by (27) on choosing $r = 1$. Assume that (29) holds with the constants $D, E > 0$, i.e.,

$$\begin{aligned}
|T(f, g; u)| &\leq \frac{LD}{u(b) - u(a)} \int_a^b \left| t - \frac{a+b}{2} \right| \\
&\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t) \\
&\leq \frac{LE(b-a)}{u(b) - u(a)} \int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| du(t).
\end{aligned} \tag{30}$$

Consider the functions $f = g$, where $f : [a, b] \rightarrow \mathbf{R}$, $f(t) = t$ and u is as given by (25). Then, obviously, f is Lipschitzian with the constant $L = 1$, g is continuous and u is monotonic nondecreasing on $[a, b]$.

Since, we know, for these functions

$$T(f, g; u) = \frac{(b-a)^2}{4},$$

and

$$\int_a^b \left| t - \frac{a+b}{2} \right| \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right| du(t) = \frac{(b-a)^2}{2},$$

$$\int_a^b \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right| du(t) = b-a,$$

then by (30) we deduce

$$\frac{(b-a)^2}{4} \leq \frac{D}{2} \cdot \frac{(b-a)^2}{2} \leq \frac{E(b-a)^2}{2}$$

giving $D \geq 1$ and $E \geq \frac{1}{2}$.

Another natural possibility to obtain bounds for the functional $T(f, g; u)$, where u is Lipschitzian with the constant $K > 0$, is embodied in the following theorem [10].

Theorem 6 *Assume that $f : [a, b] \rightarrow \mathbf{R}$ is of r -Hölder type on $[a, b]$. If $g : [a, b] \rightarrow \mathbf{R}$ is Riemann integrable on $[a, b]$ and $u : [a, b] \rightarrow \mathbf{R}$ is Lipschitzian with the constant $K > 0$ and $u(a) \neq u(b)$, then one has the inequalities:*

$$\begin{aligned} |T(f, g; u)| &\leq \frac{HK}{|u(b)-u(a)|} \int_a^b \left| t - \frac{a+b}{2} \right|^r \\ &\quad \times \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right| dt \\ &\leq \begin{cases} \frac{HK(b-a)^{r+1}}{2^{r(r+1)}|u(b)-u(a)|} \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right\|_{\infty}; \\ \frac{HK(b-a)^{r+\frac{1}{q}}}{2^{r(qr+1)\frac{1}{q}}|u(b)-u(a)|} \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right\|_p \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{HK(b-a)^r}{2^r|u(b)-u(a)|} \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right\|_1. \end{cases} \end{aligned} \quad (31)$$

Proof. Using the identity (21), we have successively

$$\begin{aligned}
|T(f, g; u)| &\leq \frac{K}{|u(b) - u(a)|} \int_a^b \left| f(t) - f\left(\frac{a+b}{2}\right) \right| \\
&\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\
&\leq \frac{KH}{|u(b) - u(a)|} \int_a^b \left| t - \frac{a+b}{2} \right|^r \\
&\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt
\end{aligned} \tag{32}$$

and the first inequality in (31) is proved.

Since

$$\begin{aligned}
&\int_a^b \left| t - \frac{a+b}{2} \right|^r \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\
&\leq \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_{\infty} \int_a^b \left| t - \frac{a+b}{2} \right|^r dt \\
&= \frac{(b-a)^{r+1}}{2^r(r+1)} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_{\infty},
\end{aligned}$$

then by (32) we deduce the first part in the second inequality in (31).

By Hölder's integral inequality we have

$$\begin{aligned}
&\int_a^b \left| t - \frac{a+b}{2} \right|^r \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\
&\leq \left(\int_a^b \left| t - \frac{a+b}{2} \right|^{qr} dt \right)^{\frac{1}{q}} \left(\int_a^b \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right|^p dt \right)^{\frac{1}{p}} \\
&= \left[\frac{(b-a)^{qr+1}}{2^{qr}(qr+1)} \right]^{\frac{1}{q}} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_p \\
&= \frac{(b-a)^{r+\frac{1}{q}}}{2^r(qr+1)^{\frac{1}{q}}} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_p.
\end{aligned}$$

Using (32), we deduce the second part of the second inequality in (31).

Finally, since

$$\left| t - \frac{a+b}{2} \right|^r \leq \left(\frac{b-a}{2} \right)^r, \quad t \in [a, b],$$

we deduce

$$\begin{aligned}
&\int_a^b \left| t - \frac{a+b}{2} \right|^r \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\
&\leq \frac{(b-a)^r}{2^r} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_1
\end{aligned}$$

and the theorem is completely proved.

The following particular case is useful in applications [10].

Corollary 3 *If f is Lipschitzian with the constant L and g and u are as in Theorem 6, then we have the inequalities:*

$$\begin{aligned}
|T(f, g; u)| &\leq \frac{LK}{|u(b) - u(a)|} \int_a^b \left| t - \frac{a+b}{2} \right| \\
&\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\
&\leq \begin{cases} \frac{LK(b-a)^2}{4|u(b) - u(a)|} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_\infty; \\ \frac{LK(b-a)^{1+\frac{1}{q}}}{2(q+1)^{\frac{1}{q}} |u(b) - u(a)|} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_p \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{LK(b-a)}{2|u(b) - u(a)|} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_1. \end{cases} \quad (33)
\end{aligned}$$

The first inequality in (33) is sharp.

