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S–ORTHOGONALITY ON THE SEMICIRCLE[1]

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Abstract. In this paper s–orthogonal polynomials on the semicircle are considered. Generalizing the previous works of Gautschi, Milovanović and Landau [5, 4, 8, 7, 9, 3] we transfer concept of s–orthogonality (see [6, 10]) to the unit semicircle in the complex plane, with respect to the complex-valued inner product $(f, g) = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\,d\theta$. A detailed study is made of the s–orthogonal polynomials on the semicircle in the case of the Chebyshev weight of the first kind.

1. INTRODUCTION

For $s \in \mathbf{N}_0$, s–orthogonal polynomials $\pi_n = \pi_{n,s}$ (π_n–monic of degree n) on $\mathbf{R}$ with respect to the measure $d\lambda(t)$ are polynomials which satisfy the conditions

$$\int_{\mathbf{R}} [\pi_{n,s}(t)]^{2s+1} t^k \, d\lambda(t) = 0 \quad (k = 0, 1, \dots, n-1). \tag{1}$$

In the case $d\lambda(t) = w(t)\,dt$ on $[a, b]$ these polynomials are known as s–orthogonal polynomials in the interval $[a, b]$ with respect to the weight function w. For $s = 0$ they reduce to the standard orthogonal polynomials.

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It is known (see [6, 10]) that several properties of standard orthogonal polynomials on real line are also valid for s–orthogonal polynomials on real line (see [1, 2]).

Theorem 1. *There exists a unique monic polynomial* $\pi_{n,s}$ *for which* (1) *is satisfied, and* $\pi_{n,s}$ *has* n *distinct real zeros which are all contained in the open interval* (a, b).

Theorem 2. *For every* $s \in \mathbf{N}_0$, *the zeros of* $\pi_{n,s}$ *and* $\pi_{n+1,s}$ *mutually separate each other.*

The paper is organized as follows. Some basic facts on the orthogonal polynomials on the semicircle are given in Section 2. In Section 3 we transfer the concept of s–orthogonality to the unit semicircle in the complex plane. We introduce the s–orthogonal polynomials on the semicircle, give a method for their construction and study the case of the Chebyshev weight function of the first kind.

2. POLYNOMIALS ORTHOGONAL ON THE SEMICIRCLE

Polynomials orthogonal on the semicircle were introduced by Gautschi and Milovanović [5].

Let w be a weight function which is positive and integrable on the open interval $(-1, 1)$, though possibly singular at the endpoints, and which can be extended to a function $w(z)$ holomorphic in the half disc

$$D_+ = \{z \in \mathbf{C} : |z| < 1, \ \mathrm{Im}\ z > 0\}.$$

Consider the following inner product

$$[f, g] = \int_\Gamma f(z)g(z)w(z)(iz)^{-1}\, dz = \int_0^\pi f(e^{i\theta})g(e^{i\theta})w(e^{i\theta})\, d\theta, \tag{2}$$

where Γ is the circular part of ∂D_+ and all integrals are assumed to exist (possibly) as appropriately defined improper integrals.

This inner product (2) is not Hermitian and the existence of the corresponding orthogonal polynomials, therefore, is not guaranteed.

We call a system of complex polynomials $\{\pi_k\}$ *orthogonal on the semicircle* if

$$[\pi_k, \pi_l] \begin{cases} = 0 & \text{if } k \neq l, \\ \neq 0 & \text{if } k = l, \end{cases} \quad k, l = 0, 1, 2, \ldots; \tag{3}$$

we assume π_k monic of degree k.

