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SMOOTH STRUCTURES ON π–MANIFOLDS

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Abstract. In this paper, we give a simple classification of smooth structures on closed $(m-1)-$connected $2m-$dimensional $\pi-$manifold M^{2m}. We will use the above to show that the number of smooth structures on $M^{2m} \times S^{2m}$ changes significantly.

INTRODUCTION

The notions of smooth manifold and diffeomorphism go back to Poincare. In his famous paper 'Analysis situs' [14], Poincare introduced the study of smooth manifolds under the equivalence relation of diffeomorphism. Poincare used the word homeomorphism to mean what is today called diffeomorphism (smooth). Of course the subject has considerably developed since Poincare. Major contributions have been made by Whitney [23], Pontrjagin [15], Lefschets [7], J. H. C Whitehead [22], Newman, and Alexander among others.

In the 20th century, some of the finest mathematicians have contributed to this area and it still remains the focus of mathematicians and mathematical physicists of different background. J. Munkres [11], S. Smale [19], S. P. Novikov [13], Antoni

A.Kosinski [2], C. T. C. Wall [21], Morris Hirsh [12], John Milnor [8,9,10], E. C. Zeeman [24]are among the 20th century mathematicians who contributed immensely to the development of the theory of differentiable (smooth) manifolds and in particular classification of smooth manifolds.

Manifolds are considered equivalent if they are diffeomorphic, i.e., there exists a differentiable map from one to the other with differentiable inverse. Given two smooth manifolds M and M', how can we decide whether or not there exists a diffeomorphism from M to M'. The general problem is to classify up to orientation-preserving diffeomorphism, those smooth manifolds homeomorphic to a given manifold. The first case to be treated in detail was of course the sphere. In [10], J. Milnor constructed a manifold which is homeomorphic to a 7-sphere but not diffeomorphic to it. The next case considered was a product of two spheres where [17] R. Schultz, [3] R. Desapio and [6] K. Kawakubo independently gave a complete classification of smooth structures on product of two spheres. This author in [1] extended the result of Desapio, Schultz and Kawakubo to structures of product of three spheres. It was later generalized to product of any number of spheres.

By studying diffeomorphism of product of two spheres, Edward C. Turner in [20] and Hajime Sato in [16] independently classified manifolds which can be expressed as union of two handlebodies along their boundaries. We will here give a classification of smooth structures on a $2m-$dimensional $\pi-$manifolds.

1. PRELIMINARIES

In [3] R. de Sapio gave a classification of smooth (differentiable) structures on product of spheres of the form $S^k \times S^p$ where $2 \leq k \leq p$, $k + p \geq 6$. In [1], this author extended R. de Sapio's result to smooth structures on product of spheres of the form $S^p \times S^q \times S^r$ where $2 \leq p \leq q \leq r$. In [4], R.de Sapio showed that when a type of $\pi-$manifold is embedded in Euclidean space, it is diffeomorphic to some connected sum of product of spheres. We will apply the above techniques and results to give a simple classification of smooth structures on $\pi-$manifold M^{2m} which is $(m-1)-$connected, where m=2,4,6(mod 8). We will also give a classification of a $4m-$manifold of the form $M^{2m} \times S^{2m}$. S^n denotes the unit $n-$sphere with the usual smooth (differential) structure in the Euclidean (n+1)-sphere R^{n+1}. A homotopy $n-$sphere is a closed $n-$manifold which has the homotopy type of the standard $n-$sphere S^n. Σ^n will denote an homotopy $n-$sphere. Θ^n denotes the group of $h-$cobordism classes of homotopy $n-$sphere under the connected sum operation. $\#$ means connected sum along boundary as defined by J. Milnor and M. Kervaire[9]. $H(p,k)$ denote the subset of θ^p consisting of those homotopy $p-$spheres Σ^p such that $\Sigma^p \times S^k$ is diffeomorphic to $S^p \times S^k$. By [3] Lemma 4, $H(p,k)$ is a subgroup and it is not always zero. Infact in [1], we showed that if $p-3 \leq k \leq p$ then $H(p,k) = \theta^p$. However by [5]. Corollary 1.5, $H(16,12) = 0$.