The constants $\frac{1}{4}$ and $\frac{1}{2}$ in the second branch of the second inequality cannot be replaced by smaller constants, respectively.

Proof. The inequality (33) follows obviously from (31) on choosing $r = 1$.

Now, assume that the following inequalities hold

$$\begin{aligned}
|T(f, g; u)| &\leq \frac{CLK}{|u(b) - u(a)|} \int_a^b \left| t - \frac{a+b}{2} \right| \\
&\quad \times \left| g(t) - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right| dt \\
&\leq \begin{cases} \frac{DLK(b-a)^2}{|u(b) - u(a)|} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_\infty; \\ \frac{ELK(b-a)^{1+\frac{1}{q}}}{(q+1)^{\frac{1}{q}} |u(b) - u(a)|} \left\| g - \frac{1}{u(b) - u(a)} \int_a^b g(s) du(s) \right\|_p \\ \quad \text{if } p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases} \quad (34)
\end{aligned}$$

with $C, D, E > 0$.

Consider the functions $f, g, u : [a, b] \rightarrow \mathbf{R}$, defined by $f(t) = t - \frac{a+b}{2}$, $u(t) = t$ and

$$g(t) = \begin{cases} -1 & \text{if } t \in \left[a, \frac{a+b}{2} \right], \\ 1 & \text{if } t \in \left(\frac{a+b}{2}, b \right]. \end{cases}$$

Then both f and u are Lipschitzian with the constant $L = K = 1$ and g is Riemann integrable on $[a, b]$.

We obviously have

$$\begin{aligned} |T(f, g; u)| &= \frac{1}{b-a} \int_a^b f(t) g(t) dt - \frac{1}{b-a} \int_a^b f(t) dt \cdot \frac{1}{b-a} \int_a^b g(t) dt \\ &= \frac{b-a}{4}, \end{aligned}$$

$$\begin{aligned} \int_a^b \left| t - \frac{a+b}{2} \right| \left| g(t) - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right| dt &= \frac{(b-a)^2}{4} \\ \left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right\|_{\infty} &= \|g\|_{\infty} = 1 \end{aligned}$$

and

$$\left\| g - \frac{1}{u(b)-u(a)} \int_a^b g(s) du(s) \right\|_p = \|g\|_p = (b-a)^{\frac{1}{p}}.$$

Consequently, by (34), one has

$$\frac{b-a}{4} \leq \frac{C}{b-a} \frac{(b-a)^2}{4} \leq \begin{cases} \frac{D(b-a)^2}{b-a} \cdot 1 \\ \frac{E(b-a)^2}{(q+1)^{\frac{1}{q}}(b-a)} \end{cases}$$

giving

$$\frac{1}{4} \leq \frac{C}{4} \leq \begin{cases} D \\ \frac{E}{(q+1)^{\frac{1}{q}}}, \quad q > 1. \end{cases}$$

From the first inequality we obtain $C \geq 1$. Also, we get $D \geq \frac{1}{4}$ and $E \geq \frac{(q+1)^{\frac{1}{q}}}{4}$. Letting $q \rightarrow 1+$, we deduce $E \geq \frac{1}{2}$ and the corollary is proved.

3. A QUADRATURE FORMULA

Let us consider the partition of the interval $[a, b]$ given by

$$I_n : a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b. \quad (35)$$

Denote $v(I_n) := \max \{h_i | i = \overline{0, n-1}\}$ where $h_i := x_{i+1} - x_i$, $i = \overline{0, n-1}$.

If $f : [a, b] \rightarrow \mathbf{R}$ is continuous on $[a, b]$ and if we define

$$\begin{aligned} M_i &:= \sup_{t \in [x_i, x_{i+1}]} f(t), \quad m_i := \inf_{t \in [x_i, x_{i+1}]} f(t), \quad \text{and} \\ v(f, I_n) &= \max_{i=\overline{0, n-1}} (M_i - m_i), \end{aligned}$$

then, obviously, by the continuity of f on $[a, b]$, for any $\varepsilon > 0$, we may find a division I_n with norm $v(I_n) < \delta$ such that $v(f, I_n) < \varepsilon$.

Consider now the quadrature rule

$$S_n(f, g; u, I_n) := \sum_{i=0}^{n-1} \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} f(t) du(t) \cdot \int_{x_i}^{x_{i+1}} g(t) du(t) \quad (36)$$

provided $f, g \in C[a, b]$, $u \in BV[a, b]$ and $u(x_{i+1}) \neq u(x_i)$, $i = 0, \dots, n-1$.

