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ON A SIX-PARAMETER GENERALIZED LOGISTIC DISTRIBUTION

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Abstract. Because of their flexibility, much attention has been given to the study of generalized models in recent times. In particular, many authors have investigated the properties of the generalized logistic distribution and its application in analysing bioassay and quantal response data.

Recently, Wu Jong-Wuu et al (2000) developed a method which was used to obtain the density function of a five-parameter generalized logistic distribution. He investigated some of its properties and used it to analyse some bioassay data.

In this present paper, we take a step forward by defining a suitable random variable that enables us to obtain a six-parameter generalized logistic distribution.

The cumulants of the distribution are derived and it is indicated how an approximation to the c.d.f. can be obtained. Finally we state and prove some theorems that relate the distribution to some other common statistical distributions.

1. INTRODUCTION

The role of the logistic model in analysing bioassay and quantal response data are well known (see Berkson 1944, Cox 1970, Johnson and Kotz 1970, Ojo 1989), to mention a few. However, in general, generalized models being more flexible than ordinary single models, are usually prefered in analysing most data sets. This has prompted several authors to embark upon investigating the properties and applications of generalized models. In particular, because of its roles in analysing bioassay and quantal response experiments, a lot of research has been reported in literature in studying the properties and applications of the generalized logistic models (see El-Saidi et al 1990, George and Ojo 1980, Meenakshi et al 1993, Aranda-Ordaz 1982, Zelterman 1989, Patil and Taillie 1994, Ojo 2002, Ojo 2003).

Wu Jong-Wuu et al (2000) made an extension to the usual four-parameter generalized logistic distribution. He developed a method of deriving a five-parameter generalized logistic distribution and discussed some of its properties and applications.

In this present paper, a step forward is taken. By defining a suitable random variable, the density function of a six-parameter generalized logistic distribution is derived. The cumulants of the distribution are obtained and it is indicated how its c.d.f can be approximated. Finally, we establish some relationships between the distribution and some other well known statistical distributions.

2. THE SIX-PARAMETER GENERALIZED LOGISTIC DISTRIBUTION, ITS MOMENTS AND CUMULATIVE DISTRIBUTION FUNCTION

2.1. THE SIX-PARAMETER GENERALIZED LOGISTIC DISTRIBUTION

Without loss of generality we shall assume throughout the standardized form for all the distributions under consideration. That is, we shall assume that the location and scale parameters take the values 0 and 1 respectively.

Theorem 2.1. *Let X be a random variable that has a generalized beta type II distribution with parameters λ, p and q whose density is defined as*

$$f(x;0,1,\lambda,p,q)=\frac{\lambda^q}{B(p,q)}\frac{x^{p-1}}{(\lambda+x)^{p+q}},\quad 0<x<\infty,\ \lambda>0,\ p>0,\ q>0,\tag{2.1}$$

(see Patil and Taillie 1994). Then, the random variable $Y=\ln(X/\beta)$ has the six-parameter generalized logistic distribution with parameters $(0,1,\lambda,\beta,p,q)$.

Proof. By using the transformation $y = \ln(x/\beta)$ in 2.1 above we readily have

$$g(y; 0, 1, \lambda, \beta, p, q) = \frac{\lambda^q \beta^p}{B(p,q)} \frac{e^{py}}{(\lambda + \beta e^y)^{p+q}}, \quad -\infty < y < \infty. \tag{2.2}$$

This distribution is symetric only when $\lambda = \beta$ and $p = q$ and assymetric for all other combinations of the parameters. It reduces to the five-parameter generalized logistic of Wu Jong-Wuu et al (2000) when $\beta = 1$ and reduces to the four-parameter generalized logistic when $\lambda = \beta = 1$. Of course when $(\lambda, \beta, p, q) = (1, 1, 1, 1)$ we have the famous logistic distribution which had been studied and used by several authors.

