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AHLFORS-SCHWARZ LEMMA AND CURVATURE

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Abstract In this note we give short review of known results and announce new results (see below Theorem 8 and Theorem 6 and its generalizations) . In the first part of this review paper , we focus on ultrahyperbolic metric and Ahlfors lemmas and to the estimate opposite to Ahlfors-Schwarz lemma proved by the author(Theorem 5-6). The second part is devoted to Ahlfors-Schwarz lemma for harmonic-quasiregular maps and some results obtained in [AMM].

Introduction

In Section 1 , of this review paper , we focus on ultrahyperbolic and pseudohermitian metrics , Ahlfors lemmas and to the estimate opposite to Ahlfors-Schwarz lemma proved by the author(Theorem 5-6).
Section 2 is devoted to Ahlfors-Schwarz lemma for harmonic-quasiregular maps .

In [W], Wan showed that every harmonic quasi-conformal diffeomorphism f from the unit disk Δ onto itself with respect to *Poincaré* metric is a quasi-isometry of *Poincaré* disk.
Let $\rho_0 = \sigma \circ f\,|f_z|$ and $K_0 = K_{\rho_0}$ the Gaussian curvature of the metric ρ_0. In his proof Wan [W] used the method of sub-solutions and super-solutions and the fact that ρ_0 is complete metric.
We will show in a forthcoming paper that we can use Ahlfors-Schwarz lemma and the estimate opposite to Ahlfors-Schwarz lemma (Theorem 6) instead of the method of sub-solutions and super-solutions to prove Wan's result and get further generalizations of it .
Also , we announce the following result which we call *Ahlfors-Schwarz lemma for harmonic-quasiregular maps* (see also Theorem 8 below) :

Theorem A . *Let R be hyperbolic surfaces with Poincare metric densities $\lambda$and S be another with Poincare metric densities σ and let the Gaussian curvature of the metric $ds^2 = \rho(w)|dw|^2$ be uniformly bounded from above on S by the negative constant $-a$. Then any harmonic k-quasiregular map f from R into S decreases distances up to a constant depending only on a and k .*
Let $\rho_0 = \sigma \circ f\,|p|$ and $K_0 = K_{\rho_0}$ the Gaussian curvature of the metric ρ_0.
A proof of Theorem A can be based on the estimate of the curvature $K_0 = K_S\,(1 - |\mu|^2)$ and Ahlfors-Schwarz lemma.
In Section 3, we discuss some results obtained in [AMM] .
An uniform estimate of radius of maximal φ-disks of the Hopf differential of a quasiregular harmonic map with respect to strongly negatively curved metric(see below Theorem 9) is proved. As an application we show that the Hopf differential of a quasiregular harmonic map with respect to strongly negatively curved metric belongs to Bers space.
Finally ,in Section 4 , we state several dimensional generalization of Schwarz lemma due to Yau and Royden .

1. Ahlfors-Schwarz lemma

Hyperbolic distance and Schwarz lemma . By Δ we denote the unit disk . Let B be the disk with center at z_0 and radius r. Using the conformal automorphisms $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$, $a \in \Delta$, of Δ, one can define pseudo-hyperbolic distance on Δ by

$$\delta(a,b) = |\phi_a(b)|,\ a,b \in \Delta\,.$$

Next, using the conformal map $A(\zeta) = \frac{\zeta - z_0}{r}$ from B onto Δ, one can define pseudo-hyperbolic distance on B by

$$\delta_B(z,w) = \delta(A(z), A(w))$$

and the *hyperbolic metric* on B by

$$\lambda(z,w) = \log \frac{1 + \delta_B(z,w)}{1 - \delta_B(z,w)}$$

for $z, w \in B$.
In particular , hyperbolic distance on the unit disk Δ is

$$\lambda(z,\omega) = \ln \frac{1 + |\frac{z-\omega}{1-z\bar{\omega}}|}{1 - |\frac{z-\omega}{1-z\bar{\omega}}|}\ .$$

The classic Schwarz lemma states : If $f\,:\,\Delta \to \Delta$ is an analytic function , and if $f(0) = 0$, then $|f(z)| \le |z|$ and $|f'(0)| \le 1$. Equality $|f(z)| = |z|$ with $z \ne 0$ or $|f'(0)| = 1$ can occur only for $f(z) = e^{i\alpha}z$, α is a real constant .
It was noted by Pick that result can be expressed in invariant form . We refer the following result as Schwarz-Pick lemma .

