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QUATERNIONS AND LIE GROUPS ON S^2

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Abstract. Pfaff [1] using quaternion product gave some propoties of commutative multiplication of number triplets of R^3. We [2] gave a new explanation of multiplication of number triplets by representation matrix. In this paper, we show that, with the product, the great circle on S^2 are one parameter Lie groups. Furthermore, we optain that all the circles on S^2 are Lie groups.

1 PRELIMINARIES

A quaternion is defined depending on four units $1, i, j, k$:

$$q = a1 + bi + cj + dk,$$

where a, b, c, d are real numbers and i, j, k are arbitrary "units" which satisfy the relations $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$.

We now replace each quadruplet $x = a1 + bi + cj + 0k = (a, b, c, 0)$ by the corresponding triplet (a, b, c) so that $1 = (1, 0, 0)$, $i = (0, 1, 0)$ and $j = (0, 0, 1)$, and a general trucated quaternion is written as $x = a1 + bi + cj$. The subspace containing the x-axis is called a leaf [1]. If $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ in the same leaf, then

$$x \times y = (x_1y_1 - x_2y_2 - x_3y_3)1 + (x_1y_2 + y_1x_2)i + (x_1y_3 + y_1x_3)j, \quad (1)$$

where $\times$ is quaternion multiplication [1]. If x and y are on different leaves, then the procedure for multiplication of x and y is as follows:

(i) The plane determined by x and y intersect the (x, y)-plane in a line passing through the origin which makes an angle, say θ, with the positive x direction.

(ii) Rotate the plane of x and y about z axis, bringing its line of intersection with the (x, y)-plane into coincidence with x axis and carrying the vector x and y to $x^{'}$ and $y^{'}$.

(iii) The plane of $x^{'}$ and $y^{'}$ is a leaf so we can use $\times$ product from (1). Thus we can obtain a $z^{'}$ vector $z^{'} = x^{'} \times y^{'}$.

(iv) Rotate the plane of $x^{'}$ and $y^{'}$ and $z^{'}$ back to the original plane of x and y using the inverse of the rotation in used step (ii) . This rotation sends $x^{'}$ and $y^{'}$ to x and y , respectively , and sends $z^{'}$ to a vector z , which we call the $\otimes$ product of x and y and write $z = x \otimes y$ [1] .

Pfaff [1] gave some algebraic properties of this $\otimes$ product. It is shown that, this $\otimes$ product is not distributive. So the resulting structure is not an algebra.

In this study, we obtain a new representation of $\otimes$ product. We show that, if M is the plane passing through the origin then M is a division algebra. Furthermore, we give some geometrical interpretations of $\otimes$ product.

2 NEW EXPLANATION OF $\otimes$ PRODUCT

Let $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ be any linear independent vectors in R^3. The plane determined by x and y intersects the xy-plane in a line passing through the origin which makes an angle, say θ, with the positive x direction. We can take the line $l = (\cos\theta, \sin\theta, 0)$ as the direction vector of intersection. If $A(\theta)$ is rotation matrix about the z-axis, then

$$\begin{aligned} z &= x \otimes y \\ z &= A^{-1}(\theta)\,[A(\theta)x \times A(\theta)y] \end{aligned} \tag{2}$$

or using (1) we find,

$$\begin{aligned} z = {} & [[(x_1\cos\theta + x_2\sin\theta)\,y_1 + (x_1\sin\theta - x_2\cos\theta)\,y_2 + (-x_3\cos\theta)\,y_3]\,1 + \\ & [(x_2\cos\theta - x_1\sin\theta)\,y_1 + (x_1\cos\theta + x_2\sin\theta)\,y_2 + (-x_3\sin\theta)]\,i + \\ & [(x_3\cos\theta)\,y_1 + (x_3\sin\theta)\,y_2 + (x_1\cos\theta + x_2\sin\theta)\,y_3]\,j] \end{aligned} \tag{3}$$

It is easy to see that, for $\theta = 0$, $x \otimes y = x \times y$, i.e., the $\otimes$ product of x and y reduces to the $\times$ quaternion product. If $\theta = 0$ and $x_3 = y_3 = 0$ then the $\otimes$ product is complex product.