2.2. THE CUMULANTS OF THE DISTRIBUTION

The moment generating function is given as

$$\begin{aligned} \phi(t) &= \frac{\lambda^q \beta^p}{B(p,q)} \int_{-\infty}^{\infty} \frac{e^{(p+t)y}}{(\lambda + \beta e^y)^{p+q}} dy = \frac{\lambda^q \beta^p}{B(p,q)} \int_0^{\infty} \frac{y^{p+t-1}}{(\lambda + \beta y)^{p+q}} dy \\ &= \frac{\lambda^q \beta^p}{B(p,q)} \int_0^{\infty} \frac{u^{p+t-1}}{(1+u)^{p+q}} du \\ &= \lambda^t \beta^{-t} \frac{\Gamma(p+t)}{\Gamma(p)} \frac{\Gamma(q-t)}{\Gamma(q)}. \end{aligned} \tag{2.3}$$

The corresponding characteristic function is given as

$$\phi(it) = \lambda^{it} \beta^{-it} \frac{\Gamma(p+it)}{\Gamma(p)} \frac{\Gamma(q-it)}{\Gamma(q)}. \tag{2.4}$$

The cumulant generating function is given as

$$\ln\phi(t) = t\ln(\lambda/\beta) + \ln\Gamma(p+t) + \ln\Gamma(q-t) - \ln\Gamma(p) - \ln\Gamma(q).$$

The r^{th} cumulant is obtained as

$$\kappa_r(y) = \frac{d^r}{dt^r}[t\ln(\frac{\lambda}{\beta})]_{t=0} + \frac{d^r}{dt^r}[\ln\Gamma(p+t) + \ln\Gamma(q-t)]_{t=0}. \tag{2.5}$$

The 2^{nd} or the 3^{rd} term of the right hand side of equation (2.5) is called di-gamma function. The series expansion of this function (see Copson 1962) is given as

$$\psi^{r-1}(x) = (r-1)![(-1)^r \sum_{j=0}^{\infty} (j+x)^{-r}]_{r \geq 2}$$

and

$$\psi(x) = \sum_{j=0}^{\infty}(j+x)^{-1},\ r=1.$$

Therefore

$$\kappa_r(Y) = (r-1)!(-1)^r[\sum_{j=0}^{\infty}(j+p)^{-r} + (-1)^r\sum_{j=0}^{\infty}(j+q)^{-r}]_{r\geq 2} \tag{2.6}$$

and

$$\kappa_r(Y) = \ln\frac{\lambda}{\beta} + \sum_{j=0}^{\infty}(j+q)^{-1} - \sum_{j=0}^{\infty}(j+p)^{-1},\ r=1. \tag{2.7}$$

2.3. THE CUMULATIVE DISTRIBUTION FUNCTION AND ITS APPROXIMATION

Let $F(y)$ denote the c.d.f of Y

$$\begin{aligned} F(y) &= \frac{\lambda^q\beta^p}{B(p,q)}\int_{-\infty}^{y}\frac{e^{px}}{(\lambda+\beta e^x)^{p+q}}dx \\ &= \frac{1}{B(p,q)}\int_0^{\frac{\beta}{\lambda}e^y/(1+\frac{\beta}{\lambda}e^y)} t^{p-1}(1-t)^{q-1}dt = I_u(p,q). \end{aligned}$$

Where $u = \frac{\beta/\lambda e^y}{1+\beta/\lambda e^y}$ and where $I(\,.\,,\,.)$ denotes the incomplete beta function.

Thus, the c.d.f is an incomplete beta function which has earlier on been successfully approximated by the $t-$distribution (Ojo 1988). For the purpose of this paper we hereby quote the result of the approximation as follows.

$$Pr[Y \leq \log\frac{u}{(1-u)}] \doteq Pr[t_\nu \leq c(\log(\frac{u}{(1-u)}) - \kappa_1)]$$

and in terms of percentile

$$u = [1 + exp - (\frac{t_\nu}{c} + \kappa_1)]$$

where t_ν denotes the lower percentile of the t distribution with ν degrees of freedom, $c = \frac{\sigma t}{\kappa_2^{1/2}}$, $\sigma t^2 = \nu/\nu - 2$ and κ_i is the i^{th} cumulant of Y. The appropriate values of ν are obtained by matching kurtosis. That is by equating the coefficient of kurtosis of the $t-$distribution to that of the generalized logistic distribution (see the details of this approximation in Ojo 1988).