Gautschi, Landau and Milovanović [4] have established the existence of orthogonal polynomials $\{\pi_k\}$ assuming only that

$$\text{Re } [1, 1] = \text{Re } \int_0^\pi w(e^{i\theta})\, d\theta \neq 0. \tag{4}$$

They have represented π_n as a linear (complex) combination of p_n and p_{n-1} ($\{p_k\}$ is corresponding ordinary orthogonal polynomials sequence (real) with respect to the same weight function w in the interval $[-1, 1]$):

$$\pi_n(z) = p_n(z) - i\theta_{n-1} p_{n-1}(z), \quad n = 0, 1, 2, \ldots .$$

Polynomials orthogonal on the semicircle also satisfy the tree-term recurrence relation:

$$\pi_{k+1}(z) = (z - i\alpha_k)\pi_k(z) - \beta_k \pi_{k-1}(z), \quad k = 0, 1, 2, \ldots,$$

with initial conditions $\pi_{-1}(z) = 0, \quad \pi_0(z) = 1$.

Under certain conditions all zeros of polynomials orthogonal on the semicircle are in D_+ (see [3, 4, 5, 8, 9]).

3. S–ORTHOGONALITY ON THE SEMICIRCLE

Denote, as in Section 2,

$$D_+ = \{z \in \mathbf{C} : |z| < 1, \text{ Im } z > 0\}, \quad \Gamma = \{z \in \mathbf{C} : |z| = 1, \text{ Im } z > 0\} .$$

We consider the case of the Chebyshev weight function of the first kind

$$w(z) = (1 - z^2)^{-1/2}.$$

Using Cauchy's theorem, it is easy to see that the following lemma holds:

Lemma 1. *For any polynomial g we have*

$$\int_\Gamma \frac{g(z)w(z)}{iz}\,dz - \pi g(0)w(0) + \frac{1}{i}\fint_{-1}^{1} \frac{g(x)w(x)}{x}\,dx = 0,$$

i.e.,

$$\fint_{-1}^{1} \frac{g(x)}{x\sqrt{1-x^2}}\,dx = i\pi g(0) - \int_\Gamma \frac{g(z)}{z\sqrt{1-z^2}}\,dz. \tag{5}$$

Let $\pi_{n,s}$ be monic polynomials of degree n which satisfy the s–orthogonality conditions:

$$\int_\Gamma \pi_{n,s}^{2s+1}(z) z^k (1-z^2)^{-1/2}\,\frac{dz}{iz} = 0 \qquad (k = 0, 1, \ldots, n-1). \tag{6}$$

For $s = 0$ we have polynomials orthogonal on the semicircle (see Section 2).

According to (5), we have:

(i) For $k = 0$:

$$\fint_{-1}^{1} \frac{\pi_{n,s}^{2s+1}(x)}{x}\,\frac{dx}{\sqrt{1-x^2}} = i\pi\,\pi_{n,s}^{2s+1}(0)\,. \tag{7}$$

(ii) For $k = 1, 2, \ldots, n-1$:

$$\fint_{-1}^{1} \pi_{n,s}^{2s+1}(x) x^\nu\,\frac{dx}{\sqrt{1-x^2}} = 0 \qquad (\nu = 0, 1, \ldots, n-2)\,. \tag{8}$$

From (8) we see that the polynomial $\pi_{n,s}^{2s+1}$ is orthogonal to all polynomials of degree at most $n-2$ with respect to $w(x)$ in the interval $[-1, 1]$. Because of that the polynomial $\pi_{n,s}^{2s+1}$ may be represented in the form:

$$\pi_{n,s}^{2s+1}(z) = C_{n-1}T_{n-1}(z) + C_n T_n(z) + \ldots + C_{(2s+1)n}T_{(2s+1)n}(z), \tag{9}$$

where T_i are the Chebyshev polynomials of the first kind and $C_{(2s+1)n} = \dfrac{1}{2^{(2s+1)n-1}}$.