We may now state the following result in approximating the Stieltjes integral

$$\int_a^b f(t) g(t) du(t).$$

Theorem 7 *Let $f, g \in C[a, b]$ and $u \in BV[a, b]$. If I_n is a division of the interval $[a, b]$ and $u(x_{i+1}) \neq u(x_i)$, $i = 0, \dots, n-1$, then we have:*

$$\int_a^b f(t) g(t) du(t) = S_n(f, g; u, I_n) + R_n(f, g; u, I_n), \quad (37)$$

where $S_n(f, g; u, I_n)$ is as defined in (36) and the remainder $R_n(f, g; u, I_n)$ satisfies the estimate

$$\begin{aligned} |R_n(f, g; u, I_n)| &\leq \frac{1}{2} v(f, I_n) \\ &\times \max_{i=\overline{0, n-1}} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) du(s) \right\|_{[x_i, x_{i+1}], \infty} \bigvee_a^b(u). \end{aligned} \quad (38)$$

The constant $\frac{1}{2}$ is sharp in (38) in the sense that it cannot be replaced by a smaller constant.

Proof. Applying the inequality (8) on the intervals $[x_i, x_{i+1}]$, $i = 0, \dots, n-1$, we have

$$\begin{aligned} & \left| \int_{x_i}^{x_{i+1}} f(t) g(t) du(t) \right. \\ & \quad \left. - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} f(t) du(t) \cdot \int_{x_i}^{x_{i+1}} g(t) du(t) \right| \\ & \leq \frac{1}{2} (M_i - m_i) \sup_{t \in [x_i, x_{i+1}]} \left| g(t) - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) du(s) \right| \bigvee_{x_i}^{x_{i+1}}(u). \end{aligned} \quad (39)$$

Summing the inequalities (39) over i from 0 to $n-1$, and using the generalised triangle inequality, we have

$$\begin{aligned} & |R_n(f, g; u, I_n)| \\ & \leq \frac{1}{2} \sum_{i=0}^{n-1} (M_i - m_i) \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) du(s) \right\|_{[x_i, x_{i+1}], \infty} \\ & \quad \times \bigvee_{x_i}^{x_{i+1}}(u) \\ & \leq \frac{1}{2} v(f, I_n) \max_{i=0, n-1} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) du(s) \right\|_{[x_i, x_{i+1}], \infty} \\ & \quad \times \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(u) \\ & = \frac{1}{2} v(f, I_n) \max_{i=0, n-1} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) du(s) \right\|_{[x_i, x_{i+1}], \infty} \\ & \quad \times \bigvee_a^b(u), \end{aligned} \quad (40)$$

and the estimate (38) is obtained.

Remark 1 *Similar results may be stated for either u monotonic or Lipschitzian. We omit the details.*

We may now state another result in approximating the Stieltjes integral

$$\int_a^b f(t) g(t) du(t).$$

Theorem 8 *Let $f, g : [a, b] \rightarrow \mathbf{R}$ be such that f is of r - H -Hölder type on $[a, b]$ (see Theorem 4), g is continuous on $[a, b]$, I_n is as above and $u : [a, b] \rightarrow \mathbf{R}$ is of*

bounded variation on $[a, b]$ with $u(x_{i+1}) \neq u(x_i)$, $i = 0, \dots, n-1$. Then we have the representation

$$\int_a^b f(t)g(t)du(t) = S_n(f, g; u, I_n) + R_n(f, g; u, I_n), \quad (41)$$

where the quadrature $S_n(f, g; u, I_n)$ is as defined in (36) and the remainder $R_n(f, g; u, I_n)$ satisfies the estimate

$$\begin{aligned} |R_n(f, g; u, I_n)| &\leq \frac{H}{2^r} [v(I_n)]^r \\ &\times \max_{i=0, n-1} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) du(s) \right\|_{[x_i, x_{i+1}], \infty} \bigvee_a^b(u), \end{aligned} \quad (42)$$

where $v(I_n) := \max \{h_i | i = \overline{0, n-1}\}$.

Proof. Applying the inequality (20) on the interval $[x_i, x_{i+1}]$ to get

$$\begin{aligned} &\left| \int_{x_i}^{x_{i+1}} f(t)g(t)du(t) \right. \\ &\quad \left. - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} f(t)du(t) \cdot \int_{x_i}^{x_{i+1}} g(t)du(t) \right| \\ &\leq \frac{Hh_i^r}{2^r} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(t)du(t) \right\|_{[x_i, x_{i+1}], \infty} \bigvee_{x_i}^{x_{i+1}}(u), \end{aligned} \quad (43)$$

for each $i \in \{0, \dots, n-1\}$.

Summing the inequalities (43) over i from 0 to $n-1$, and using the generalised triangle inequality, we have

$$\begin{aligned}
& |R_n(f, g; u, I_n)| \tag{44} \\
& \leq \frac{H}{2^r} \sum_{i=0}^{n-1} h_i^r \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(t) du(t) \right\|_{[x_i, x_{i+1}], \infty} \bigvee_{x_i}^{x_{i+1}}(u) \\
& \leq \frac{H}{2^r} [v(f)]^n \max_{i=0, n-1} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(t) du(t) \right\|_{[x_i, x_{i+1}], \infty} \sum_{i=0}^{n-1} \bigvee_{x_i}^{x_{i+1}}(u) \\
& = \frac{H}{2^r} [v(f)]^n \max_{i=0, n-1} \left\| g - \frac{1}{u(x_{i+1}) - u(x_i)} \int_{x_i}^{x_{i+1}} g(s) du(s) \right\|_{[x_i, x_{i+1}], \infty} \bigvee_a^b(u),
\end{aligned}$$

and the inequality (42) is obtained.