3. RELATIONSHIPS WITH SOME OTHER DISTRIBUTIONS

In what follows, we state and prove a few number of theorems which relate the six-parameter generalized logistic distribution to some other distributions.

Theorem 3.1. *Let X be a beta random variable with parameters p and q. Let the random variable Y be defined as*

$$Y = \ln\left(\frac{\lambda X}{\beta(1-X)}\right).$$

Then the random variable Y has a six-parameter generalized logistic distribution with parameters $(0, 1, \lambda, \beta, p, q)$.

Proof. The characteristic function of Y is given as

$$\begin{aligned}
\phi_Y(it) &= E(e^{itY}) = E\left[\left(\frac{\lambda X}{\beta(1-X)}\right)^{it}\right] \\
&= \frac{\lambda^{it}}{\beta^{it}}\frac{1}{B(p,q)}\int_0^1 x^{p+it-1}(1-x)^{q-it-1}dx \\
&= \lambda^{it}\beta^{-it}\frac{\Gamma(p+it)\Gamma(q-it)}{\Gamma(p)\Gamma(q)}
\end{aligned}$$

by virtue of equation (2.4), the theorem is proved.

Theorem 3.2. *Let X_1 and X_2 be random variables having the generalized Gumbel distributions with densities defined as*

$$g_1(x_1) = \frac{\lambda^{-p}}{\Gamma(p)}e^{px_1}e^{-1/\lambda}e^{x_1}, \quad -\infty < x_1 < \infty$$

and

$$g_2(x_2) = \frac{\beta^{-q}}{\Gamma(q)}e^{qx_2}e^{-1/\beta}e^{x_2}, \quad -\infty < x_2 < \infty.$$

The random variable $Y = X_1 - X_2$ has the six-parameter generalized logistic if X_1 and X_2 are independent.

Proof. The characteristic function of X_1 is given as

$$\begin{aligned}
\phi_{X_1}(it) &= \frac{\lambda^{-p}}{\Gamma(p)}\int_{-\infty}^{\infty} e^{px_1}e^{-1/\lambda e^{x_1}} \cdot e^{itx_1}dx_1 \\
&= \frac{\lambda^{-p}}{\Gamma(p)}\int_0^{\infty} u^{p+it-1}e^{-u}du = \frac{\lambda^{it}\Gamma(p+it)}{\Gamma(p)}.
\end{aligned}$$

Similarly, the c.f of $-X_2$ is given as

$$\phi_{-X_2}(it) = \frac{\beta^{-it}\Gamma(q-it)}{\Gamma(q)}.$$

Since X_1 and X_2 are independent, then the c.f of $Y = X_1 - X_2$ is simply

$$\phi_Y(it) = \lambda^{it}\beta^{-it}\frac{\Gamma(p+it)\Gamma(q-it)}{\Gamma(p)\Gamma(q)}.$$

Since this is exactly the c.f of the six-parameter generalized logistic, the theorem is established.

Theorem 3.3. *Let X be an $F-$random variable with $(2p, 2q)$ degrees of freedom. Define the random variable $Y = \ln(\frac{\lambda p}{\beta q}X)$, λ, $\beta > 0$. Then the random variable Y has the six-parameter generalized logistic distribution.*

Proof. The density function of an $F(2p, 2q)$ random variable is given as

$$f(x) = \frac{1}{B(p,q)}\frac{(\frac{p}{q})^p x^{p-1}}{(1+\frac{p}{q}x)^{p+q}}, \quad 0 < x < \infty.$$

Now define the transformation $y = \ln(\frac{\lambda p}{\beta q}x)$. The Jacobian of the inverse tranformation is given as

$$|J| = \frac{\beta q}{\lambda p}e^y.$$

Thus the density function of Y is

$$g(y) = \frac{\lambda^q\beta^p}{B(p,q)}\frac{e^{py}}{(\lambda+\beta e^y)^{p+q}}, \quad -\infty < y < \infty.$$

This completes the proof.

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