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CONDITIONS FOR INVARIANT SUBMANIFOLD OF A MANIFOLD WITH THE (φ, ξ, η, G)-STRUCTURE

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Abstract. We shall investigate a necessary and sufficient condition for a submanifold immersed in an almost paracontact Riemannian manifold to be invariant and show further properties of invariant submanifold in a manifold with the (φ, ξ, η, G)-structure.

1. (φ, ξ, η, G)-structure

Let $\bar{\mathcal{M}}$ be an m-dimensional differentiable manifold. If there exist on $\bar{\mathcal{M}}$ a $(1,1)$-tensor field φ, a vector field ξ and a 1-form η satisfying

$$\eta(\xi) = 1, \qquad \varphi^2 = I - \eta \otimes \xi, \tag{1.1}$$

where I is the identity, then $\bar{\mathcal{M}}$ is said to be an almost paracontact manifold. In the almost paracontact manifold, the following relations hold good:

$$\varphi\xi = 0, \qquad \eta \circ \varphi = 0, \qquad \text{rank } (\varphi) = m - 1. \tag{1.2}$$

Every almost paracontact manifold has a positive definite Riemannian metric G such that

$$\eta(\bar{X}) = G(\xi, \bar{X}), G(\varphi\bar{X}, \varphi\bar{Y}) = G(\bar{X}, \bar{Y}) - \eta(\bar{X})\eta(\bar{Y}), \bar{X}, \bar{Y} \in \mathcal{X}(\bar{\mathcal{M}}), \tag{1.3}$$

where $\mathcal{X}(\bar{\mathcal{M}})$ denotes the set of differentiable vector fields on $\bar{\mathcal{M}}$. In this case, we say that $\bar{\mathcal{M}}$ has an almost paracontact Riemannian structure (φ, ξ, η, G) and $\bar{\mathcal{M}}$ is said to be an almost paracontact Riemannian manifold [1]. From (1.3) we can easily get the relation

$$G(\varphi\bar{X}, \bar{Y}) = G(\bar{X}, \varphi\bar{Y}). \tag{1.4}$$

Hereafter, we assume that $\bar{\mathcal{M}}$ is an almost paracontact Riemannian manifold with a structure (φ, ξ, η, G). It is clear that the eigenvalues of the matrix (φ) are 0 and ± 1, where the multiplicity of 0 is equal to 1.

Let $\mathcal{M}$ be an n-dimensional differentiable manifold $(m - n = s)$ and suppose that $\mathcal{M}$ is immersed in the almost paracontact Riemannian manifold $\bar{\mathcal{M}}$ by the immersion $i : \mathcal{M} \to \bar{\mathcal{M}}$. We denote by B the differential of the immersion i. The induced Riemannian metric g of $\mathcal{M}$ is given by $g(X, Y) = G(BX, BY)$, $X, Y \in \mathcal{X}(\mathcal{M})$, where $\mathcal{X}(\mathcal{M})$ is the set of differentiable vector fields on $\mathcal{M}$. We denote by $T_p(\mathcal{M})$ the tangent space of $\mathcal{M}$ at $p \in \mathcal{M}$, by $T_p(\mathcal{M})^\perp$ the normal space of $\mathcal{M}$ at p and by $\{N_1, N_2, \ldots, N_s\}$ an orthonormal basis of the normal space $T_p(\mathcal{M})^\perp$.

The transform φBX of $X \in T_p(\mathcal{M})$ by φ and φN_i of N_i by φ can be respectively written in the next forms:

$$\varphi BX = B\psi X + \sum_{i=1}^{s} u_i(X) N_i, \qquad X \in \mathcal{X}(\mathcal{M}), \tag{1.5}$$

$$\varphi N_i = BU_i + \sum_{i=1}^{s} \lambda_{ij} N_j, \tag{1.6}$$

where ψ, u_i, U_i and λ_{ij} are respectively a induced $(1,1)$-tensor, 1-forms, vector fields and functions on $\mathcal{M}$ and Latin indices h, i, j, k, l run over the range $\{1, 2, \ldots, s\}$. The vector field ξ can be expressed as follows:

$$\xi = BV + \sum_{i=1}^{s} \alpha_i N_i, \tag{1.7}$$

where V and α_i are respectively a vector field and functions on $\mathcal{M}$. Using (1.5) and (1.6) for $X, Y \in \mathcal{X}(\mathcal{M})$

$$g(\psi X, Y) = G(B\psi X, BY) = G(\varphi BX, BY) = G(BX, \varphi BY) = G(BX, B\psi Y).$$