Remark 2 *Similar results may be stated if one uses Theorem 5 and Theorem 6. We omit the details.*

4. SOME PARTICULAR CASES

For $f, g, w : [a, b] \rightarrow \mathbf{R}$, integrable and with the property that $\int_a^b w(t) dt \neq 0$, reconsider the weighted Čebyšev functional

$$\begin{aligned}
T_w(f, g) & := \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) g(t) dt \tag{45} \\
& \quad - \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) f(t) dt \cdot \frac{1}{\int_a^b w(t) dt} \int_a^b w(t) g(t) dt.
\end{aligned}$$

1. If $f, g, w : [a, b] \rightarrow \mathbf{R}$ are continuous and there exists the real constants m, M such that

$$m \leq f(t) \leq M \text{ for each } t \in [a, b], \tag{46}$$

then one has the inequality

$$\begin{aligned}
|T_w(f, g)| & \leq \frac{1}{2} (M - m) \frac{1}{\left| \int_a^b w(s) ds \right|} \tag{47} \\
& \quad \times \left\| g - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right\|_{[a, b], \infty} \int_a^b |w(s)| ds.
\end{aligned}$$

The proof follows by Theorem 1 on choosing $u(t) = \int_a^t w(s) ds$.

2. If f, g, w are as in **1** and $w(s) \geq 0$ for $s \in [a, b]$, then one has the inequality

$$|T_w(f, g)| \leq \frac{1}{2}(M - m) \frac{1}{\int_a^b w(s) ds} \times \int_a^b \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right| w(s) ds. \quad (48)$$

The proof follows by Theorem 2 on choosing $u(t) = \int_a^t w(s) ds$.

3. If f, g are Riemann integrable on $[a, b]$ and f satisfies (46), and w is continuous on $[a, b]$, then one has the inequality

$$|T_w(f, g)| \leq \frac{1}{2} \|w\|_{[a,b],\infty} (M - m) \frac{1}{\left| \int_a^b w(s) ds \right|} \times \int_a^b \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right| ds. \quad (49)$$

The proof follows by Theorem 5 on choosing $u(t) = \int_a^t w(s) ds$.

4. If $f, g, w : [a, b] \rightarrow \mathbf{R}$ are continuous and f is of $r - H$ -Hölder type (see Theorem 4), then one has the inequality

$$|T_w(f, g)| \leq \frac{H |b - a|^r}{2^r} \cdot \frac{1}{\left| \int_a^b w(s) ds \right|} \times \left\| g - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right\|_{[a,b],\infty} \int_a^b |w(s)| ds.$$

The proof follows by Theorem 4 on choosing $u(t) = \int_a^t w(s) ds$.

5. If f, g, w are as in **4** and $w(s) \geq 0$ for $s \in [a, b]$, then one has the inequality

$$\begin{aligned} |T_w(f, g)| & \leq \frac{H}{\int_a^b w(s) ds} \int_a^b \left| t - \frac{a+b}{2} \right|^r \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right| w(s) ds \\ & \leq \frac{H (b-a)^r}{2^r \int_a^b w(s) ds} \int_a^b \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right| w(s) ds. \end{aligned} \quad (50)$$

The proof follows by Theorem 5 on choosing $u(t) = \int_a^t w(s) ds$.

6. If f is of $r - H$ -Hölder type, g are Riemann integrable on $[a, b]$ and w is continuous

on $[a, b]$, then one has the inequality

$$\begin{aligned}
& |T_w(f, g)| \\
& \leq \frac{H \|w\|_{[a,b],\infty}}{\left| \int_a^b w(s) ds \right|} \int_a^b \left| t - \frac{a+b}{2} \right|^r \left| g(t) - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right| dt \\
& \leq \begin{cases} \frac{H \|w\|_{[a,b],\infty} (b-a)^{r+1}}{2^{r(r+1)} \left| \int_a^b w(s) ds \right|} \left\| g - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right\|_{[a,b],\infty} ; \\ \frac{H \|w\|_{[a,b],\infty} (b-a)^{r+\frac{1}{q}}}{2^{r(qr+1)^{\frac{1}{q}}} \left| \int_a^b w(s) ds \right|} \left\| g - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right\|_{[a,b],p} , \\ \qquad \qquad \qquad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{H \|w\|_{[a,b],\infty} (b-a)^r}{2^r \left| \int_a^b w(s) ds \right|} \left\| g - \frac{1}{\int_a^b w(s) ds} \int_a^b g(s) w(s) ds \right\|_{[a,b],1} . \end{cases} \tag{51}
\end{aligned}$$

The proof follows by Theorem 6 on choosing $u(t) = \int_a^t w(s) ds$.

5. OTHER INEQUALITIES FOR STIELTJES INTEGRAL

In [11], the authors have considered the following functional

$$D(f; u) := \int_a^b f(x) du(x) - [u(b) - u(a)] \cdot \frac{1}{b-a} \int_a^b f(t) dt,$$

provided that the involved integrals exist.