Let bP_{n+1} denote the subgroup of θ^n consisting of those homotopy $n-$spheres which bound parallelizable manifolds. It is known that if $\Sigma^n \in bP_{n+1}$ then Σ^n embeds in R^{n+2} with trivial normal bundle, hence $\Sigma^n \in H(n,k)$ where $k \geq 2$. Furthermore, by [9], if $n = 7, 11$, $bP_{n+1} = \theta^n$ and so $H(n,k) = \theta^n$ in these cases. All manifolds are here understood to be smooth (differentiable), compact and oriented. Smooth or differentiable will always mean of class C^∞. The notation D^n will be used for the standard unit $n-$disc in Euclidean space R^n and ∂M denotes the boundary of manifold M. -M is the manifold M with the orientation reversed. Recall that a $\pi-$manifold (or stably parallelizable or $s-$parallelizable manifold) is a manifold which has the property that the Whitney sum of its tangent bundle with a trivial line bundle is trivial. A framed $n-$manifold is a pair (M^n, f) where M^n is a $\pi-$manifold and $f : M \to ESO(n+1)$ is a fixed trivialization of the stable tangent bundle of M^n. Such an f is called a framing of M^n. In Theorem 2.1, we perform a framed spherical modification on a framed manifold (M^n, f). The technique is fully discussed in [8] and [9]. Framed corbordism is closely related to this technique. Note that two framed manifolds $({M_1}^n, f_1)$ and $({M_2}^n, f_2)$ where ${M_1}^n$ and ${M_2}^n$ are closed $\pi-$manifolds, are framed cobordant if there is an $(n+1)-$manifold W^{n+1} with a trivialization $F : W^{n+1} \to ESO(n+1)$ of its tangent bundle such that $\partial(W^{n+1}) = {M_1}^n \cup (-{M_2}^n)$ disjoint union and $F|_{M_1^n} = f_i, (i = 1, 2)$. Recall that in [8], theorem 1, it was proved that two-framed manifolds are framed cobordant if and only if we may obtain one from the other by a sequence of framed spherical modifications.

2. CLASSIFICATION

Theorem 2.1. *Let M^{2m} be an $(m-1)-$connected closed $\pi-$manifold $m \geq 3$. If $m = 2, 4, 6$, mod(8), then the number of distinct smooth structures on M^{2m} is in one-to-one correspondence with group θ^{2m}.*

Proof. By [4], Theorem B, manifold M^{2m} satisfying the above condition is diffeomorphic to the manifold of the form $(\#_{i=1}^{s} S^m \times S^m)\#\Sigma^{2m}$ where Σ^{2m} is unique and $r = 2s =$ rank of $H_m(M^{2m}, Z)$. From [1] and [3], it is easily deduced that any $2m-$manifold homeomorphic to $S^m \times S^m$ is almost diffeomorphic to it. We now show that any two manifolds satisfying the above conditions are almost diffeomorphic.

Let M_1^{2m} and M_2^{2m} satisfy the above conditions of the theorem, then by [4] there exists uniquely, homotopy spheres Σ_1^{2m},Σ_2^{2m} such that M_1^{2m} is diffeomorphic to $\#_{i=1}^{s}(S^m \times S^{2m})_i\#\Sigma_1^{2m}$ and M_2^{2m} is diffeomorphic to $\#_{i=1}^{s}(S^m \times S^{2m})_i\#\Sigma_2^{2m}$.

However,

$$\begin{aligned} M_1^{2m}\#(-\Sigma_1^{2m}\#\Sigma_2^{2m}) &= \#_{i=1}^{s}(S^m \times S^m)\#\Sigma_1^{2m}\#(-\Sigma_1^{2m}\#\Sigma_2^{2m}) \\ &= \#_{i=1}^{s}(S^m \times S^{2m})_i(\#\Sigma_1^{2m}\# - \Sigma_1^{2m})\#\Sigma_2^{2m} \\ &= \#_{i=1}^{s}(S^m \times S^{2m})_i\#\Sigma_2^{2m} \\ &= M_2^{2m} \end{aligned}$$