We can calculate matrix representation with respect to basis $\{1, i, j\}$ of the $\otimes$ product. $z = x \otimes y = A_x(y)$, where the matrix of A_x is

$$A_x = \begin{bmatrix} x_1\cos\theta + x_2\sin\theta & -x_2\cos\theta + x_1\sin\theta & -x_3\cos\theta \\ x_2\cos\theta - x_1\sin\theta & x_1\cos\theta + x_2\sin\theta & -x_3\sin\theta \\ x_3\cos\theta & x_3\sin\theta & x_1\cos\theta + x_2\sin\theta \end{bmatrix} \tag{4}$$

Let M be the plane passing through the origin and $E = M \cap Sp\{i, j\}$. Let $l = (\cos\theta, \sin\theta, 0)$ be the unit vector of the direction E, where θ is the angle between the line E and x-axis. Let be $\Phi = (l, N)$ orthonormal basis of M, where $n\Lambda l = N$ and n is unit normal vector of M.

The $\otimes$ product on the plane M is

$$\begin{array}{rll} \Phi: & M \times M & \to M \\ & (x, y) & \to x \otimes y = A_x y = (l\Lambda x)\Lambda y + \langle l, x \rangle \, y \end{array} \tag{5}$$

Let be x unit vector in M. Then $A_x : M \to M$ is isometric mapping. A_x is not orthogonal matrix, but A_x is orthogonal matrix with respect to basis Φ. Thus A_x is a rotating matrix in plane M

From(4) , $A_x A_x^t \neq I_3$. Thus A_x is not orthogonal with respect to Ψ basis of R^3.

In terms of equation (5) ,we can write x and y vectors with respect to Φ basis of M,

Thus,

$$x = \cos v_1 l + \sin v_1 N \; and \; y = \cos v_2 l + \sin v_2 N \tag{6}$$

where $l \otimes l = l, \; l \otimes N = N, \; N \otimes N = -l.$

Thus, from (5)

$$A_x y = \cos (v_1 + v_2) \, l + \sin (v_1 + v_2) \, N$$

$$[A_x]_\Psi = \begin{bmatrix} \cos v_1 & -\sin v_1 \\ \sin v_1 & \cos v_1 \end{bmatrix} \tag{7}$$

$$[A_x y]_\Psi = [A_x]_\Psi \, [y]_\Psi$$

$(M, +, \otimes)$ is a field.

$(M, +)$ is a abelian group $(M is a subvectorspace in R^3)$.

$(i)\otimes$ is associative $[2]$.

$(ii) \otimes$ is commutative.

$$x \otimes y = A^{-1}(\theta)\,[A(\theta)x \times A(\theta)y]$$

$$x \otimes y = A^{-1}(\theta)\,[A(\theta)y \times A(\theta)x]$$

$$= y \otimes x$$

where $\times$ is commutative on leaf.

(iii) $x = (x_1, x_2, x_3) \in M$. If x unit vector then

$$x^{-1} = (x_1 \cos 2\theta + x_2 \sin 2\theta, x_1 \sin 2\theta - x_2 \cos 2\theta, -x_3)\,.$$

For spacial case $\theta = 0$

$$x^{-1} = (x_1, -x_2, -x_3).$$

This is inverse of x quaternion on the leaf.

$\det\left(l, x, x^{^{-1}}\right) = 0$ then x^{-1} in M.

(iv) l is unit element.

$M^* = M - \{0\}$ is a Lie group.

$$\begin{array}{lll} \otimes : & M^* \times M^* & \rightarrow M^* \\ & (x, y) & \rightarrow x \otimes y = A_x y \end{array}$$

$\otimes$ is smooth.