In the same paper, the following result in estimating the above functional has been obtained.

Theorem 9 *Let $f, u : [a, b] \rightarrow \mathbf{R}$ be such that u is Lipschitzian on $[a, b]$, i.e.,*

$$|u(x) - u(y)| \leq L|x - y| \quad \text{for any } x, y \in [a, b] \quad (L > 0) \tag{52}$$

and f is Riemann integrable on $[a, b]$. If $m, M \in \mathbf{R}$ are such that

$$m \leq f(x) \leq M \quad \text{for any } x, y \in [a, b], \tag{53}$$

then we have the inequality

$$|D(f; u)| \leq \frac{1}{2}L(M - m)(b - a). \quad (54)$$

The constant $\frac{1}{2}$ is sharp in the sense that it cannot be replaced by a smaller constant.

In [12], the following result complementing the above one was obtained.

Theorem 10 *Let $f, u : [a, b] \rightarrow \mathbf{R}$ be such that $u : [a, b] \rightarrow \mathbf{R}$ is of bounded variation in $[a, b]$ and $f : [a, b] \rightarrow \mathbf{R}$ is K -Lipschitzian ($K > 0$). Then we have the inequality*

$$|D(f; u)| \leq \frac{1}{2}K(b - a) \bigvee_a^b(u). \quad (55)$$

The constant $\frac{1}{2}$ is sharp in the above sense.

In this section further similar results will be pointed out.

The following identity is interesting in itself.

Lemma 1 *Let $f, u : [a, b] \rightarrow \mathbf{R}$ be such that the Stieltjes integral $\int_a^b f(t) du(t)$ and the Riemann integral $\int_a^b f(t) dt$ exist. Then we have the identity*

$$\begin{aligned} D(f; u) &= \frac{1}{b-a} \int_a^b \Phi(t) df(t) = \frac{1}{b-a} \int_a^b \Gamma(t) df(t) \\ &= \frac{1}{b-a} \int_a^b (t-a)(b-t) \Delta(t) df(t), \end{aligned} \quad (56)$$

where

$$\begin{aligned} \Phi(t) &:= \frac{(t-a)u(b) + (b-t)u(a)}{b-t} - u(t), \quad t \in [a, b], \\ \Gamma(t) &:= (t-a)[u(b) - u(t)] - (b-t)[u(t) - u(a)], \quad t \in [a, b], \end{aligned}$$

and

$$\Delta(t) := [u; b, t] - [u; t, a], \quad t \in (a, b),$$

where $[u; \alpha, \beta]$ is the divided difference, i.e., we recall it

$$[u; \alpha, \beta] := \frac{u(\alpha) - u(\beta)}{\alpha - \beta}.$$

Proof. We observe that

$$\begin{aligned}
\int_a^b \Phi(t) df(t) &= \int_a^b \left[\frac{(t-a)u(b) + (b-t)u(a)}{b-t} - u(t) \right] df(t) \\
&= \left[\frac{(t-a)u(b) + (b-t)u(a)}{b-t} - u(t) \right] f(t) \Big|_a^b \\
&\quad - \int_a^b f(t) d \left[\frac{(t-a)u(b) + (b-t)u(a)}{b-t} - u(t) \right] \\
&= [u(b) - u(b)] - [u(a) - u(a)] - \int_a^b f(t) \left[\frac{u(b) - u(a)}{b-a} dt - du(t) \right] \\
&= \int_a^b f(t) du(t) - \frac{u(b) - u(a)}{b-a} \int_a^b f(t) dt
\end{aligned}$$

and the first identity in (56) is proved.

The second and third identities are obvious.

Remark 3 If u is an integral, i.e., $u(t) = \int_a^t g(s) ds$, then from (56) we deduce Cerone's result in [1]

$$T(f, g) = \frac{1}{(b-a)^2} \int_a^b \Psi(t) df(t), \quad (57)$$

where

$$\begin{aligned}
\Psi(t) &= \frac{t-a}{b-t} \int_a^b g(s) ds - \int_a^t g(s) ds \quad (t \in [a, b]) \\
&= (t-a) \int_t^b g(s) ds - (b-t) \int_a^t g(s) ds \quad (t \in [a, b]) \\
&= (t-a)(b-t) \left[\frac{\int_t^b g(s) ds}{b-t} - \frac{\int_a^t g(s) ds}{t-a} \right] \quad (t \in (a, b)).
\end{aligned}$$

If $w : [a, b] \rightarrow \mathbf{R}$ is integrable and $\int_a^b w(t) dt \neq 0$, then the choice

$$u(t) := \frac{\int_a^t w(s) g(s) ds}{\int_a^t w(s) ds}, \quad t \in [a, b], \quad (58)$$

will produce

$$\begin{aligned}
D(f; u) &= \frac{\int_a^b w(s) f(s) g(s) ds}{\int_a^b w(s) ds} - \frac{\int_a^b w(s) g(s) ds}{\int_a^b w(s) ds} \cdot \frac{1}{b-a} \int_a^b f(t) dt \\
&=: E(f, g; w).
\end{aligned}$$

The following corollary is thus a natural application of the above Lemma 1.