Therefore we have $g(\psi X, Y) = g(X, \psi Y)$. Furthermore, from $G(\varphi BX, N_i) = G(BX, \varphi N_i)$ and $G(\varphi N_i, N_j) = G(N_i, \varphi N_j)$, we can respectively get the equations $u_i(X) = g(U_i, X)$, $\lambda_{ij} = \lambda_{ji}$.

Lemma 1.1. *In a submanifold $\mathcal{M}$ immersed in an almost paracontact Riemannian manifold $\bar{\mathcal{M}}$, the following equations hold good:*

$$\psi^2 X = X - v(X)V - \sum_{i=1}^{s} u_i(X)U_i \text{ or } \psi^2 = I - v\otimes V - \sum_{i=1}^{s} u_i \otimes U_i, \; X \in \mathcal{X}(\mathcal{M}), \tag{1.8}$$

$$u_j(\psi X) + \sum_{i=1}^{s} \lambda_{ji} u_i(X) + \alpha_j v(X) = 0, \tag{1.9}$$

$$\psi U_j + \sum_{i=1}^{s} \lambda_{ji} U_i + \alpha_j V = 0, \tag{1.10}$$

$$u_k(U_j) = \delta_{kj} - \alpha_k \alpha_j - \sum_{i=1}^{s} \lambda_{ki} \lambda_{ji}, \tag{1.11}$$

where v is a 1*-form on $\mathcal{M}$ and $v(X) = g(V, X)$.*

Proof. From (1.5), we have

$$\begin{aligned} \varphi^2 BX &= \varphi B \psi X + \sum_i u_i(X) \varphi N_i \\ &= B\psi^2 X + \sum_i u_i(\psi X) BU_i + \sum_i u_i(X) \sum_j \lambda_{ij} N_j. \end{aligned}$$

On the other hand, since we have

$$\varphi^2 BX = BX - \eta(BX)\xi = BX - v(X)i - v(X) \sum_j \alpha_j N_j,$$

we get (1.8) and (1.9). Similarly, from (1.6) we have (1.10) and (1.11).

Equations (1.9) and (1.10) are equivalent.

Lemma 1.2. *In a submanifold $\mathcal{M}$ immersed in an almost paracontact Riemannian manifold $\bar{\mathcal{M}}$, the following equations hold good:*

$$\psi V + \sum_{i=1}^{s} \alpha_i U_i = 0, \qquad u_i(V) + \sum_{j=1}^{s} \alpha_j \lambda_{ji} = 0, \tag{1.12}$$

$$v(V) = 1 - \sum_{i=1}^{s} \alpha_i^2, \tag{1.13}$$

$$g(\psi X, \psi Y) = g(X, Y) - v(X)v(Y) - \sum_{i=1}^{s} u_i(X) u_i(Y), \tag{1.14}$$

where $X, Y \in \mathcal{X}(\mathcal{M})$.

Proof. From (1.7),

$$\varphi\xi = \varphi BV + \sum_i \alpha_i \varphi N_i = B\psi V + \sum_i u_i(V)N_i + \sum_i \alpha_i(BU_i + \sum_j \lambda_{ij}N_j).$$

By means of $\varphi\xi = 0$, we have (1.12). Similarly, from $\eta(\xi) = 1$ we get (1.13). And using $G(\varphi BX, \varphi BY) = G(BX, BY) - \eta(BX)\eta(BY)$, we have (1.14).