Furthermore,

$$Sp\,\{l, N\}\mid_l = T_l M^*.$$

Let be x_1 and x_2 left invariant vector fields of M^*. For the components of these vector fields at the point $p = a_0 + a_1 i + a_2 j$, we have

$$(x_1)\mid_p = p \otimes l = p, \quad (x_2)\mid_p = p \otimes N. \tag{8}$$

If we wish to find the structure of the Lie algebra of the group M^*.We compute $[x_1, x_2] = 0$ (M^* is abelian).

The values at the point $p = l$ we obtain

$$(x_1)\mid_l = l, \quad (x_2)\mid_l = N_l and \quad [l, N] = 0.$$

3 One Parameter Subgroups of M*

Let be $\alpha(v) = M^* \cap S^2$.

Thus we can write $\alpha(v) = \cos vl + \sin vN$. $\alpha(v)$ is a great circle in S^2 (geodesics in S^2). Then

$$\alpha(v) = \cos v(\cos\theta, \sin\theta, 0) + \sin v\,(n_1, n_2, n_3)\,,\ \theta = constant. \tag{9}$$

For $N = (0, 0, 1)$, $\alpha(v)$ is meridian (longitude). The $\otimes$ product indeed on $\alpha(v)$. Then

$$\begin{array}{rll} \otimes : & \alpha(v) \times \alpha(v) & \to \alpha(v) \\ & (\alpha(v_1), \alpha(v_2)) & \to \alpha(v_1) \otimes \alpha(v_2) \end{array}$$

$(i)\, \alpha(v_1) \otimes \alpha(v_2) = \alpha(v_1 + v_2)$

$(ii)\, [\alpha(v_1)]^{-1} = \alpha(-v_1)$

$(iii)\, \alpha(0) = l$

$(iv)\, \alpha(v_1) \otimes \alpha(v_2) = \alpha(v_2) \otimes \alpha(v_1)$

Thus $\alpha(v)$ is a one parameter subgroup of M^*.For the components of x_2 left invariant vector field at the point p of $\alpha(v)$

$$(x_2)\mid_p = p \otimes N$$

and

$$(x_2)\mid_l = N_l.$$

Spacial case 1 $N = (0,0,1)$, $\alpha(v)$ is a meridian. Then

$\alpha(v) = (\cos v \cos\theta, \cos v \sin\theta, \sin v)$ is a Lie group. Its Lie algebra is z-axis passing from l point.

Spacial case 2 Let be θ change and v constant. $N = (0,0,1)$ and $v = 0$,

$0 \leq \theta \leq 2\pi$ then $\alpha(\theta) = (\cos\theta, \sin\theta, 0)$ is equvator. $\otimes$ product is complex product.

Example Let the equation of plane M be $x - y + z = 0$. Therefore $\theta = \pi/4$,

$l = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)$, $n = \frac{1}{\sqrt{3}}(1,-1,1)$ and $N = n\Lambda l = \frac{1}{\sqrt{6}}(-1,1,2)$

Thus it follows that

$\alpha(v) = \cos v(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0) + \sin v \frac{1}{\sqrt{6}}(-1,1,2)$ $(\alpha(v), *)$ is a one parameter Lie group of M^*.

Let be β any circle on S^2, i.e. $\beta = T \cap S^2$ such that T is a plane. If p point on T is center of circle β curve and r is radius of β then we can give $*$ product,

$$\begin{array}{lll} * : & \beta \times \beta & \rightarrow \beta \\ & (x,y) & \rightarrow x * y = \frac{1}{r} x' \otimes y' - C \end{array} \tag{10}$$

where $C = \overrightarrow{PO}$ is translation vector and vectors $x' = x + C$, $y' = y + C$ are on plane the passing through the origin that the plane parallel to T. Thus $(\beta, *)$ is a Lie group.

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