So let $\Sigma^{2m} = -\Sigma_1^{2m}\#\Sigma_2^{2m}$, then it follows that $M_1^{2m}\#\Sigma^{2m}$ is diffeomorphic to M_2^{2m} It then follows that any two manifolds that satisfy the conditions of the theorem are almost diffeomorphic. To prove the theorem, it then suffices to show that the inertial group of M^{2m} is trivial. Wall [21] showed this for $m = 3, 5, 7$(mod8). Here we will take $m = 2, 4, 6$(mod8) and use framed surgery to give a proof. Since M^{2m} is a closed $\pi-$manifold, its index is zero, so it follows from [9] Lemma 7.3 that M^{2m} can be reduced by framed surgery to an homotopy $2m-$sphere Σ^{2m}. Consider $M^{2m}\#(-\Sigma^{2m})$.

If necessary, then we can assume that M^{2m} can be reduced to S^{2m} by framed surgery. So M^{2m} is framed cobordant to S^{2m}. M^{2m} bounds a $\pi-$manifold, in fact a parallelizable manifold. However by assumption, M^{2m} is diffeomorphic to $M^{2m}\#\Sigma^{2m}$, it follows that $M^{2m}\#\Sigma^{2m}$ bounds a $\pi-$manifold W^{2m+1} say. Let $F : W^{2m+1} \to ESO(2m+1)$ be the framing of a tangent bundle of W^{2m+1} and let $f' = F|_{M^{2m}\#\Sigma^{2m}}$. By frame surgery, $(M^{2m}\#\Sigma^{2m}, f')$ can be reduced to (Σ^{2m}, h) for some framing h of Σ^{2m}. This means that it is framed cobordant to Σ^{2m}. Let W'^{2m+1} be the cobordism manifold. Since M^{2m} is diffeomorphic to $(\#_{i=1}^{s}(S^m \times S^m)_i)\#\Sigma'^{2m}$,using Milnors notation in [8], then all the framed surgeries are of the type $(m+1, m)$ and since $m = 2, 4, 6(\text{mod}8)$, the framed surgeries on $M^{2m}\#\Sigma^{2m}$ do not depend on framing of the stable tangent bundle of $M^{2m}\#\Sigma^{2m}$. This is because obstruction to extending any framing f of $M^{2m}\#\Sigma^{2m}$ to a trivialization of the tangent bundle of W'^{2m+1} is in $\pi_m SO(2m)$. But for $m = 2, 4, 6(\text{mod } 8)$, $\pi_m SO(2m) = 0$. So these surgeries on $M^{2m}\#\Sigma^{2m}$ may be framed with respect to any framing (particularly with respect to f'). Hence glueing W^{2m+1} and W'^{2m+1} along their common boundary $M^{2m}\#\Sigma^{2m}$, we get a framed manifold W''^{2m+1} with boundary Σ^{2m}, hence Σ^{2m} bounds a $\pi-$manifold W''^{2m+1}. This implies $\Sigma^{2m} \in bP_{2m+1}$, but by [9] Theorem 5, $bP_{2m+1} = 0$ hence $\Sigma^{2m} = S^{2m}$. Thus the inertial group $I(M^{2m})$ of M^{2m} is trivial hence the number of smooth structures on M^{2m} is θ^{2m}.

We will use the above to classify a $4m-$manifold of the form $M^{2m} \times S^{2m}$ where M^{2m} is a $\pi-$manifold and $(m-1)-$connected. We will apply the obstruction theory of Munkres [11]. If two $n-$manifolds M and N are piecewise linear homeomorphic, by [11] (Theorem 2.8), there is a diffeomorphism modulo L of M onto N, where L is the $(n-1)-$skeleton of a triangulation of M. Suppose $f : M^n \to N^n$ is a diffeomorphism modulo $m-$skeleton $m < n$, the obstruction to deforming f to $g : M^n \to N^n$, a diffeomorphism modulo $(m-1)-$skeleton is an element $\lambda_m(f) \in H_m(M, \Gamma^{n-m})$ where Γ^{n-m} is a group of diffeomorphism of S^{n-m-1} modulo those that are extendable to diffeomorphism of D^{n-m} (g is called the smoothing of f). If $\lambda_m(f) = 0$ then by [11] Section 4, the smoothing g exists. We will apply this theory to show the following.