Let $\{N_1, N_2, \ldots, N_s\}$ be an orthonormal basis of the normal space $T_p(\mathcal{M})^\perp$ at $p \in \mathcal{M}$ [1]. We assume that $\{\bar{N}_1, \bar{N}_2, \ldots, \bar{N}_s\}$ is the another orthonormal basis of $T_p(\mathcal{M})^\perp$ and we put

$$\bar{N}_i = \sum_{l=1}^{s} k_{li} N_l. \tag{1.15}$$

By means of $G(\bar{N}_i, \bar{N}_j) = \sum_{l=1}^{s} k_{li}k_{lj} = \delta_{ij}$, from which $\sum_{h=1}^{s} k_{ih}k_{jh} = \delta_{ij}$. Consequently a matrix (k_{ij}) is an orthogonal matrix. Thus from (1.15), we have $N_j = \sum_{l=1}^{s} k_{jl}\bar{N}_l$.

Making use of (1.15), equations (1.5), (1.6) and (1.7) are respectively written in the following form:

$$\varphi BX = B\psi X + \sum_{l=1}^{s} \bar{u}_l(X)\bar{N}_l,$$

$$\varphi \bar{N}_l = B\bar{U}_l + \sum_{h=1}^{s} \bar{\lambda}_{lh}\bar{N}_h, \tag{1.16}$$

$$\xi = BV + \sum_{l=1}^{s} \bar{\alpha}_l \bar{N}_l,$$

where

$$\bar{u}_l(X) = \sum_{i=1}^{s} k_{il}u_i(X), \quad \bar{U}_l = \sum_{i=1}^{s} k_{il}U_i, \tag{1.17}$$

$$\bar{\lambda}_{lh} = \sum_{i,j=1}^{s} k_{il}\lambda_{ij}k_{jh}, \quad \bar{\lambda}_{lh} = \bar{\lambda}_{hl}, \tag{1.18}$$

$$\bar{\alpha}_l = \sum_{i=1}^{s} k_{il}\alpha_i.$$

By virtue of (1.17), the linear independence of vectors $U_i (i = 1, 2, \ldots, s)$ is invariant under the transformation (1.15) of the orthonormal basis $\{N_1, N_2, \ldots, N_s\}$.

Furthermore, because λ_{ij} is symmetric in i and j, from (1.18) we can find that under a suitable transformation (1.15) λ_{ij} reduces to $\bar{\lambda}_{ij} = \lambda_i \delta_{ij}$, where $\lambda_i (i = 1, 2, \ldots, s)$ are eigenvalues of matrix (λ_{ij}). In this case, (1.16) and (1.11) are respectively written in the next forms:

$$\begin{aligned} \varphi \bar{N}_l &= B\bar{U}_l + \lambda_l \bar{N}_l, \\ \bar{u}_k(\bar{U}_j) &= \delta_{kj} - \bar{\alpha}_k \bar{\alpha}_j - \lambda_k \lambda_j \delta_{kj}, \end{aligned} \tag{1.19}$$

from which we have $\bar{u}_j(\bar{U}_j) = 1 - \bar{\alpha}_j^2 - \lambda_j^2$ and $\bar{u}_k(\bar{U}_j) = -\bar{\alpha}_k \bar{\alpha}_j (k \neq j)$.

2. Invariant submanifolds of an almost paracontact Riemannian manifold

Let $\mathcal{M}$ be a submanifold immersed in an almost paracontact Riemannian manifold $\bar{\mathcal{M}}$. If $\varphi(B(T_p(\mathcal{M}))) \subset T_p(\mathcal{M})$ for any point $p \in \mathcal{M}$, then $\mathcal{M}$ is called an invariant submanifold [4]. In an invariant submanifold $\mathcal{M}$, (1.5), (1.6) and (1.7) are respectively written in the following forms:

$$\varphi BX = B\psi X, \qquad X \in \mathcal{X}(\mathcal{M}), \tag{2.1}$$

$$\varphi N_i = \sum_{j=1}^{s} \lambda_{ij} N_j, \tag{2.2}$$

$$\xi = BV + \sum_{i=1}^{s} \alpha_i N_i. \tag{2.3}$$

Furthermore, from Lemma 1.1 and Lemma 1.2, we have the following lemmas.