On the other hand, we have

$$\pi_{n,s}(z) = a_0T_0(z) + a_1T_1(z) + \cdots + a_nT_n(z)\,, \qquad a_n = \frac{1}{2^{n-1}}\,. \tag{10}$$

Relations (9) and (10) give

$$(a_0T_0 + a_1T_1 + \cdots + a_nT_n)^{2s+1} = C_{n-1}T_{n-1} + \cdots + C_{(2s+1)n}T_{(2s+1)n}\,. \tag{11}$$

Using the formula:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum \frac{n!}{k_1!k_2!\ldots k_m!}\, x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m}\,,$$

where the sum is computed for all m-tuples of nonnegative integers $(k_1, k_2, \ldots, k_m)$ satisfying $k_1 + k_2 + \cdots + k_m = n$, and the property of the Chebyshev polynomials

$$T_m \cdot T_n = \frac{1}{2}\left(T_{m+n} + T_{|m-n|}\right),$$

the left hand side of (11) may be written in the form

$$(a_0T_0 + a_1T_1 + \cdots + a_nT_n)^{2s+1} = C_0T_0 + C_1T_1 + \cdots + C_{(2s+1)n}T_{(2s+1)n}. \tag{12}$$

Now, because of (11), we conclude that $C_0 = C_1 = \ldots = C_{n-2} = 0$. These equations together with (7) give a system of $n - 1 + 1 = n$ equations for determining the unknown coefficients a_ν $(\nu = 0, 1, \ldots, n-1)$:

$$\begin{cases} C_\nu = 0 \quad (\nu = 0, 1, \ldots, n-2) \\[2ex] \displaystyle\fint_{-1}^{1} \frac{\pi_{n,s}^{2s+1}(x)}{x\sqrt{1-x^2}}\, dx = i\pi\, \pi_{n,s}^{2s+1}(0)\,. \end{cases} \tag{13}$$

3.1. CASE $s = 1$

For $s = 1$, the relation (11) gives

$$(a_0T_0 + a_1T_1 + \cdots + a_nT_n)^3 = C_{n-1}T_{n-1} + \cdots + C_{3n}T_{3n}\,. \tag{14}$$

On the left hand side of (14) we have addends of form

$$\begin{aligned} &a_iT_i^3 && (i = 0, 1, \ldots, n), \\ &3a_ia_j^2T_iT_j^2 && (i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, n\,;\ i \neq j), \\ &6a_ia_ja_kT_iT_jT_k && (i = 0, 1, \ldots, n;\ j = 1, 2, \ldots, n;\ k = 2, 3, \ldots, n;\ \ i, j, k \text{ different})\,, \end{aligned}$$

for which hold

$$T_i^3 = \frac{1}{4}T_{3i} + \frac{3}{4}T_i \qquad (i = 0, 1, \ldots, n),$$

$$T_iT_j^2 = \frac{1}{4}T_{2j+i} + \frac{1}{4}T_{|2j-i|} + \frac{1}{2}T_i \qquad (i = 0, 1, \ldots, n;\ j = 0, 1, \ldots, n;\ \ i \neq j),$$

$$T_iT_jT_k = \frac{1}{4}T_{i+j+k} + \frac{1}{4}T_{|j+k-i|} + \frac{1}{4}T_{i+|j-k|} + \frac{1}{4}T_{||j-k|-i|} \quad (i = 0, \ldots, n;\ j = 1, \ldots, n;\ k = 2, \ldots, n;\ i, j, k \text{ different}).$$

It is known that for all $k \in \mathbf{N}$, $T_{2k+1}(0) = 0$ and $T_{2k}(0) = (-1)^k$. Using these relations we get

$$\pi_{n,1}^3(0) = C_0 - C_2 + C_4 - C_6 + \cdots .$$

1° Case $n = 1$.