Theorem 2.2. *Let M^{4m} be a $4m$ manifold of the form $M^{2m} \times S^{2m}$ where M^{2m} is a closed, $(m-1)-$connected $\pi-$manifold, then there exists homotopy spheres Σ^{3m} and Σ^{4m} such that M^{4m} is diffeomorphic to*

$$\{S^m \times \{(S^m \times S^{2m})\#\Sigma^{3m}\}\}\#\{(\#_1^{S-1}(S^m \times S^m)_i) \times S^{2m}\}\#\Sigma^{4m}$$

Proof. By [4] (Theorem B), M^{2m} is diffeomorphic to $\#_{i=1}^{S}(S^m \times S^m)_i \# \Sigma^{2m}$, where Σ^{2m} is unique, hence $M^{2m} \times S^{2m}$ is diffeomorphic to $[(\#_1^S(S^m \times S^m)_i)\}\#\Sigma^{2m}] \times S^{2m}$ But by [1] Lemma 2.1.1, $\Sigma^{2m} \times S^{2m}$ is diffeomorphic with $S^{2m} \times S^{2m}$, hence it follows that $M^{2m} \times S^{2m}$ is diffeomorphic with $(\#_1^S(S^m \times S^m)_i) \times S^{2m}$. It then follows that the number of smooth structures on $M^{2m} \times S^{2m}$ is the number of smooth stuctures on $(\#_1^S(S^m \times S^m)_i) \times S^{2m}$. Notice that the homology $H_*(M^{2m} \times S^{2m}, Z)$ is non-zero only on dimensions m, $2m$, $3m$ and $4m$.

Let $h : M^{4m} \to M^{2m} \times S^{2m}$ be the given homeomorphism, using Munkre's theory [11], there is no obstruction to deforming h to a diffeomorphism modulo $3m-$skeleton. This is because $H_i(M^{2m} \times S^{2m}, Z) = 0$ for $3m < i < 4m$. The obstruction to deforming h to a diffeomorphism modulo $(3m-1)-$skeleton is $\lambda_{3m}(h) \in H_{3m}(M^{2m} \times S^{2m}, \Gamma^m) = \Gamma^m \oplus \Gamma^m$.

Let $\lambda_{3m}(h) = \varphi_1 + \varphi_2$, where each $\varphi_i : S^{m-1} \to S^{m-1}$ is a diffeomorphism. We define $\Sigma^m = D_1^m \cup_{\varphi_1\varphi_2} D_2^m$, and let $j : S^m \to \Sigma^m$ be a homeomorphism defined by $j : D_1^m \cup_{id} D_2^m \to D_1^m \cup_{\varphi_1\varphi_2} D_2^m$ where

$$j(z) = \begin{cases} z & \text{if } z \in int(D_1^m) \\ |z|(\varphi_1\varphi_2)^{-1}(\frac{z}{|z|}) & \text{if } z \in D_2^m \end{cases}$$

j is a piecewise linear homeomorphism and the obstruction to deforming j to a diffeomorphism is $[\phi_1\phi_2^{-1}] = -\lambda_m(h)$. From j, we define a map $j \times id : S^m \times S^m \to \Sigma^m \times S^m$ which is a piecewise linear homeomorphism and its obstruction to a diffeomorphism is $-\lambda_m(h)$. However, by [1] and [3], $\Sigma^m \times S^m$ is diffeomorphic to $S^m \times S^m$ and so $j \times id : S^m \times S^m \to S^m \times S^m$ is a piecewise linear homeomorphism with obstruction to a diffeomorphism is also $-\lambda_m(h)$.