Theorem 1. (*Schwarz − Pick lemma*) *. Let F be an analytic function from a disk B to another disk U. Then F does not increase the corresponding hyperbolic (pseudo-hyperbolic) distances.*

Curvature. A Riemannian metric given by the fundamental form

$$ds^2 = \rho^2(dx^2 + dy^2)$$

or $ds = \rho|dz|$, $\rho > 0$, is conformal with euclidian metric.

If $\rho > 0$ is a C^2 function on Δ ,the Gaussian curvature of a Riemannian metric ρ on Δ is expressed by the formula

$$K = K_\rho = -\rho^{-2}\mathbf{\Delta} \ln \rho \ .$$

Also we write $K(\rho)$ instead of K_ρ .
Recall that a pseudohermitian metric on Δ is a non-negative upper semicontinuous function ρ such the set $\rho^{-1}(0)$ is discrete in Δ .

If u is an upper semicontinuous function , the *lower generalized Laplacian* of u is defined by ([AP], see also [GeVi])

$$\mathbf{\Delta}_L\, u(\omega) = 4 \liminf_{r\to 0} \frac{1}{r^2}\Big(\frac{1}{2\pi}\int_0^{2\pi} u(\omega + re^{it})dt - u(\omega)\Big).$$

When u is a C^2 function , then the lower generalized Laplasian of u reduces to the usual Laplacian

$$\mathbf{\Delta} u = u_{xx} + u_{yy} \ .$$

The Gaussian curvature of a pseudohermitian metric ρ on Δ is defined by the formula

$$K = K_\rho = -\rho^{-2}\mathbf{\Delta_L} \ln \rho.$$

For all $a > 0$ define the family of functions λ_a

$$\lambda_a(z) = \frac{2}{a(1 - |z|^2)} \ .$$

Also, it is convenient to write λ instead of λ_1 .The Gaussian curvature of λ_a is $K(\lambda_a) = -a$. This family of Hermitian metrics on Δ is of interest because it allows an ordering of all pseudohermitian metrics on Δ in the sence of the following ([AP]).

Theorem 2. *Let ρ be a pseudohermitian metric on Δ such that*

$$K_\rho(z) \le -a$$

for some $a > 0$. Then $\rho \le \lambda_a$.

This kind of estimate is similar to Ahlfors-Schwarz lemma .
Ahlfors lemma can be found in Ahlfors [Ah].

Ahlfors-Schwarz lemma

A metric ρ is said to be ultrahyperbolic in a region Ω if it has the following properties :

(a) ρ is upper semicontinuous ; and
(b) at every z_0 there exits a supporting metric ρ_0 , defined and class C^2 in a neighborhood V of z_0 , such that $\rho_0 \le \rho$ and $K_{\rho_0} \le -1$ in V , while $\rho_0(z_0) = \rho(z_0)$.

Theorem 3. *(Ahlfors Lemma 1). Suppose ρ is an ultrahyperbolic metric on Δ. Then $\rho \le \lambda$.*

The version presented in [Ga] has a slightly modified definition of supporting metric.This modification and formulation is due to Earle . This version has been used (see [Ga]) to prove that *Teichmüller* distance is less than equal to *Kobayashi's* on *Teichmüller* space .
Ahlfors [Ah] proved a stronger version of Schwarz's lemma and Ahlfors lemma 1 .

Theorem 4. *(Ahlfors lemma 2) . Let f be an analytic mapping of Δ into a region on which there is given ultrahyperbolic metric ρ . Then $\rho[f(z)]\,|f'(z)| \le \lambda$.*

The proof consists of observation that $\rho[f(z)]\,|f'(z)|$ is ultrahyperbolic metric on Δ .Observe that the zeros of $f'(z)$ are singularities of this metric.

Note that if f is the identity map on Δ we get Theorem 3 (Ahlfors lemma 1)from Theorem 4 .

The notation of an ultrahyperbolic metric makes sense , and the theorem remains valid if Ω is replaced by a Riemann surface .

In a plane region Ω whose complement has at least two points , there exists a unique maximal ultrahyperbolic metric ,and this metric has constant curvature -1 .

The maximal metric is called the *Poincaré metric* of Ω , and we denote it by λ_Ω . It is maximal in the sense that every ultrhyperbolic metric ρ satisfies $\rho \le \lambda_\Omega$ throughout Ω .

The hyperbolic metric of a disk $|z| < R$ is given by

$$\lambda_R(z) = \frac{2R}{R^2 - |z|^2} \ .$$

If ρ is ultrhyperbolic in $|z| < R$, then $\rho \le \lambda_R$. In particular , if ρ is ultrhyperbolic in the whole plane , then $\rho = 0$. Hence there is no ultrahyperbolic metric in the whole plane .