In this case $\pi_{1,1}(z) = T_1 + a_0 T_0$ and

$$\begin{aligned} \pi_{1,1}^3(z) &= T_1^3(z) + 3a_0 T_0(z) T_1^2(z) + 3a_0^2 T_0^2(z) T_1(z) + a_0^3 T_0^3(z) \\ &= \frac{1}{4} T_3(z) + \frac{3}{2}\, a_0 T_2(z) + \left(\frac{3}{4} + 3a_0^2\right) T_1(z) + \frac{3}{2} a_0 + a_0^3. \end{aligned}$$

Therefore,

$$\pi_{1,1}^3(0) = -\frac{3}{2}\, a_0 + \frac{3}{2}\, a_0 + a_0^3 = a_0^3\,.$$

The coefficient a_0 may be determined from (7), i.e., from

$$\frac{1}{4} \fint_{-1}^{1} \frac{T_3(x)}{x\sqrt{1-x^2}}\, dx + \frac{3}{2}\, a_0 \fint_{-1}^{1} \frac{T_2(x)}{x\sqrt{1-x^2}}\, dx + \left(\frac{3}{4} + 3a_0^2\right) \fint_{-1}^{1} \frac{T_1(x)}{x\sqrt{1-x^2}}\, dx$$

$$+ \left(\frac{3}{2}\, a_0 + a_0^3\right) \fint_{-1}^{1} \frac{dx}{x\sqrt{1-x^2}} = i\pi\, a_0^3.$$

Hence, a_0 satisfies

$$-\frac{\pi}{4} + \left(\frac{3}{4} + 3a_0^2\right) \pi = i\pi\, a_0^3,$$

i.e.,

$$1 + 6a_0^2 - 2ia_0^3 = 0\,.$$

The previous equation has three pure imaginary solutions

$$0.3843671526\, i, \quad 0.4421253017\, i, \quad -2.9422418510\, i.$$

Thus, there are three polynomials of degree one which satisfy the s–orthogonality condition on the semicircle (6).

2° Case $n = 2$.

According to (10) we write

$$\pi_{2,1}(z) = a_2 T_2(z) + a_1 T_1(z) + a_0 T_0(z)\,.$$

Since $a_2 = \dfrac{1}{2}$, using the previous described procedure, we obtain

$$C_0 = \frac{3}{8}\,a_1^2 + a_0^3 + \frac{3}{2}\,a_0 a_1^2 + \frac{3}{8}\,a_0,$$

$$C_1 = \frac{3}{4}\,a_1^3 + 3a_0^2 a_1 + \frac{3}{8}\,a_1 + \frac{3}{2}\,a_0 a_1,$$

$$C_2 = \frac{3}{2}\,a_0 a_1^2 + \frac{3}{32} + \frac{3}{2}\,a_0^2 + \frac{3}{4}\,a_1^2,$$

$$C_3 = \frac{1}{4}\,a_1^3 + \frac{3}{16}\,a_1 + \frac{3}{2}\,a_0 a_1,$$

$$C_4 = \frac{3}{8}\,a_0 + \frac{3}{8}\,a_1^2,$$

$$C_5 = \frac{3}{16}\,a_1,$$

$$C_6 = \frac{1}{32}.$$

Therefore,

$$\pi_{2,1}^3(0) = C_0 - C_2 + C_4 - C_6 = a_0^3 + \frac{3}{4}\,a_0 - \frac{3}{2}\,a_0^2 - \frac{1}{8}\,.$$

Unknown coefficients a_0 and a_1 are determined from the system of equations

$$\begin{cases} C_0 = 0, \\[2ex] \displaystyle\fint_{-1}^{1} \frac{\pi_{2,1}^3(x)}{x\sqrt{1-x^2}}\,dx = i\pi\,\pi_{2,1}^3(0)\,. \end{cases} \tag{15}$$

Using the representation

$$\pi_{2,1}^3(x) = C_1 T_1(x) + C_2 T_2(x) + \cdots + C_6 T_6(x)$$

and knowing that

$$\fint_{-1}^{1} \frac{T_{2k}(x)}{x\sqrt{1-x^2}}\,dx = 0 \quad \text{for all } k \in \mathbf{N}\,,$$

the second equation from (15) is equivalent to equation

$$C_1 \fint_{-1}^{1} \frac{T_1(x)}{x\sqrt{1-x^2}}\,dx + C_3 \fint_{-1}^{1} \frac{T_3(x)}{x\sqrt{1-x^2}}\,dx + C_5 \fint_{-1}^{1} \frac{T_5(x)}{x\sqrt{1-x^2}}\,dx = i\pi\,\pi^3_{2,1}(0),$$