Let $S^m \times S^m - int(D^{2m}) = (S^m \times S^m)_0$. We define a map

$g : (S^m \times S^m)\#_{i=1}^{S-1} S^m \times S^m \to S^m \times S^m \#_{i=1}^{S-1} S^m \times S^m$

$$g(z) = \begin{cases} j \times id(z) & \text{if } z \in (S^m \times S^m)_0 \\ z & \text{elsewhere} \end{cases}$$

and $g \times id : (\#_{i=1}^{S} S^m \times S^m) \times S^{2m} \to (\#_{i=1}^{S} S^m \times S^m) \times S^{2m}$ whose obstruction to a diffeomorphism is $-\lambda_m(h)$. The obstruction to deforming the composite $M^{4m} \to^h M^{2m} \times S^{2m} \to^{g \times id} M^{2m} \times S^{2m}$ to a diffeomorphism modulo $(3m-1)-$skeleton is $\lambda_m((g \times id)h) = -\lambda_m(h) + \lambda_m(h) = 0$. Hence $h' = (g \times id)h : M^{4m} \to M^{2m} \times S^{2m}$ is a diffeomorphism modulo $(3m-1)-$skeleton. Since $H_i(M^{2m} \times S^{2m}, Z) = 0$ for $2m < i < 3m$, there is no obstruction to deforming h' to a diffeomorphism modulo $2m-$skeleton. The obstruction to deforming h' to a diffeomorphism modulo $(2m-1)-$skeleton is $\lambda_{2m}(h') \in H_{2m}(M^{2m} \times S^{2m}, \Gamma^{2m}) = \Gamma^{2m} \oplus \Gamma^{2m}$.

Let $\lambda_{2m}(h') = \psi_1 + \psi_2$ where $\psi_i : S^{2m-1} \to S^{2m-1}$ is a diffeomorphism for $i = 1, 2$. Again we define $\Sigma^{2m} = D_1^{2m} \cup_{\psi_1\psi_2} D_2^{2m}$ and $j : S^{2m} \to \Sigma^{2m}$ defined by $j : D_1^{2m} \cup_{id} D_2^{2m} \to D_1^{2m} \cup_{id} D_2^{2m}$ where

$$j(z) = \begin{cases} z & \text{if } z \in int(D_1^{2m}) \\ |z|(\psi_1\psi_2)^{-1}(\frac{z}{|z|}) & \text{if } z \in D_2^{2m} \end{cases}$$

j is a piecewise linear homeomorphism and the obstruction to deforming j to a diffeomorphism is $[(\psi_1\psi_2)^{-1}] = -\lambda_{2m}(h')$. From j, we build a map $j' : S^m \times S^m \# S^{2m} \to S^m \times S^m \# \Sigma^{2m}$ where

$$j'(z) = \begin{cases} z & \text{for } z \in (S^m \times S^m)_0 = S^m \times S^m - int(D^{2m}) \\ j(z) & \text{for } z \in D^{2m} \end{cases}$$

Obstruction to deforming j' to a diffeomorphism is $-\lambda_{2m}(h')$. From j' we build a map j'' by taking connected sum with $\#_{i=1}^{S-1} S^m \times S^m$ on both sides to have $j'' : (\#_{i=1}^{S-1} S^m \times S^m)\# S^m \times S^m \to (\#_{i=1}^{S-1} S^m \times S^m)\# S^m \times S^m \# \Sigma^{2m}$ which is the same as $j'' : \#_{i=1}^{S} S^m \times S^m \to (\#_{i=1}^{S-1} S^m \times S^m)\#\Sigma^{2m}$ and the obstruction to deforming j'' to a diffeomorphism is $-\lambda_{2m}(h')$. Similarly, the map

$j'' \times id : (\#_{i=1}^{S-1}(S^m \times S^m)_i) \times S^{2m} \to (\#_{i=1}^{S-1}(S^m \times S^m)_i \# \Sigma^{2m}) \times S^{2m}$

is a piecewise linear homeomorphism with obstruction to a diffeomorphism which is $-\lambda_{2m}(h')$.

Consider the composite

$M^{4m} \to^{h'} (\#_{i=1}^{S-1}(S^m \times S^m)_i) \times S^{2m} =$

$$M^{2m} \times S^{2m} \to^{j'' \times id} (\#_{i=1}^{S}(S^m \times S^m)_i \# \Sigma^{2m}) \times S^{2m}$$