Corollary 4 *If w, f, g are Riemann integrable on $[a, b]$ and $\int_a^b w(t) dt \neq 0$, then*

$$\begin{aligned} E(f, g; w) &= \int_a^b \Phi_w(t) df(t) = \frac{1}{b-a} \int_a^b \Gamma_w(t) df(t) \\ &= \frac{1}{b-a} \int_a^b (t-a)(b-t) \Delta_w(t) df(t), \end{aligned} \quad (59)$$

where

$$\begin{aligned} \Phi_w(t) &= \left(\frac{t-a}{b-t} \right) \cdot \frac{\int_a^b w(s) g(s) ds}{\int_a^b w(s) ds} - \frac{\int_a^t w(s) g(s) ds}{\int_a^b w(s) ds}, \\ \Gamma_w(t) &= (t-a) \frac{\int_a^b w(s) g(s) ds}{\int_a^b w(s) ds} - (b-t) \frac{\int_a^t w(s) g(s) ds}{\int_a^b w(s) ds}, \\ \Delta_w(t) &= \frac{\int_t^b w(s) g(s) ds}{(b-t) \int_a^b w(s) ds} - \frac{\int_a^t w(s) g(s) ds}{(t-a) \int_a^b w(s) ds}. \end{aligned}$$

The following general result in bounding the functional $D(f; u)$ may be stated.

Theorem 11 *Let $f, u : [a, b] \rightarrow \mathbf{R}$.*

(i) *If f is of bounded variation and u is continuous on $[a, b]$, then*

$$|D(f; u)| \leq \begin{cases} \sup_{t \in [a, b]} |\Phi(t)| V_a^b(f), \\ \frac{1}{b-a} \sup_{t \in [a, b]} |\Gamma(t)| V_a^b(f), \\ \frac{1}{b-a} \sup_{t \in (a, b)} [(t-a)(b-t) |\Delta(t)|] V_a^b(f). \end{cases} \quad (60)$$

(ii) *If f is L -Lipschitzian and u is Riemann integrable on $[a, b]$, then*

$$|D(f; u)| \leq \begin{cases} L \int_a^b |\Phi(t)| dt, \\ \frac{L}{b-a} \int_a^b |\Gamma(t)| dt, \\ \frac{L}{b-a} \int_a^b (t-a)(b-t) |\Delta(t)| dt. \end{cases} \quad (61)$$

(iii) If f is monotonic nondecreasing on $[a, b]$ and u is continuous on $[a, b]$, then

$$|D(f; u)| \leq \begin{cases} \int_a^b |\Phi(t)| df(t), \\ \frac{1}{b-a} \int_a^b |\Gamma(t)| df(t), \\ \frac{1}{b-a} \int_a^b (t-a)(b-t) |\Delta(t)| df(t). \end{cases} \quad (62)$$

Proof. Follows by Lemma 1 on taking into account that

$$\left| \int_c^d p(t) dv(t) \right| \leq \sup_{t \in [a, b]} |p(t)| \bigvee_c^d(t)$$

if p is continuous on $[c, d]$ and v is of bounded variation,

$$\left| \int_c^d p(t) dv(t) \right| \leq L \int_c^d |p(t)| dt;$$

if v is L -Lipschitzian on $[c, d]$ and p is Riemann integrable on $[c, d]$ and

$$\left| \int_c^d p(t) dv(t) \right| \leq \int_c^d |p(t)| dt$$

if p is continuous on $[c, d]$ and v is monotonic nondecreasing on $[c, d]$.

It is natural to consider the following corollaries, since they provide simpler bounds for the functional $D(f; u)$ in terms of Δ defined in Lemma 1.

Corollary 5 If f is of bounded variation and u is continuous on $[a, b]$, then

$$\begin{aligned} |D(f; u)| &\leq \frac{1}{b-a} \sup_{t \in [a, b]} [(t-a)(b-t)\Delta(t)] \bigvee_a^b(f) \\ &\leq \frac{b-a}{4} \|\Delta\|_\infty \bigvee_a^b(f). \end{aligned} \quad (63)$$

Corollary 6 If f is L -Lipschitzian and u is Riemann integrable on $[a, b]$, then

$$\begin{aligned} |D(f; u)| &\leq \frac{L}{b-a} \int_a^b (t-a)(b-t) |\Delta(t)| dt \\ &\leq \begin{cases} \frac{1}{6} L (b-a)^2 \|\Delta\|_\infty, \\ L (b-a)^{1+\frac{1}{q}} [B(q+1, q+1)]^{\frac{1}{q}} \|\Delta\|_p, \quad p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{4} L (b-a) \|\Delta\|_1, \end{cases} \end{aligned} \quad (64)$$