i.e.,

$$C_1\,\pi - C_3\,\pi + C_5\,\pi = i\pi\,\pi^3_{2,1}(0).$$

Finally, the second equation in (15) is equivalent to

$$\frac{1}{2}\,a_1^3 + 3a_0^2 a_1 + \frac{3}{8}\,a_1 = i\left(a_0^3 + \frac{3}{4}\,a_0 - \frac{3}{2}\,a_0^2 - \frac{1}{8}\right).$$

Therefore, (15) is equivalent to

$$\begin{cases} 3a_1^2 + 8a_0^3 + 12a_0a_1^2 + 3a_0 = 0, \\ 4a_1^3 + 24a_0^2a_1 + 3a_1 = i(8a_0^3 + 6a_0 - 12a_0^2 - 1). \end{cases} \tag{16}$$

From the first equation in (16) we obtain

$$a_1^2 = \frac{-3a_0 - 8a_0^3}{3 + 12a_0}. \tag{17}$$

From the second equation in (16) and (17) we get a_0. Knowing a_0 from (17) we find a_1.

The next 9 polynomials satisfy the system of equations (15):

$0.5\,(2z^2 - 1) - 1.255041679\,i\,z - 1.2273728340,$
$0.5\,(2z^2 - 1) + 1.410610320\,i\,z - 0.2958451886,$
$0.5\,(2z^2 - 1) - 0.2280444577\,i\,z + 0.06474935317,$
$0.5\,(2z^2 - 1) - (0.3573159034 + 0.7273322236\,i)\,z - 0.2570758901 - 0.1246144376\,i,$
$0.5\,(2z^2 - 1) + (0.3573159034 - 0.7273322236\,i)\,z - 0.2570758901 + 0.1246144376\,i,$
$0.5\,(2z^2 - 1) + (0.1321606183 + 0.3519722699\,i)\,z + 0.1404065541 - 0.2708240895\,i,$
$0.5\,(2z^2 - 1) - (0.1321606183 - 0.3519722699\,i)\,z + 0.1404065541 + 0.2708240895\,i,$
$0.5\,(2z^2 - 1) - (0.2511981689 + 0.2567189698\,i)\,z + 0.1887254527 - 0.5743377565\,i,$
$0.5\,(2z^2 - 1) + (0.2511981689 - 0.2567189698\,i)\,z + 0.1887254527 + 0.5743377565\,i.$

Zeros of these polynomials are respectively

$$\begin{array}{lcl}
-1.154811859 + 0.627520840\,i & \text{and} & 1.154811859 + 0.627520840\,i, \\
-0.546250693 - 0.705305160\,i & \text{and} & 0.546250693 - 0.705305160\,i, \\
-0.649807339 + 0.114022229\,i & \text{and} & 0.649807339 + 0.114022229\,i, \\
-0.646294092 + 0.209379493\,i & \text{and} & 1.003609995 + 0.517952731\,i, \\
-1.003609995 + 0.517952731\,i & \text{and} & 0.646294092 + 0.209379493\,i, \\
-0.689477474 - 0.411857056\,i & \text{and} & 0.557316855 + 0.059884785\,i, \\
-0.557316855 + 0.059884785\,i & \text{and} & 0.689477474 - 0.411857056\,i, \\
-0.578687886 - 0.302275647\,i & \text{and} & 0.829886055 + 0.558994616\,i, \\
-0.829886055 + 0.558994616\,i & \text{and} & 0.578687886 - 0.302275647\,i.
\end{array}$$

For all computations we used program package MATHEMATICA.

Therefore, for $n = 2$ there are 9 monic polynomials which satisfy s–orthogonality conditions on the semicircle (6).

From these examples we see that s–orthogonal polynomials on the semicircle are not unique. Furthermore, zeros are not all in D_+.

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