The obstruction to deforming $j'' \times id)h'$ to a diffeomorphism modulo $(2m-1)-$skeleton is $\lambda_{2m}((j'' \times id)h') = \lambda_{2m}(j'' \times id) + \lambda_{2m}(h') = -\lambda_{2m}(h') + \lambda_{2m}(h') = 0$. Let $h'' = (j'' \times id)h'$, hence $h'' : M^{4m} \to (\#_{i=1}^{S}(S^m \times S^m)_i \# \Sigma^{2m}) \times S^{2m}$ is a diffeomorphism modulo $(2m-1)-$skeleton. Recall that $\Sigma^{2m} \times S^{2m}$ is diffeomorphic with $S^{2m} \times S^{2m}$ and so $(\#_{i=1}^{S}(S^m \times S^m)_i \# \Sigma^{2m}) \times S^{2m} = (\#_{i=1}^{S}(S^m \times S^m)_i) \times S^{2m} = M^{2m} \times S^{2m}$, hence $h'' : M^{4m} \to M^{2m} \times S^{2m}$ is a diffeomorphism modulo $(2m-1)-$skeleton. Since $H_i(M^{4m}, Z) = 0$ for $m < i < 2m$, there is no obstruction to deforming h'' to a diffeomorphism modulo $m-$skeleton. The obstruction to deforming h'' to a diffeomorphism modulo $(m-1)-$skeleton is $\lambda_m(h'') \in H_m(M^{4m}, \Gamma^{3m}) = \Gamma^{3m} \oplus \Gamma^{3m}$. Let $\lambda_m(h'') = \alpha_1 + \alpha_2$ where $\alpha_i : S^{3m-1} \to S^{3m-1}$, $i = 1, 2$ is a diffeomorphism $\alpha_i \in \Gamma^{3m}$. Let $\Sigma^{3m} = D_1^{3m} \cup_{\alpha_1\alpha_2} D_2^{3m}$ and define

$j : S^{3m} \to \Sigma^{3m}$ $j : D_1^{3m} \cup_{id} D_2^{3m} \to D_1^{3m} \cup_{\alpha_1\alpha_2} D_2^{3m}$ where

$$j(z) = \begin{cases} z & \text{if } z \in int(D_1^{3m}) \\ |z|(\alpha_1\alpha_2)^{-1}(\frac{z}{|z|}) & \text{if } z \in D_2^{3m} \end{cases}$$

The obstruction to deforming j to a diffeomorphism is $\lambda_m(\alpha_1\alpha_2)^{-1} = -\lambda_m(h'')$. By taking the connected sum with $S^m \times S^{2m}$, we have a map say

$g : S^m \times S^{2m} \# S^{3m} \to S^m \times S^{2m} \# \Sigma^{3m}$, where

$$g(z) = \begin{cases} z & \text{if } z \in (S^m \times S^{2m})_0 = S^m \times S^{2m} - int(D^{3m}) \\ j & \text{elsewhere} \end{cases}$$

Thus the obstruction to deforming g to a diffeomorphism is still $-\lambda_m(h'')$ but $S^m \times S^{2m} \# S^{3m} = S^m \times S^{2m}$, hence we have $g : S^m \times S^{2m} \to S^m \times S^{2m} \# \Sigma^{3m}$. Consider the map $id \times g : S^m \times S^m \times S^{2m} \to S^m \times (S^m \times S^{2m} \# \Sigma^{3m})$, we then add to both sides, by connected sum, the manifold $(\#_{i=1}^{S-1}(S^m \times S^m)_i) \times S^{2m}$ to get

$g' : S^m \times S^m \times S^{2m} \#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m} \to$

$$S^m \times (S^m \times S^{2m} \# \Sigma^{3m} \#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m}$$

This gives
$g' : \#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m} \to S^m \times (S^m \times S^{2m}\#\Sigma^{3m}\#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m}$ and the obstruction to a diffeomorphism of g' is also $-\lambda_m(h'')$. The obstruction to deforming the composite

$$g'h'' : M^{4m} \to^{h''} \#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m} \to^{g'} S^m \times (S^m \times S^{2m}\#\Sigma^{3m})\#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m}$$