The same is true of the punctured plane $C^* = \{z : z \ne 0\}$. Indeed, if ρ were ultrahyperbolic metric in the whole plane, then $\rho(e^z)\ |\ e^z\ |$ would be ultrahyperbolic in the hole plane. These are only cases in which ultrahyperobolic metric fails to exist .
Ahlfors [Ah] used Theorem 4 to prove Bloch and the Picard theorems.Ultrahypebolic metrics (without the name) were introduced by Ahlfors . They found many applications in the theory of several complex variables .

An inequality opposite to Ahlfors-Schwarz lemma

Mateljević [Ma] proved an estimate opposite to Ahlfors-Schwarz lemma .

A metric $H|dz|$ is said to be superhyperbolic in a region Ω if it has the following properties :

(a) H is continuous (more general, lower semicontinuous) on Ω .

(b) at every z_0 there exists a supporting metric H_0 ,defined and class C^2 in a neighborhood V of z_0 , such that $H_0 \geq H$ and $K_{H_0} \geq -1$ in V , while $H_0(z_0) = H(z_0)$.

Theorem 5. ([Ma]).*Suppose H is a superhyperbolic metric on Δ for which*
(c) $H(z)$ tends to $+\infty$ when $|z|$ tends to 1_-
Then $\lambda \leq H$.

By applying a method developed by Yau in [Ya1] (or by generalized maximum principle of Cheng and Yau [CYa]), it follows that this result holds if we suppose instead of (c) that

(d) H is a complete metric on Δ .

Theorem 6. . *If ρ and σ are two metrics on Δ, σ complete and $0 > K_\sigma \geq K_\rho$ on Δ , then $\sigma \geq \rho$.*

This theorem remains valid if ρ is ultrahyperbolic metric and σ superhyperbolic metric on Δ .Also , we can get further generalizations if Δ is replaced by a Riemann surface .

The method of sub-solutions and super-solutions have been used in study harmonic maps between surfaces . We will show in a forthcoming paper that we can use Theorem 6 instead of the method of sub-solutions and super-solutions .

2. Schwarz lemma for harmonic and quasiconformal maps

Wan [W] showed that

Theorem 7. (Wan) . *Every harmonic quasi-conformal diffeomorphism from Δ onto itself with respect to Poincaré metric is a quasi-isometry of Poincaré disk.*

Let $\rho_0 = \sigma \circ f\,|f_z|$ and $K_0 = K_{\rho_0}$ the Gaussian curvature of ρ. In his proof Wan [W] used the method of sub-solutions and super-solutions and the fact that ρ_0 is complete metric. Recall ,we will show in a forthcoming paper that we can use Ahlfors-Schwarz lemma and Theorem 6 instead of the method of sub-solutions and super-solutions and, in particular , that a proof of Wan's result can be based on these results .

Definition and properties of Harmonic and quasiregular maps

Let R and S be two surfaces. Let $\sigma(z)|dz|^2$ and $\rho(w)|dw|^2$ be the metrics with respect to the isothermal coordinate charts on R and S respectively, and let f be a C^2-map from R to S.

It is convenient to use notation in local coordinates $df = p\,dz + q\,d\overline{z}$, where $p = f_z$ and $q = f_{\overline{z}}$. Also we introduce the complex (Beltrami) dilatation

$$\mu_f = Belt[f] = \frac{q}{p}$$

where it is defined.

The energy integral of f is

$$E(f,\rho) = \int_R \rho \circ f\,(|p|^2 + |q|^2)\,dxdy\,.$$

A critical point of the energy functional is called a harmonic mapping. The Euler-Lagrange equation for the energy functional is

$$\tau(f) = f_{z\overline{z}} + (\log\rho)_w \circ f\,p\,q = 0\,.$$

Thus, we say that a C^2-map f from R to S is harmonic if f satisfies the above equation. For basic properties of harmonic maps and for further information on the literature we refer to Jost [Jo] and Schoen-Yau [SYa3].

The following facts and notation are important in our approach:

A1 If f is a harmonic mapping then

$$\varphi\,dz^2 = \rho \circ f\,p\,\overline{q}\,dz^2$$

is a quadratic differential on R, and we say that φ is the *Hopf differential* of f and we write $\varphi =$Hopf(f).