Lemma 2.1. *In an invariant submanifold $\mathcal{M}$ immersed in an almost paracontact Riemannian manifold $\bar{\mathcal{M}}$, the following equations hold good:*

$$\psi^2 = I - v \otimes V, \tag{2.4}$$

$$\alpha_i V = 0, \tag{2.5}$$

$$\delta_{kj} - \alpha_k \alpha_j - \sum_{i=1}^{s} \lambda_{ki} \lambda_{ji} = 0, \tag{2.6}$$

$$\psi V = 0, \tag{2.7}$$

$$\sum_{i=1}^{s} \alpha_i \lambda_{ij} = 0, \tag{2.8}$$

$$v(V) = 1 - \sum_{i=1}^{s} \alpha_i^2, \tag{2.9}$$

$$g(\psi X, \psi Y) = g(X, Y) - v(X)v(Y), \qquad X, Y \in \mathcal{X}(\mathcal{M}). \tag{2.10}$$

From (2.5) and (2.9), we get the following two cases:

When $V = 0$ (or $\sum_i \alpha_i^2 = 1$), that is, ξ is normal to $\mathcal{M}$, since from (2.4) and (2.10) we have $\psi^2 = I$, $g(\psi X, \psi Y) = g(X, Y)$, (ψ, g) is an almost product Riemannian structure whenever ψ is non-trivial.

When $V \neq 0$ (or $\alpha_i = 0$), that is, ξ is tangent to $\mathcal{M}$, by means of (2.4), (2.9), (2.10) and $v(X) = g(V, X)$, (ψ, V, v, g) is an almost paracontact Riemannian structure. From [2], [3] we have

Theorem 2.1. *Let $\mathcal{M}$ be an invariant submanifold immersed in an almost paracontact Riemannian manifold $\mathcal{M}$ with a structure (φ, ξ, η, G). Then one of the following cases occurs.*

1.) ξ is normal to $\mathcal{M}$. In this case, the induced structure (ψ, g) on $\mathcal{M}$ is an almost product Riemannian structure whenever ψ is non-trivial.

2.) ξ is tangent to $\mathcal{M}$. In this case, the induced structure (ψ, V, v, g) is an almost paracontact Riemannian structure.

Furthermore, we have the following theorems.

Theorem 2.2. *In order that, in an almost paracontact Riemannian manifold $\bar{\mathcal{M}}$ with a structure (φ, ξ, η, G), the submanifold $\mathcal{M}$ of $\bar{\mathcal{M}}$ is invariant, it is necessary and sufficient that the induced structure (ψ, g) on $\mathcal{M}$ is an almost product Riemannian structure whenever ψ is non-trivial or the induced structure (ψ, V, v, g) on $\mathcal{M}$ is an almost paracontact Riemannian structure.*

Proof. From Theorem 2.1, the necessity is evident. Conversely, we first assume that the induced structure (ψ, g) is an almost product Riemannian structure. Then from (1.8) we have $v(X)V + \sum_i u_i(X)U_i = 0$, from which $g(v(X)V + \sum_i u_i(X)U_i, X) = 0$, that is, $v(X)^2 + \sum_i u_i(X)^2 = 0$. Consequently, since we get $v(X) = u_i(X) = 0$ $(i = 1, 2, \ldots, s)$, the submanifold $\mathcal{M}$ is invariant and ξ is normal to $\mathcal{M}$.

Next, we assume that the induced structure (ψ, V, v, g) is an almost paracontact Riemannian structure. Then from (1.8) we have $\sum_i u_i(X)U_i = 0$, from which $u_i(X) = 0$ $(i = 1, 2, \ldots, s)$ and from (1.9) we get $\alpha_i = 0$. Thus $\mathcal{M}$ is invariant and ξ is tangent to $\mathcal{M}$.

Theorem 2.3. *In order that, in an almost paracontact Riemannian manifold $\bar{\mathcal{M}}$ with a structure (φ, ξ, η, G), the submanifold $\mathcal{M}$ of $\bar{\mathcal{M}}$ is invariant, it is necessary and sufficient that the normal space $T_p(\mathcal{M})^{\perp}$ at every point $p \in \mathcal{M}$ admits an orthonormal basis consisting of eigenvectors of the matrix (φ).*

Proof. We assume that $\mathcal{M}$ is invariant.

