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AREA AND LENGTH OF SPHERICAL CYCLOID

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Abstract. Formulas for area and length of spherical cycliods are presented.

Roulette $\mathcal{R}(c, g, P)$ is curve traced out by a point P in fixed position with respect to a rolling curve c which rolls without slipping along a fixed base curve g. Let S^2 be a sphere of radius R in Euclidean space E^3. The curvature measure κ_c on some spherical curve c is defined at regular points by the geodesic curvature functional $\kappa(X)$, while at singular points the point-measure is equal to the geodesic point-curvature. So the total geodesic curvature of c is $\kappa_c^T = \int_c d\kappa$. The *centroid* B of curvature of some spherical curve c is barycenter of c, assuming c is weighted by the curvature measure κ_c and considered as a curve in Euclidean space. The *intrinsic centroid* $\hat{B}$ of curvature of c is intrinsic barycenter of c considered in S^2. Therefore $\hat{B} \in S^2$ and $B \in O\hat{B}$. If c is a curve in Euclidean plane centroids $B = \hat{B}$ become Steiner point of c. For our purposes c should be weighted by $\kappa_c - \kappa_g$. Let $c \subset S^2$ be a circle of spacial radius r, let g be a great circle—geodesic line on S^2, and $P \in c$.

Theorem. *Area bounded by cycloid and base line is*

$$\operatorname{area}\mathcal{R}(c,g,P) = 2\pi r^2 + \pi(r^2 - v^2)\cos\alpha,$$

$$v = R - \sqrt{R^2 - r^2}, \qquad \cos\alpha = \sqrt{R^2 - r^2}/R;$$

and

$$\operatorname{length}\mathcal{R}(c,g,P) = 4r\cos\alpha\left[1 + \frac{R^2 - r^2}{2Rr}\ln\frac{R+r}{R-r}\right].$$

Proof. Apply the next two formulas [1] for closed rolling curves, which are addapted for spherical case

$$\operatorname{area}\mathcal{R}(c,g,P) = \operatorname{area}\mathcal{R}(c,g,\hat{B}) + \frac{1}{2}(\kappa_c^T - \kappa_g^T)(BP^2 - B\hat{B}^2),$$

$$\operatorname{length}\mathcal{R}(c,g,P) = \int_c PX\sqrt{1 - (PX/2R)^2}\, d(\kappa_c - \kappa_g), \qquad X \in c.$$

Barycenters B and $\hat{B}$ are center of c its central projection on S^2 respectively. The roulette $\mathcal{R}(c,g,\hat{B})$ is cyclic arc parallel to g, which with base polode and two spherical radii of c, bounds a segment of spherical zone with area $2\pi r^2$. Further $\kappa_c = \frac{1}{r}\cos\alpha$, $\kappa_c^T = 2\pi\cos\alpha$, and $\kappa_g^T = 0$, as well as $BP = r$ and $B\hat{B} = v$. $\square$

In the asymptotic case $R \to \infty$, or $r \to 0$, we obtain Euclidean cycloid with the area $3\pi r^2$ and the length $8r$ since

$$\frac{R+r}{R}\frac{R-r}{2r}\ln\left(1 + \frac{2r}{R-r}\right) \to 1, \quad R \to \infty. \tag{1}$$

For the other asymptotic case $r \uparrow R$, the area bounded by the cycloid arises up to hemisphere $2\pi R^2$, and the length vanishes, what can be verified in (1). For length we have

$$\int_c PX\sqrt{1-(PX/2R)^2}\frac{\cos\alpha}{r}\,ds = \frac{\cos\alpha}{r}\int_0^{2\pi} 2r\sin\frac{\varphi}{2}\sqrt{1-(PX/2R)^2}r\,d\varphi.$$

References

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