A2 The Gaussian curvature on S is given by

$$K_S = -\frac{1}{2}\frac{\boldsymbol{\Delta}\ln\rho}{\rho}\,.$$

A3 We will use the following notation $\mu = Belt[f] = \frac{q}{p}$ and $\tau = \log\frac{1}{|\mu|}$ and *Bochner* formula (see [SYa3])

$$\boldsymbol{\Delta}\ln|\partial f| = -K_S\,J(f) + K_R\,,$$

$$\boldsymbol{\Delta}\ln|\bar{\partial} f| = K_S\,J(f) + K_R\,,$$

$$\boldsymbol{\Delta}\tau = -K_S\,|\varphi|\,\sinh\tau\,.$$

A4 **Definition of quasiregular function.** Let R and S be two Riemann surfaces and $f : R \to S$ be a C^2-mapping. If P is a point on R, $\tilde{P} = f(P) \in S$, ϕ

a local parameter on R defined near P and ψ a local parameter on S defined near $\tilde{P}$, then the map $w = h(z)$ defined by $h = \psi \circ f \circ \phi^{-1}|_V$ (V is a sufficiently small neighborhood of P) is called a local representer of f at P. The map f is called k-quasiregular if there is a constant $k \in (0,1)$ such that for every representer h, at every point of R, $|h_{\overline{z}}| \leq k|h_z|$.

Ahlfors-Schwarz lemma for harmonic-quasiregular maps

Let $\rho_0 = \sigma \circ f\,|p|$ and $K_0 = K_{\rho_0}$ the Gaussian curvature of ρ.

Using that $K_0 = K_S\,(1 - |\mu|^2)$ and Ahlfors-Schwarz lemma we can prove the following result .

Theorem 8. . *Let R be hyperbolic surfaces with Poincare metric densities $\lambda$and S be another with Poincare metric densities σ and let the Gaussian curvature of the metric $ds^2 = \rho(w)|dw|^2$ be uniformly bounded from above on S by the negative constant $-a$. Then any harmonic k-quasiregular map f from R into S decreases distances up to a constant depending only on a and k .*

3. Applications

Uniformly bounded maximal φ-disks, Bers space and harmonic maps
Let φ be an analytic function on the unit disk Δ. Then φ belongs to *Bers space* $Q = Q(\Delta)$ if

$$\operatorname{ess\,sup} \omega(z)^2 |\varphi(z)| < +\infty \ ,$$

where $\omega(z) = 1 - |z|^2$.

In this section we will give an uniform estimate of radius of maximal φ-disks of the Hopf differential of a quasiregular harmonic map with respect to strongly negatively curved metric (see below Theorem 9).As an application we show that the Hopf differential of a quasiregular harmonic map with respect to strongly negatively curved metric belongs to Bers space. First we define maximal φ-disks.

Maximal φ-disk. Let φ be an analytic function on the unit disk Δ and let z_0 be a regular point of φ, i.e. $\varphi(z_0) \neq 0$. Let Φ_0 be a single valued branch of

$$w = \Phi(z) = \int \sqrt{\varphi(z)}\, dz$$

near z_0, $\Phi(z_0) = 0$. There is a neighborhood U of z_0 which is mapped one-to-one conformally onto an open set V in the w-plane. We can assume, by restriction, that V is a disk around $w = 0$. The inverse Φ_0^{-1} is a conformal homeomorphism of V into Δ and evidently there is a largest open disk V_0 around $w = 0$ such that the analytic continuation of Φ_0^{-1} (which is still denoted by Φ_0^{-1}) is homeomorphic, and that $\Phi_0^{-1}(V_0) \subset \Delta$. The image $U_0 = \Phi_0^{-1}(V_0)$ is called the *maximal φ-disk* around z_0; its φ-radius (injectivity radius) r_0 is the Euclidean radius of V_0.

For the definition of φ-disks and a discussion of their important role in the theory of holomorphic quadratic differentials we refer the interested reader to Strebel's book [St].

Theorem 9. ([AMM]) *Let ρ be the metric on Δ with Gaussian curvature K uniformly bonded from above on Δ by the negative constant $-a$,and let f be a harmonic k-quasiregular map from Δ into itself with respect to the metric ρ. If $R = R_z$ is the radius of the maximal φ-disk around z, where $\varphi =$*Hopf(f)*, then R is bounded from above by the constant C which depends only on k and a.*