Corollary 7 *If f is monotonic nondecreasing and g is continuous, then*

$$\begin{aligned}
 |D(f; u)| &\leq \frac{1}{b-a} \int_a^b (t-a)(b-t) |\Delta(t)| dt \\
 &\leq \begin{cases} \frac{1}{4}(b-a) \int_a^b |\Delta(t)| df(t), \\ \frac{1}{b-a} \left(\int_a^b [(b-t)(t-a)]^q df(t) \right)^{\frac{1}{q}} \left(\int_a^b |\Delta(t)|^p df(t) \right)^{\frac{1}{p}}, \\ \frac{1}{b-a} \|\Delta\|_\infty \int_a^b (t-a)(b-t) df(t). \end{cases} \quad (65)
 \end{aligned}$$

$p > 1, \frac{1}{p} + \frac{1}{q} = 1;$

Remark 4 *If one chooses in Corollaries 5 – 7, $u(t) = \int_a^t g(s) ds$, then the result incorporated in Theorems 4 – 6 of [1] are recaptured.*

Finally, the following result on the positivity of the functional $D(f; u)$ holds.

Theorem 12 *Let f be a monotonic nondecreasing function on $[a, b]$. If u is such that*

$$\Delta(t) = \Delta(u; a, t, b) := [u; b, t] - [u; t, a] \geq 0 \quad (66)$$

for any $t \in (a, b)$, then we have the inequality

$$D(f; u) \geq \frac{1}{b-a} \left| \int_a^b (t-a)(b-t) [|u; b, t] - [|u; t, a]| df(t) \right| \geq 0. \quad (67)$$

The proof is similar to the case in Theorem 3 of [2] and we omit the details.

Remark 5 *It is easy to see that a sufficient condition for (66) to hold is that $u : [a, b] \rightarrow \mathbf{R}$ is a convex function on $[a, b]$.*

Remark 6 *Similar results for composite rules in approximating the Stieltjes integral may be stated but we omit the details.*

6. SOME RESULTS FOR MONOTONIC INTEGRATORS

The following result holds.

Theorem 13 *Let $f : [a, b] \rightarrow \mathbf{R}$ be L -Lipschitzian on $[a, b]$ and u monotonic non-decreasing on $[a, b]$. Then we have the inequality*

$$\begin{aligned} |D(f; u)| &\leq \frac{1}{2}L(b-a)[u(b) - u(a) - K(u)] \\ &\leq \frac{1}{2}L(b-a)[u(b) - u(a)], \end{aligned} \quad (68)$$

where

$$\begin{aligned} K(u) &:= \frac{4}{(b-a)^2} \int_a^b u(x) \left(x - \frac{a+b}{2}\right) dx \\ &= \frac{4}{(b-a)^2} \int_a^b \left[u(x) - u\left(\frac{a+b}{2}\right)\right] \left(x - \frac{a+b}{2}\right) dx \geq 0. \end{aligned} \quad (69)$$

The constant $\frac{1}{2}$ in both inequalities is sharp in the sense that it cannot be replaced by a smaller constant.

Proof. As u is monotonic nondecreasing on $[a, b]$, then

$$\begin{aligned} &\left| \int_a^b f(x) du(x) - \frac{u(b) - u(a)}{b-a} \int_a^b f(t) dt \right| \\ &= \left| \int_a^b \left(f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right) du(x) \right| \\ &\leq \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| du(x). \end{aligned} \quad (70)$$

Taking into account that f is L -Lipschitzian, we have the following Ostrowski type inequality (see for example [7])

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq L \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] (b-a) \quad (71)$$

for all $x \in [a, b]$, from where we deduce

$$\begin{aligned} &\int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| du(x) \\ &\leq L(b-a) \int_a^b \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] du(x). \end{aligned} \quad (72)$$

Now, observe that, by the integration by parts formula for the Stieltjes integral, we have

$$\begin{aligned} \int_a^b \left(x - \frac{a+b}{2}\right)^2 du(x) &= u(x) \left(x - \frac{a+b}{2}\right)^2 \Big|_a^b - 2 \int_a^b u(x) \left(x - \frac{a+b}{2}\right) dx \\ &= \frac{(b-a)^2}{4} [u(b) - u(a)] - 2 \int_a^b u(x) \left(x - \frac{a+b}{2}\right) dx \end{aligned}$$

and then

$$\begin{aligned} \int_a^b \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2 \right] du(x) & \\ &= \frac{1}{2} [u(b) - u(a)] - \frac{2}{(b-a)^2} \int_a^b u(x) \left(x - \frac{a+b}{2}\right) dx. \end{aligned} \quad (73)$$

Using (70) – (73) we deduce the first part of (68).

The second part is obvious by (69) which follows by the monotonicity of u on $[a, b]$.

To prove the sharpness of the constant $\frac{1}{2}$, assume that (74) holds with the constants $C, D > 0$, i.e.,

$$|D(f; u)| \leq CL(b-a)[u(b) - u(a) - K(u)] \leq DL(b-a)[u(b) - u(a)]. \quad (74)$$

Consider the functions $f, u : [a, b] \rightarrow \mathbf{R}$ given by $f(x) = x - \frac{a+b}{2}$ and

$$u(x) = \begin{cases} 0 & \text{if } x \in [a, b) \\ 1 & \text{if } x = b. \end{cases}$$

Thus f is L -Lipschitzian with the constant $L = 1$ and u is monotonic nondecreasing.