1.) When ξ is normal to $\mathcal{M}$, at $p \in \mathcal{M}$ we consider an s-dimensional vector space $\tilde{W}$ and investigate the eigenvalues of the (s, s)-matrix (λ_{ij}).

By means of (2.8) and (2.9), it is clear that the vector $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ of the vector space $\tilde{W}$ is a unit eigenvector of the matrix (λ_{ij}) and its eigenvalue is equal to 0.

Next, suppose that a vector $(w_1, w_2, \ldots, w_s)$ satisfying $\sum_i \alpha_i w_i = 0$ is an eigenvector and its eignevalue is λ. Then we have $\sum_j \lambda_{ji} w_j = \lambda w_i$. ¿From this equation, we get $\sum_{i,j} \lambda_{ki}\lambda_{ji} w_j = \lambda \sum_i \lambda_{ki} w_i$, from which $\sum_j (\sum_i \lambda_{ki}\lambda_{ji}) w_j = \lambda^2 w_k$. Using (2.6), we have $\sum_j (\delta_{kj} - \alpha_k\alpha_j) w_j = \lambda^2 w_k$, that is, $w_k = \lambda^2 w_k$. Then we get $\lambda^2 = 1$.

Consequently, if by a suitable transformation of the orthonormal basis $\{N_1, N_2, \ldots, N_s)\}$ of $T_p(\mathcal{M})^{\perp}$, the matrix (λ_{ij}) becomes a diagonal matrix, then the diagonal components $\lambda_1, \lambda_2, \ldots, \lambda_s$ satisfy relations

$$\lambda_1^2 = \lambda_2^2 = \ldots = \lambda_{s-1}^2, \quad \lambda_s = 0.$$

In this case, if we denote by $\{\bar{N}_1, \bar{N}_2, \ldots, \bar{N}_s\}$ the orthonormal basis of $T_p(\mathcal{M})^{\perp}$, then from (1.18) we have $\varphi \bar{N}_l = \lambda_l \bar{N}_l$. Therefore, $\bar{N}_l (l = 1, 2, \ldots, s)$ are eigenvectors of the matrix (φ) and $\bar{N}_s = \xi$.

2.) When ξ is tangent to $\mathcal{M}$, from (2.6) we have $\sum_i \lambda_{ki}\lambda_{ji} = \delta_{kj}$. If we denote by $(w_1, w_2, \ldots, w_s)$ an eigenvector of matrix (λ_{ij}) and by λ its eigenvalue, then we have $\sum_j \lambda_{ji} w_j = \lambda w_i$. Consequently, we have $\sum_{i,j} \lambda_{ki}\lambda_{ji} w_j = \lambda \sum_i \lambda_{ki} w_i$, that is, $w_k = \lambda^2 w_k$, from which we get $\lambda^2 = 1$. Thus since the eigenvalues of (λ_{ij}) are ± 1, if by a suitable transformation of the orthonormal basis of $T_p(\mathcal{M})^{\perp}$, $\{N_1, N_2, \ldots, N_s\}$ becomes $\{\bar{N}_1, \bar{N}_2, \ldots, \bar{N}_s\}$, then $\bar{N}_1, \bar{N}_2, \ldots, \bar{N}_s$ are eigenvectors of matrix (φ).

Conversely, if the orthonormal basis $\{\bar{N}_1, \bar{N}_2, \ldots, \bar{N}_s\}$ of $T_p(\mathcal{M})^{\perp}$ consists of eigenvectors of (φ) and these eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_s$ satisfy $\lambda_1^2 = \lambda_2^2 = \ldots = \lambda_{s-1}^2 = 1$, $\lambda_s = \pm 1$ or 0, then we have $\varphi \bar{N}_l = \lambda_l \bar{N}_l$. Consequently, since we have $\bar{U}_l = 0$, $\mathcal{M}$ is invariant.

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