Proof. Let $R = R_z$ be the radius of the maximal φ-disk $U = U_z$ around $z \in \Delta$. Since f is k-qusiregular then $\tau \geq m$, where $m = \log \frac{1}{k}$. $m > 0$. Let $\zeta = \Phi(z)$ be the natural parameter in U and $\Phi(U) = V = B(0, R)$ With respect to the parameter ζ the Bochner formula takes the simple form

$$\boldsymbol{\Delta}\tau = -K \sinh \tau \, .$$

Since $K \leq -a$ and $\tau \geq m$, we conclude that

$$\boldsymbol{\Delta}\tau \geq \delta e^{\tau} \text{ on } V \tag{1}$$

where $\delta = \frac{a \sinh m}{e^m}$. Let $ds = \lambda(\zeta)|d\zeta|$, where $\lambda(\zeta) = \frac{2R}{R^2 - |\zeta|^2}$ is the hyperbolic metric on V and let $\tilde{\lambda}(\zeta) = \left(\frac{\delta}{2} e^{\tau(\zeta)}\right)^{\frac{1}{2}}$. From (1) we have for the Gaussian curvature of the metric $d\tilde{s} = \tilde{\lambda}(\zeta)|d\zeta|$ on V that $\tilde{K} \leq -1$ and then we can use the Ahlfors-Schwarz Lemma 1 (see also [Ah]) to obtain

$$\frac{\delta}{2k} \leq \tilde{\lambda}^2(\zeta) \leq \lambda^2(\zeta)\, . \tag{2}$$

Setting $\zeta = 0$ in (2) one obtains

$$R^2 \leq \frac{8k}{\delta}\, . \tag{3}$$

□

In [AMM],I. Anić, V. Marković and M. Mateljević characterize Bers space by means of maximal φ-disks. As an application, using Theorem 9 , they show that the Hopf differential of a quasiregular harmonic map with respect to strongly negatively curved metric belongs to Bers space. Also they give further sufficient or necessary conditions for a holomorphic function to belong to Bers space.

Let φ be a quadratic differential on a hyperbolic Riemann surface R with Poincaré metric $ds^2 = \rho(z)|dz|^2$. Let $p \in R$ and let z be a local parameter near p. We will define

$$\|\varphi\|(p) = \rho^{-1}(z(p))|\varphi(z(p))|.$$

We say that φ belongs to the *Bers space* of R (notation $Q(R)$) if $\|\varphi\|$ is a uniformly bounded function on R.

Theorem 10. ([AMM]) *Let R and S be hyperbolic surfaces with metric densities σ and ρ respectively and let the Gaussian curvature of the metric $ds^2 = \rho(w)|dw|^2$ be uniformly bounded from above on S by the negative constant $-a$. If f is a harmonic k-quasiregular map from R into S with Hopf differential φ, then $\varphi \in Q(R)$.*

Proof. Let $\tilde{f}$ be the lifting of f which maps Δ into itself and let $\tilde{\varphi}$ be the lifting of the quadratic differential φ. Let $\tilde{\rho}$ be the lifting of the density ρ. Since $\tilde{f}$ is harmonic with respect to the metric $\tilde{\rho}(\tilde{w})|d\tilde{w}|^2$ on Δ and k-quasiregular then, by Theorem 2 [AMM], $\tilde{\varphi} \in Q(\Delta)$. Hence $\varphi \in Q(R)$. □

4. Further results

There are many results related to subject of this note .We will mention only a few of them.
Yau [Ya2] proved the following generalization of Schwarz lemma .

Theorem 11. (*Yau*) . *Let M be a complete Kähler manifold with Ricci curvature bounded from below by a constant, and N be another Hermitian menifold with holomorphic bisectional curvature bounded from above by a negative constant . Then any holomorphic mapping f from M into N decrease distances up to a constant depending only on the curvature of M and N .*

Royden [Ro] improved the estimate in Yau theorem.

Theorem 12. (*Royden*) . *Let M be a complete Hermitian manifold with holomorphic sectional curvature bounded from below by a constant $k \le 0$, and N be another Hermitian menifold with holomorphic sectional curvature bounded from above by a negative constant $K < 0$. Assume either that M has Riemann sectional curvature bounded from below or that M is Kähler with holomorphic bisectional curvature bounded from below.Then any holomorphic mapping f from M into N satisfies*

$$\|df\|^2 \le \frac{k}{K} \ .$$

In [Ya2] , Yau mentioned that in order to draw a useful conclusion in the case of harmonic mappings between Riemannian manifolds, it seems that one has to assume the mapping is quasi-conformal.
Since we can consider Theorem 8 as a version of Schwarz lemma for harmonic-quasiregular maps between surfaces it seems natural to ask whether there exists a version of Yau-Royden theorem for harmonic-quasiregular maps.

Pseudoholomorphic version of the Schwarz Lemma (known as Gromov-Schwarz Lemma) is important tool in symplectic geometry .
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