We observe that

$$D(f; u) = \int_a^b f(x) du(x) = f(x)u(x) \Big|_a^b - \int_a^b u(x) dx = \frac{b-a}{2},$$

$K(u) = 0$ and $u(b) - u(a) = 1$, giving in (74)

$$\frac{b-a}{2} \leq C(b-a) \leq D(b-a)$$

and thus $C, D \geq \frac{1}{2}$ proving the sharpness of the constant $\frac{1}{2}$ in (68).

Another result of this type is the following one.

Theorem 14 Let $u : [a, b] \rightarrow \mathbf{R}$ be monotonic nondecreasing on $[a, b]$ and $f : [a, b] \rightarrow \mathbf{R}$ be of bounded variation such that the Stieltjes integral $\int_a^b f(x) du(x)$ exists. Then we have the inequality

$$\begin{aligned} |D(f; u)| &\leq [u(b) - u(a) - Q(u)] \bigvee_a^b(f) \\ &\leq [u(b) - u(a)] \bigvee_a^b(f), \end{aligned} \quad (75)$$

where

$$\begin{aligned} Q(u) &:= \frac{1}{b-a} \int_a^b \operatorname{sgn} \left(x - \frac{a+b}{2} \right) u(x) dx \\ &= \frac{1}{b-a} \int_a^b \operatorname{sgn} \left(x - \frac{a+b}{2} \right) \left[u(x) - u \left(\frac{a+b}{2} \right) \right] dx \geq 0. \end{aligned} \quad (76)$$

The first inequality in (75) is sharp in the sense that the constant $c = 1$ cannot be replaced by a smaller constant.

Proof. Since u is monotonic nondecreasing, we have (see (70)) that

$$|D(f; u)| \leq \int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| du(x). \quad (77)$$

Using the following Ostrowski type inequality obtained by the author in [8]

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| \leq \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] \bigvee_a^b(f) \quad (78)$$

for any $x \in [a, b]$, we have

$$\int_a^b \left| f(x) - \frac{1}{b-a} \int_a^b f(t) dt \right| du(x) \leq \bigvee_a^b(f) \int_a^b \left[\frac{1}{2} + \left| \frac{x - \frac{a+b}{2}}{b-a} \right| \right] du(x). \quad (79)$$

A simple calculation with the Stieltjes integral gives that

$$\begin{aligned} &\int_a^b \left| x - \frac{a+b}{2} \right| du(x) \\ &= \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x \right) du(x) + \int_{\frac{a+b}{2}}^b \left(x - \frac{a+b}{2} \right) du(x) \\ &= u(x) \left(\frac{a+b}{2} - x \right) \Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} u(x) dx \\ &\quad + \left(x - \frac{a+b}{2} \right) u(x) \Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b u(x) dx \\ &= \frac{1}{2} (b-a) [u(b) - u(a)] - \int_a^b \operatorname{sgn} \left(x - \frac{a+b}{2} \right) u(x) dx \end{aligned} \quad (80)$$

and then by (77) – (80) we deduce the first inequality in (75).

The second part of (75) follows by (76) which holds by the monotonicity property of u .

Now, assume that the first inequality in (75) holds with a constant $E > 0$, i.e.,

$$|D(f; u)| \leq E \bigvee_a^b(f) [u(b) - u(a) - Q(u)]. \quad (81)$$

Consider the mappings $f, u : [a, b] \rightarrow \mathbf{R}$, $f(x) = x - \frac{a+b}{2}$, and

$$u(x) = \begin{cases} 0 & \text{if } x \in \left[a, \frac{a+b}{2} \right], \\ 1 & \text{if } x \in \left(\frac{a+b}{2}, b \right]. \end{cases}$$

Then we have

$$\begin{aligned} D(f; u) &= \int_a^b f(x) du(x) - \frac{u(b) - u(a)}{b - a} \int_a^b f(t) dt \\ &= \int_a^b \left(x - \frac{a+b}{2} \right) du(x) = \left(x - \frac{a+b}{2} \right) u(x) \Big|_a^b - \int_a^b u(x) dx \\ &= \frac{b-a}{2} [u(b) + u(a)] = \frac{b-a}{2} \end{aligned}$$

and

$$\begin{aligned} \bigvee_a^b(f) [u(b) - u(a) - Q(u)] &= (b-a) \left[u(b) - u(a) - \left(\frac{1}{b-a} \int_a^{\frac{a+b}{2}} \operatorname{sgn} \left(x - \frac{a+b}{2} \right) u(x) dx \right. \right. \\ &\quad \left. \left. + \frac{1}{b-a} \int_{\frac{a+b}{2}}^b \operatorname{sgn} \left(x - \frac{a+b}{2} \right) u(x) dx \right) \right] \\ &= \frac{b-a}{2}. \end{aligned}$$

Thus, by (81) we obtain

$$\frac{b-a}{2} \leq E \cdot \frac{b-a}{2},$$

showing that $E \geq 1$, and the theorem is proved.

Remark 7 *Similar results for composite rules in approximating the Stieltjes integral may be stated, but we omit the details.*

For other inequalities of Grüss type, see [13]-[33].

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