to a diffeomorphism modulo $(m-1)-$skeleton is $\lambda_m(g'h'') = \lambda_m(g') + \lambda_m(h'') = -\lambda_m(h'') + \lambda_m(h'') = 0$. Thus we have a diffeomorphism $f = g'h'' : M^{4m} \to S^m \times (S^m \times S^{2m}\#\Sigma^{3m})\#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m}$ modulo $(m-1)-$skeleton. Since $H_i(M^{4m}, Z) = 0$ for $0 < i < m-1$, then there is no obstruction to deforming f to a diffeomorphism modulo zero skeleton. We can thus assume that f is a diffeomorphism modulo one point. Therefore there exists a homotopy $4m-$sphere Σ^{4m} such that M^{4m} is diffeomorphic to
$\{S^m \times \{(S^m \times S^{2m})\#\Sigma^{3m}\}\}\#\{(\#_{i=1}^{S-1}(S^m \times S^m)_i) \times S^{2m}\}\#\Sigma^{4m}$
Hence the theorem.

Theorem 2.3. *If M^{4m} is a $4m-$manifold homeomorphic to $M^{2m} \times S^{2m}$ where M^{2m} is a closed $(m-1)-$connected $\pi-$manifold, then the smooth structures on M^{4m} is in one-to-one correspondence with the group $\frac{\theta^{3m}}{H(3m,m)} \times \theta^{4m}$.*

Proof. Let O represent the standard structure on $M^{2m} \times S^{2m}$ and O_1 represents the trivial element of θ^{3m} while O_2 represents the trivial element of θ^{4m}. We define a map $f : \theta^{3m} \times \theta^{4m} \to$ Structures on $M^{2m} \times S^{2m}$ by
$f(\Sigma^{3m}, \Sigma^{4m}) = [S^m \times (S^m \times S^{2m}\#\Sigma^{3m})\#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m}]\#\Sigma^{4m}$,
f maps to (O_1, O_2) to the base point O of $M^{2m} \times S^{2m}$. If the pair $\Sigma_1^{3m}, \Sigma_2^{3m} \in \theta^{3m}$ and $\Sigma_1^{4m}, \Sigma_2^{4m} \in \theta^{4m}$ are $h-$cobordant respectively then by Smale [9], they are diffeomorphic, then $S^m \times \Sigma_1^{3m}$ is diffeomorphic to $S^m \times \Sigma_2^{3m}$, hence
$[S^m \times (S^m \times S^{2m}\#\Sigma_1^{3m})\#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m}]\#\Sigma_1^{4m}$, is diffeomorphic with $[S^m \times (S^m \times S^{2m}\#\Sigma_2^{3m})\#_{i=1}^{S-1}(S^m \times S^m)_i \times S^{2m}]\#\Sigma_2^{4m}$, and so f is well defined. f is onto because given a structure i.e. given a manifold homeomorphic to $M^{2m} \times S^{2m}$, by Theorem 2.2, there exists $(\Sigma^{3m}, \Sigma^{3m}) \in \theta^{3m} \times \theta^{4m}$ such that M^{4m} is diffeomorphic

with $[S^m \times (S^m \times S^{2m} \# \Sigma^{3m}) \#_{i=1}^{S-1} (S^m \times S^m)_i \times S^{2m}] \# \Sigma^{4m}$. We need to find kernel of f. Let $\Sigma^{3m} \in H(3m, m)$ then $\Sigma^{3m} \times S^m$ is diffeomorphic to $S^{2m} \times S^m$, hence $S^m \times (S^m \times S^{2m} \# \Sigma^{3m}) \#_{i=1}^{S-1} (S^m \times S^m)_i \times S^{2m}$, is diffeomorphic with $\#_{i=1}^{S-1} (S^m \times S^m)_i \times S^{2m}$, hence $H(3m, m) \subset$ Kernel of $f(Kerf)$. If $f(\Sigma^{3m}, O_2) = 0$ then it follows by [1] Theorem 2.2.1, that $\Sigma^{3m} \in H(3m, m)$. Similarly if $f(O_1, \Sigma^{4m}) = 0$, then by ([18] Theorem A) it follows that Σ^{4m} is diffeomorphic to S^m. Hence $H(3m, m) = Kerf$ since f is onto with Kernel $H(3m, m)$ then the number of structures on M^{4m} is in one-to-one correspondence with the group $\frac{\theta^{3m}}{H(3m,m)} \times \theta^{4m}$.

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